Plates and Shells

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Module-08
Lecture-22
Shell as Stressed Skin

Hello everybody, today I am starting module 8 and now from module 8 onwards I will be delivering the lectures on the topic of shells. So, I have covered the plates whatever is required and there are many advanced topic also in the plate and those who will be doing research in future they can take the course of advanced topics such as vibration of plates then push buckling behaviour of plates, composite plates etc.

And but the courses or the topics that I have delivered in the portion plates will be useful to follow the advanced topics. So, today introducing to you the shell structure and the first lecture will be devoted to the characteristics of the shell which can be taken as a stress scheme.

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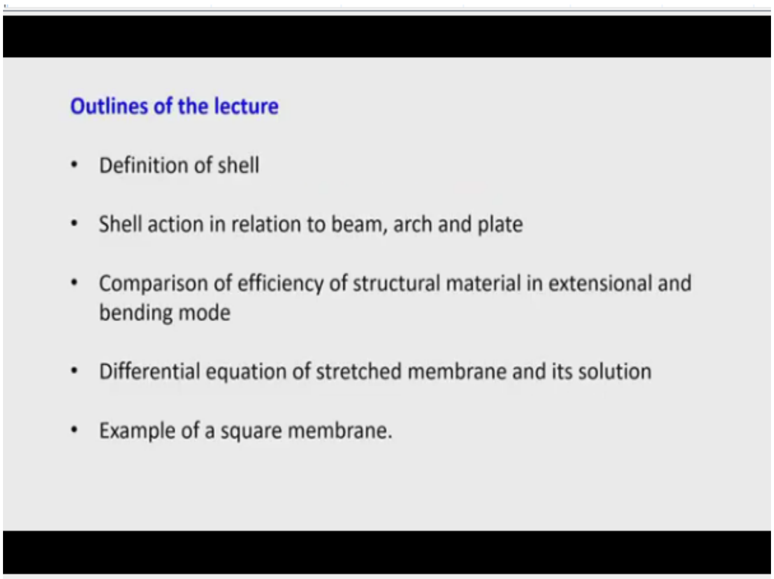

Now, outlines of this lecture is first we will define the shell; then shell action I will discuss in relation to being, arch and plate. Then I will discuss one interesting thing that how the structural material can be efficiently utilized in various member in extensional or bending mode or combined mode. So, you will get a glimpse of the structure which are most efficient in utilization of the material.

Because the shell which you know this is a thin structure that I have told you during my introductory lecture introduction in the first beginning of the module, there I have mentioned that shell is a thin structure and thin structure generally the material consumption is less. Then I will give you a differential equation of the stretched membrane because this is a very classical equation in our mathematical physics or mechanics.

And as a learners of this course plates and shell we should not miss this membrane equation. So, I will discuss this and his solution method. Then I will take an example of square membrane to illustrate the solution of the differential equation.

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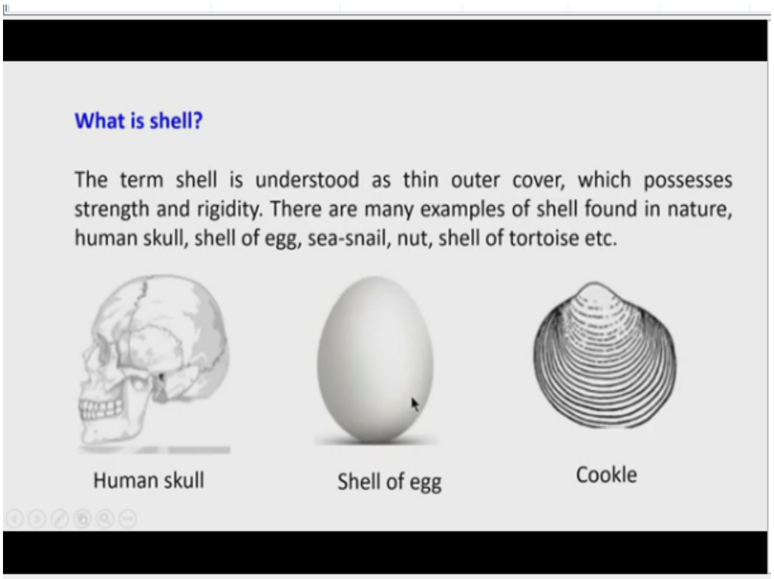

Now, you can see the term shell is generally related to a very hard covered, thin covered cover and which possesses strength and rigidity. So, in nature there are so many examples of shell. So, for example, our human skull is also a shell. Then shell of an egg this is hard covered and very thin. So, actually the shell is protecting the inner material. So, this is also a behaving like a shell.

Then some marine creature known as cookle is also having a hard outer cover which is thin and which is protecting the inner soft core. So, therefore, shell is understood as a hard thin cover, outer cover which passages strengthen rigidity. So, from this characteristic we can see there are many engineering applications of the shell specially these roofs covering that we want for covering a large area.

The shell is frequently used in there are also application of shell in pressure vessels, pipe then tank or even the space vehicles and then many other examples are there where shell is used. And this utilization of material in shell is very efficient because it is very thin and acts like a stressed skin.

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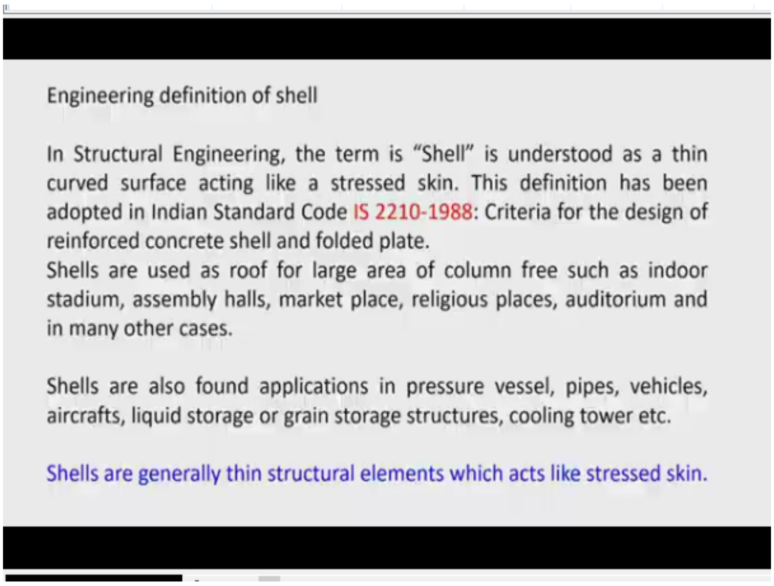

So, if I tell you the engineering definition of the shell, the term shell is understood as a thin curved surface acting like a stress skin. So, that is the engineering definition and this definition has been adopted in Indian Standard Code IS 2210-1988. This code is actually the code for the design of reinforced concrete shell and folded plate. Now, in engineering application shell can be constructed with various materials such as reinforced concrete or steel, but this code is specially devoted to the reinforced concrete shell.

Now, shell are generally used as roof in our civil engineering construction practice for large area which must be column free and also sometimes for aesthetic reason the architecture prefers shelf in large areas for example, in stadium or assembly halls, marketplace, religious places, auditorium, in many other cases, the roof in the form of shell of various geometry gives a pleasing appearance. So, therefore, shell has various application in civil engineering construction, but, it is not limited only in the civil engineering field.

Shells are also found applications in pressure vessels, pipes, vehicles, aircrafts, liquid storage or grain storage structures cooling tower etc. So, there are various application and people have seen that shell has very efficiently performing or very efficient in assisting the load specially the sulphate and other lab load. And therefore, it is used for construction of various important structures. So, in nutshell we can tell that shells are generally thin structural elements which acts like a stressed skin.

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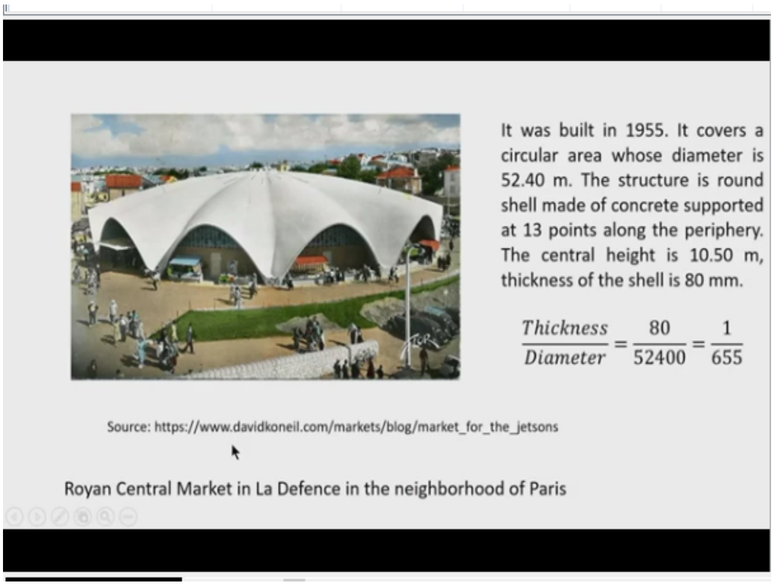

Now, let us see an example, it is a very well known example of shell construction that **is** is Royan Central Market near Paris in the place La Defence, it is in the neighbourhood of the Paris 3 kilometer distance from Paris. It was built in 1955 by renowned architectures and engineers. Now, this shell roof you are seeing here covers a area whose diameter is 52.4 meter. So, it is covering an area of 52.4 meter and structure is round shell, this thing you can see the thin structure, the thickness is very small.

And it is supported on periphery by 13 points by pillars or by short columns; the central height of the shell is 10.5 meter. So, it is almost a 3 storey building height the central height and thickness of the shell is only 80 millimeter in compared to its diameter 52.4. So, if we see the thickness diameter ratio you can see is thickness diameter ratio is very small $1/655$. So, this is the advantage of shell construction where it is used as the obstruction fee structure.

That means, in a marketplace or assembly halls or auditorium or even in the stadium, when we want that intermediate pillars should be avoided then we go for shell construction because in that case a thin structural element can be used to cover large spin. So, this example is a very renowned example of shell construction and civil engineering practice and it is actually Royan Central Market near Paris.

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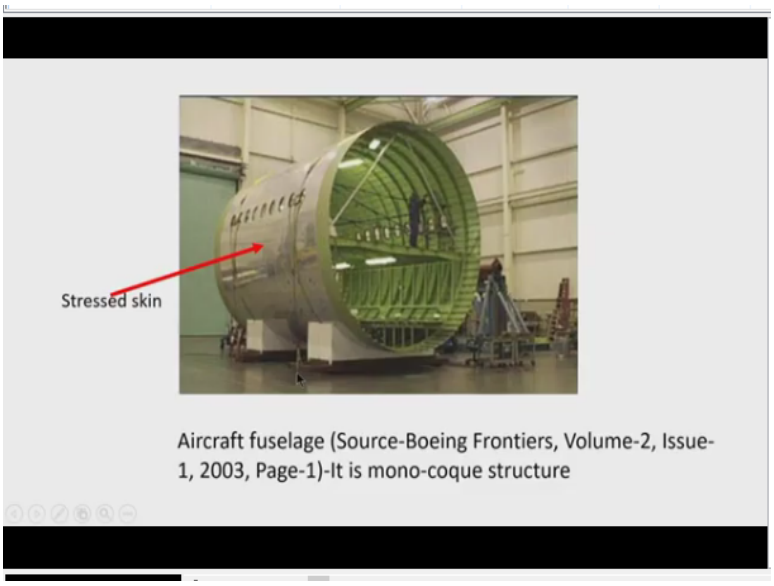

Now you can see other application of the shell, say aircraft, aircraft fuselage, the main body of the aircraft where passengers and cargo are housed. This is you can see it is a round shell. So, it is a closed shell and it acts like a stress skin, these aircraft fuselages. So, the construction of many important structure in space vehicle are also done with the help of thin elements like shell. So, therefore, shell has gained importance among the practicing engineers and designers and among researcher also because the shell analysis is complicated because of the presence of curvature.

So, that we will do here, full course will be devoted to the stresses which is producing the shell under the action of the load and therefore, we have to deal with the curve structure, but it is thin. So, this aircraft fuselage that you are seeing here, it is a mono-coque structure and it is manufactured by Boeing Company. So, you can see the vivid example of shell.

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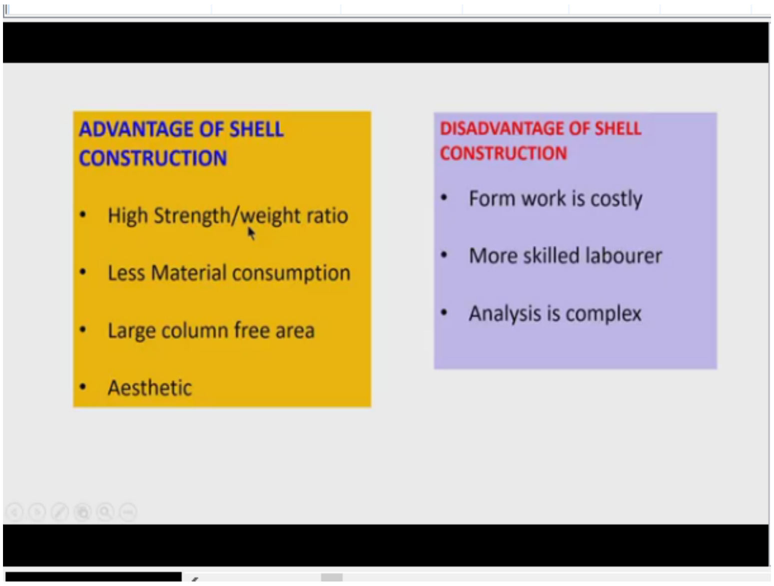

Now, if we see the advantages and disadvantages of the shell. First the high strength weight ratio is achieved in case of shell because of the reduction of thickness, the self weight becomes less. Whereas the strength and rigidity is high even the thickness is small. So, that is one important aspect of shell structures. Strength weight ratio is very high therefore, it is preferably used in the aircraft structure weight reduction is prime consideration.

Then, other advantages if we see, the less material consumption is there in case of shell, less material consumption**,** I mean that because of thickness suppose for example, a concrete shell or reinforced concrete shell which is covering the 52.4 meter span in diametrical direction then you can see that thickness requirement is only 8 centimeter. Whereas, if I construct a single span beam of 52.4 meter then simple reinforced concrete construction is prohibited because the thickness requirement will be very large.

So, in that case we can actually adopt the other type of structure that is the suspended roofs or these composite steel plate girder construction that can be also used, but material consumption in case of shell is very less and therefore, it is preferred, it becomes economy and large column free area. That is one advantage that you have seen here in the marketplace where no column is there inside and it is supported on the periphery only.

So, the users that is people will not be obstructed by intermediate column. So, that is one important aspect of the shell construction that is large column free area. Then aesthetic; aesthetic is important because it attracts many people and therefore, the aesthetic is achieved, aesthetic sense is very high in case of shell construction by giving suitable form of construction specially the curved construction.

Now, these advantages of shells are also there because the curvature that is sometimes very complicated you may have a single curve type of shell or you may have a double curve type of shell. So, therefore, formwork is very costly because it depends on the nature of the curve sometimes we want part of a circle, sometimes you want a parabola, sometimes occasionally.

So, to give a proper shape of the curve the formwork should be accurate and it is a costly, generally it is a costly thing. Then more skilled labourer are required for construction of shell to maintain the curvature and geometry. And the another important thing is that analysis is complex. Although these several advantages they are in construction, but analysis is complex compared to the flat structure.

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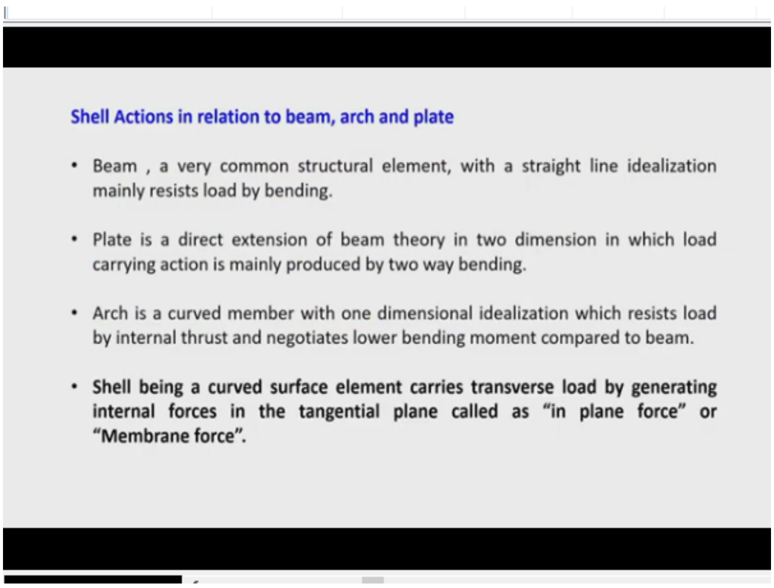

Now shall action let us see, in relation to the beam, arch and plate? Now beam actually it is a very common structural element, it is generally horizontal member with a straight line idealization that we can do in structural analysis and whenever beam is loaded in transverse direction then it raises the load mainly by bending. The normal thirst and other actions are minimal in case of beam.

Now, if we see the plate that we have covered up to the 7th module, you have found that plate is nothing but direct extension of the beam theory and in two dimension the plate equations are described and plate stresses resultant etc dependent on the 2 variables because it is a plane stress problem. Now, in which load carrying action is also produced by 2 way bending.

Now, because of 2 way of bending the plate is also economical compared to beam and this is seen in our analysis also earlier. Now, if we tell about the arch, now, what is arch actually? Arch is a carbon bar with one dimensional idealization that is there are various forms of arch there may be fixed arch were fixed at the 2 ends to supports, there may be 2 hinge arch that may hint that the 2 supports.

There are maybe 3 hinge hours that is hinge that the supports and another hinge at the intermediate point and the arch is also requiring some geometry because the shape of the arch maybe your parabolic or it may be a part of a circle or any other curve and because of the curvature of the member in this direction in the plane these internal tasks are generated because when a arch is loaded particularly downwards live load and dead load whatever may be the arch will try to spread, the member will try to spend.

But this is prevented by generating a reaction component at the support. So, this reaction component that is generated at the support is nothing but horizontal tasks. And because of that, this bending moment in the arch is reduced by some extent depending on this raise of the arch that is the height of the arch member with reference to the datum. So, that is the advantage of the arch construction that it negotiates lower bending moment and sometimes the shape of the arch can be chosen in such a way that it is subjected to only internal thirst.

For example, if I give a 3 hinge parabolic arch or an example of 2 hinge parabolic arch which is loaded by uniformly distributed load throughout the span then we have seen that bending moment at any point of the arch is 0 and therefore, arch is subjected to only mainly the internal action of the task that is the whatever thrust is produced due to support reaction, it will be transmitted at the other sections.

So, in relation to this shell can be treated as a curved surface because the arch is a line element, beam is a line element, but plate is a surface element. So, shell and plate, improved version of plate is shell. So, shell being a curved surface element carries transverse load by generating internal forces in the tangential plane called as in plane force or membrane forces.

Now, in the plate also the membrane forces generated when the plate is subjected to compression or extension in a addition to transverse loading. That is also possible but one difference between plate and shell is that the plate is a flat structure and shell is a curved surface. So, the plate also acts producing the moment of resistance in the 2 directions. Now, in the case of shell, because it is a thin structural element, the loads are mainly resisted by membrane action.

That is in plane forces are generated and that is the system. Bending moment is minimum in case of shell and only it is seen that when the shell joins a support, near the support the bending moment is predominant or significant, but away from the support the bending moment action of the bending is very negligible and in that case the load is assisted by in plane force or membrane force. So, that is the characteristics of the shell.

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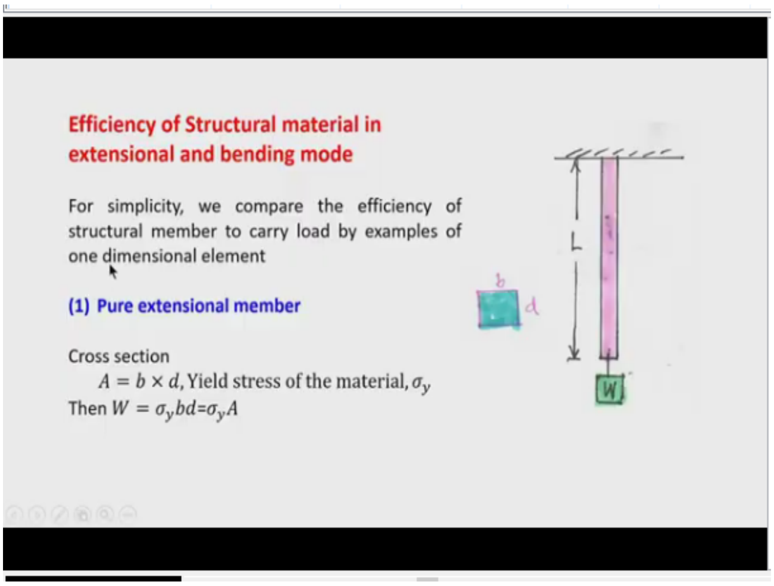

Now, if we compare the efficiency of the structural material in different modes that is one mode maybe your extensional mode or another mode maybe bending mode. Extensional mode means here the structure is this acted by a load which is along the axis. So, this is we call extensional mode and if load is in the opposite direction, that means it is compressing the structure then it is a compressional mode.

Now, in that case you are seeing a bar which is acted on by a load W and at the end, length is L will produce a stress whatever stress in the cross section say it is a σ . Now, at the L point for example, it is a steel member at the L point this dress is considered as σ_y and that will consider for example, without factor of safety, I am considering this as the limit for comparing the efficiency of the material.

So, in that case we can see $W = \sigma_y bd = \sigma_y A$, W is the load that is resisted by the structure. σ_y is the yield stress of the material and b is the width of the section and d is the depth of the section. So, course section I have taken for example, for illustrating this case is a rectangular. So, ultimately these $\frac{bd^3}{12}$. So, this is pure extensional member.

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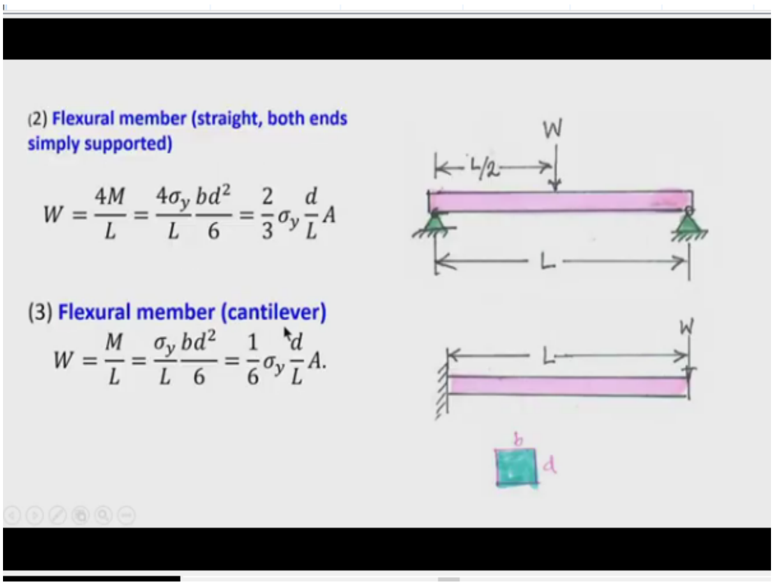

Now, next let us see the flexural member. In the flexural member, I am illustrating 2 cases one case is your simply supported beam subjected to a vertical load W at the center. So, in that case, we can see that maximum bending moment is produced at the center. So, now, if we want the bending moment M, bending moment M in terms of W it can be written $M = \frac{WL}{4}$, where W is the load, L is the span of the beam and 4 is a factor.

So, how it comes W L by 4, the reaction is W/2 and W/2 and if we take moment at the center, it will be $\frac{W}{2} \times \frac{L}{2}$ that is the bending moment at the center. So, $\frac{W}{2} \times \frac{L}{2}$ is $\frac{WL}{4}$. So, maximum bending moment is $\frac{WL}{4}$ and therefore, the maximum stress we can calculate M/z. So, now, W is nothing but $\frac{4M}{L}$. So, M is $\sigma_y bd^2/6$ and $bd^2/6$ is the section modulus of the cross section.

Cross section is a rectangular. So, moment of inertia is $\frac{bd^3}{12}$ and if you divide it by d/2, it becomes $bd^2/6$ which is known as section modulus. So, section modulus multiplied by the maximum stress will give you the bending moment M. So, W is expressed as $\frac{4\sigma_y}{L}\frac{bd^2}{6}$. After simplification it becomes $\frac{2}{3}\sigma_y\frac{d}{L}A$
.

The simple examples I have taken only to compare the efficiency of the structure and materials. That is why they extensional mode is giving you the economic option for efficient utilization of the material. Now, if I see the flexural member which is one end is free and other end is fixed that is a cantilever type of construction and here we can see the maximum bending moment occurs at a fixed end that is the root of the cantilever.

So, here maximum bending moment becomes W×L and therefore, this W can be written as M/L, where M is the maximum bending moment. Again denoting this section modulus as $bd^2/6$ and σ_y is the maximum stress then we can write that W load carrying capacity of this cantilever beam is $\frac{\sigma_y}{L}\frac{bd^2}{6}$, where the load W is acting at the free end that is the tip. So, after simplification or arranging this term we can write $W = \frac{1}{6}\sigma_y \frac{d}{L} A$

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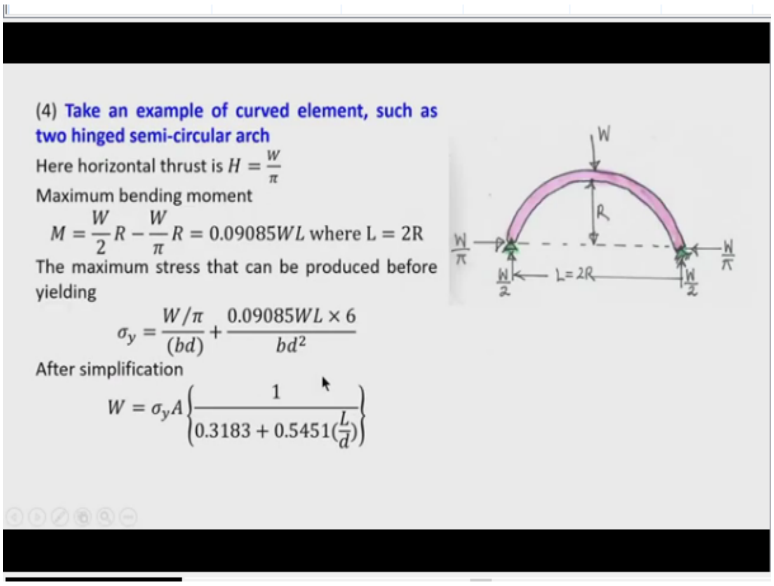

Now, let us take a curved member that is what if I told you the arch element because from beam element we now extend these to plate element. So, arch element we now extend these to shell element because various shell theories in a simplified way can be visited by the arch section, so, that we will see later. So, therefore, arch section let us see what happens. Now, it is a semicircular arch and subjected to a concentrated load at the crown and it is hinged at both ends.

So, therefore, the vertical reaction and the support will be $\frac{W}{2}$ and here also $\frac{W}{2}$ and it is a semicircular arch. So, span is 2×R, where R is the radius of the arch. Now bending moment at the crown that will be the maximum bending moment and if we calculate the bending moment at the crown then we get $\frac{W}{2}$ × R, that is the horizontal distance $\frac{W}{2}$ ×R - horizontal reaction into this distance.

Now, horizontal reaction for such type of arch is calculated and I am just quoting the result from the textbook of structural analysis. This is $\frac{W}{\pi}$ and this result can be obtained by any of the known methods say Castigliano's theorem you can apply here to find out the horizontal reaction $\frac{W}{\pi}$ because it is a single degree indeterminate structure. So, it can be very easily solved for unknown reaction here, H is the unknown reaction here because of symmetry we immediately got the particle reaction as $\frac{W}{2}$.

So, maximum bending moment is $M = \frac{W}{2}R - \frac{W}{\pi}R$. So, after simplification I am putting R = L/2 here, we ultimately get this term M = 0.09085 W L. Now, this is the bending moment of the semicircular arch to hinge subjected to a concentrated load at the crown. So, maximum stress that can be produced before yielding.

Now, this arch is subjected to, at any cross section if you see, it is subjected to a normal thrush as well as the bending moment. Now, if I go to the crown section the thrush is $\frac{W}{\pi}$ that is the horizontal thrush because the normal at the crown section is also horizontal. So, therefore, the direct stress in the section, this at the crown it will be $\frac{W}{\pi}$ × area, area was b/d because we have taken a rectangular cross section plus the stress produced due to bending. Now bending moment is this maximum bending moment.

So, maximum bending moment we have substituted here 0.09085 WL divided by section modulus. Section modulus because it is a rectangular section, so section modulus is $bd^2/6$, so 6 goes into numerator. After simplification and arranging this term we now obtain

$$W = \sigma_y A\left\{\frac{1}{0.3183+0.5451(\frac{L}{d})}\right\}$$

where L is the span of the arch and d is the depth of the section. Now generally span is large compared to the depth of the section. So therefore, L/d ratio is generally large, because d is small compared to L.

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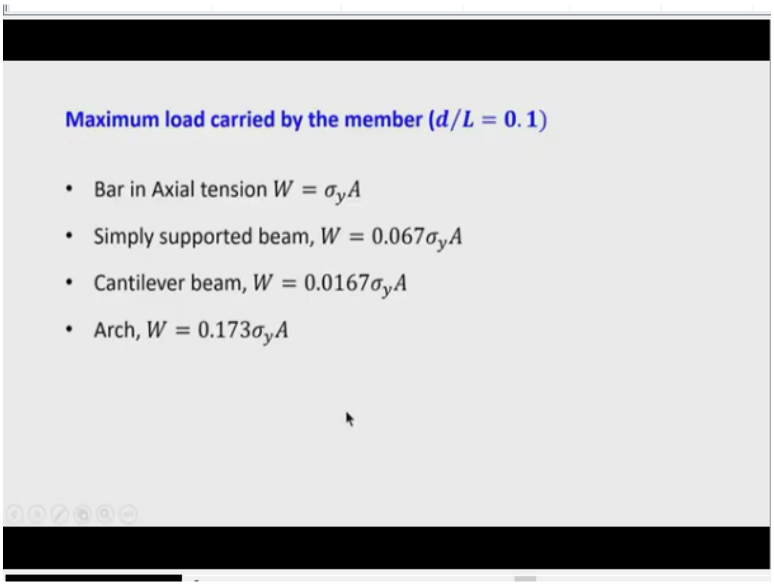

Now, let us take this d/L as 0.1, then if we compare the load carrying capacity of different members. Say, first example say W, we can see the load carrying capacity of the member is 0.67× d/L (d/L taken as 0.1). Hence 0.67×0.1 and area is b/d. So, ultimately we have tabulated this value that for this flexural member simply supported beam. We have tabulated as $\frac{n}{2a}$.

Now, for bar in axial tension that is our first element we got directly that $W = \sigma_y A$. So, therefore, this $W = \sigma_y A$. Cantilever beam we have got say $W = \frac{1}{6}\sigma_y \frac{d}{L} A$.

$\frac{d}{L}$ now we put as 0.1 and A is the area of the cross section. So, we get $\frac{1}{6}\times 0.1$. That means, $W = 0.0167\sigma_y A$

And in case of arch after substituting this d/L was 0.1. So, L by d is 10.

So, if I substitute this and calculate this factor then it becomes for arch $W = 0.173\sigma_y A$. Now comparing all these load values we can see that material is efficiently utilized. That means material is utilized to the maximum extent with the same cross section of the member and same length when it is the material is in the extensional mode or it is under in axial tension.

So, that conclusion we have obtained the most efficient utilization of the material is in the member which is under the axial tension. Next followed by the simply supported beam which is $W = 0.0167\sigma_y A$. Next followed by arch, arch will be $W = 0.173\sigma_y A$. Then we get the load carrying capacity for simply supported beam in the third position that is $W = 0.067\sigma_y A$.

And lowest is the hanging structure that is the cantilever, we get $W = 0.0167\sigma_y A$. So, from this simple analysis we can see that the maximum load for the same cross section and same material and these same member length or dimension can be achieved in case of the member which is subjected to only in the extensional mode.

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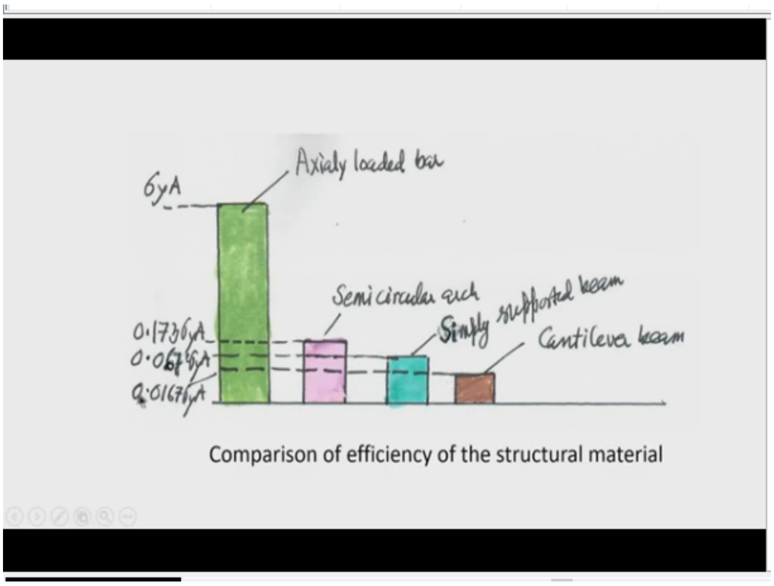

So, if we compare this pictorially then we can see the actual loaded memory is the highest in efficient utilization of the material. Then we have the arch; in the present case we have taken an example of semi circular arch. So, the capacity is $0.173\sigma_y A$. Then this simply supported beam the capacity is $0.067\sigma_y A$. Then we have the lowest is the cantilever beam $0.0167\sigma_y A$. So, this example actually is taken to show you that the extensional behaviour of the structure gives additional advantage to produce the in-plane forces and which is mostly utilized in the structure with the thin element.

So, this is possible with the help of shell structure. So, therefore, the behaviour of shell is like a stress skin and if this skin this membrane you can consider it as a membrane and it is stressed or it is acted upon by tension in uniformly in all direction for example. Then this whatever load you apply on the membrane this will be resisted by tension. So, analogous thing is the string or the cable in one dimensional element.

Therefore, I want to give you the analysis of stressed memory which is very interesting thing and equation is a very popular equation in mathematical physics or you can tell it in the structural mechanics. So, we should not miss this equation to know further analysis of the shell.

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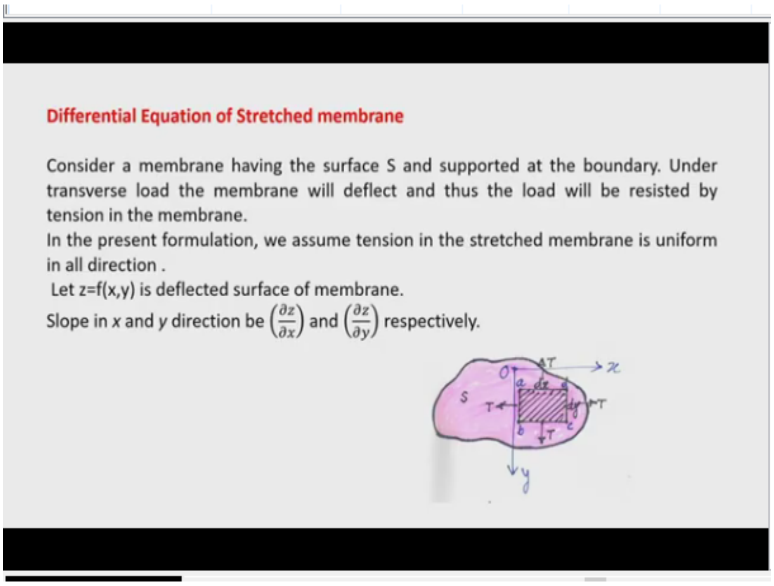

Now, let us see first the differential equation of a stress membrane. Say this is a stress membrane which has any geometry and S is the surface. Now here you can see I have taken O as the origin here and direction of x axis is here along the positive direction of x axis and along the positive direction of y axis; 2 lines are drawn and we assume that tension in the membrane is same in all directions. So, here attention is T, here also it is T in the direction along the y axis it is also T and in the end it is uniform. So, slope of the deflected surface of the membrane in x direction is $\left(\frac{\partial z}{\partial x}\right)$ and $\left(\frac{\partial z}{\partial y}\right)$.

What is z? Z is the function of x,y , is the deflected surface of the membrane. So, in the present formulation we take the tension in the stress membrane is uniform, but it is not necessary that tension must be uniform. If tension is taken non uniform then of course, the differential equation will be completed and solution will also be completed. But for simplicity let us discuss the case with a uniform tension in the membrane.

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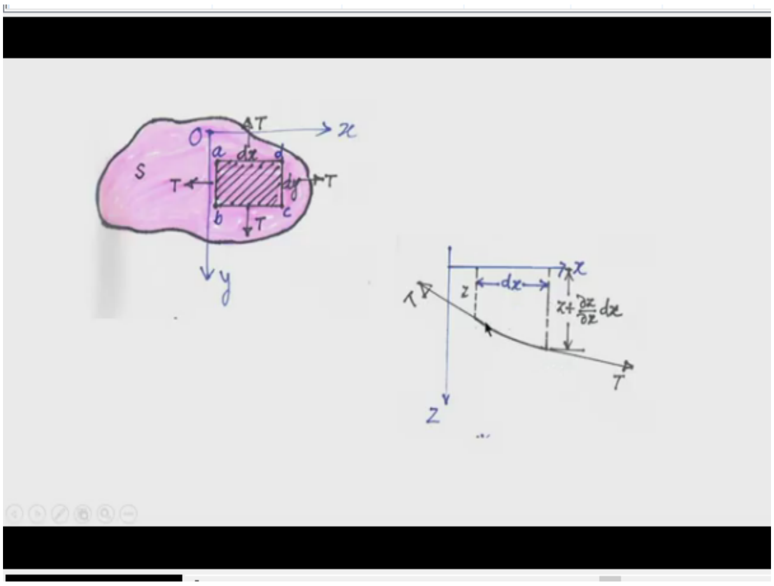

So, here is the membrane that you are seeing and suppose a deflected line is shown here and at this point these length is dx parallel to x axis. So, I have taken a section or a line along the x axis, so this is the surface or the line that you are seeing, it is a curved line actually and this is the elemental length in the projected plane is dx and z is the deflection at one end, at the other end the slope will be different.

So, therefore, this deflected value or deflected position at this point will be $Z + \frac{\partial z}{\partial x} dx$. This is the differential increment of the deflection. So, these are 2 deflection at the 2 points of the small element dx.

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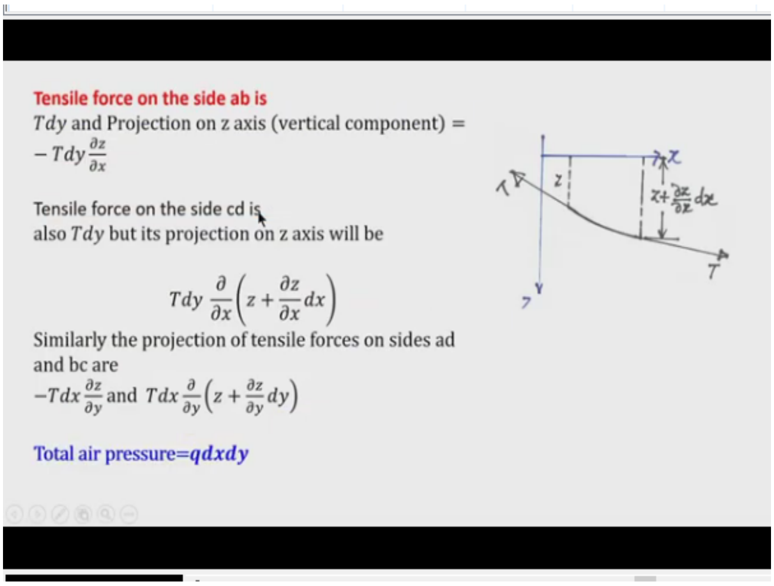

Now, if I resolve the tension in the vertical directions because we have taken this sides are say a, b, along the sides a, b total tension is T × dy, we are taking the tension is uniformly distributed and it is expressed as a tension per unit length. So, here if I say the tension on this side a, b then it is T×dy is the total tension and if our positive direction of the z axis is vertically downward then this component along the vertical direction or its projection on the z axis, z axis is vertical.

Projection on the z axis because z axis here we have taken positive downward. So, therefore, negative sign appears here (39:06). So, its component will be $-Tdy\frac{\partial z}{\partial x}$. $\frac{\partial z}{\partial x}$ is the angle. So, actually this $T\times\sin\theta$ theta but since θ is small, the slope is small we can take $\sin\theta$ or sin of slope equal to slope only. So, $-Tdy\frac{\partial z}{\partial x}$ is the projection of the tension Tdy on the side a, b; here we got it.

Now, on the other end there is at a distance dx, the tensile side is cd, side is cd and there also the tension is uniform, but you are seeing here that the direction of the tension is reversed. Now if I again project it the tensile force T×dy on the vertical axis z. Then this angle is your slope. So, then we get this, this $Tdy\dfrac{\partial}{\partial x}\left(z+\dfrac{\partial z}{\partial x}dx\right)$. So, this is the projection of this Tdy on the other end of this element of this side.

This is a, b and this is on this side cd. Similarly projection of the tensile force on the sides ad and bc, what are the sides, ad is the site, which is along the x axis. So, T is total tension along the side ad is T×dx. Similarly total tension on the side bc is T×dx. So, what I get here, the projection of the tensile force on the sides, ad and bc will be $-Tdx\dfrac{\partial z}{\partial y}$. And it will be $Tdx\dfrac{\partial}{\partial y}\left(z+\dfrac{\partial z}{\partial y}dy\right)$.

So, that we have got the production of these forces that is the vertical component of the tension, we got and let us see what will be the externally applied forces on the membrane, we will consider the air pressure on the membrane, which is uniformly distributed and its magnitude is q per unit area. So, in the small element dx dy, the total air pressure will be q×dxdy. So, if we get the total air pressure and this is the component of the tension in the vertical direction, then we can form an equilibrium equation in the z direction.

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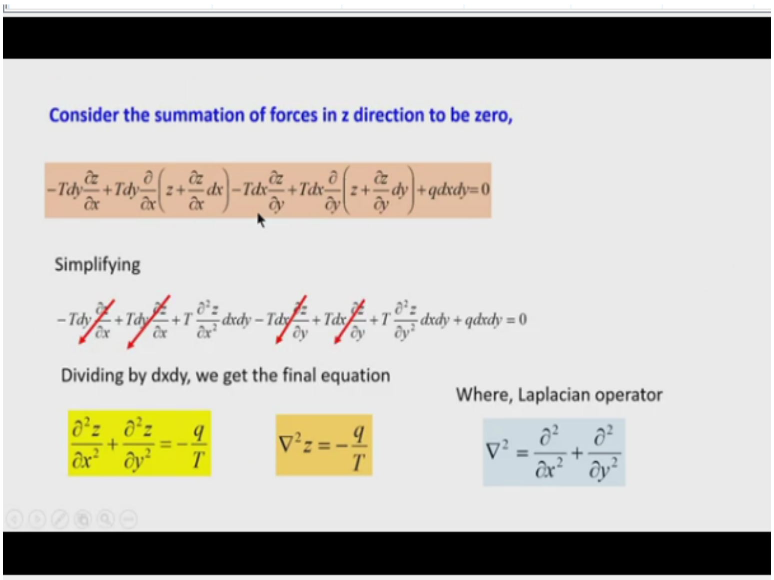

So, consider the summation of forces in the z direction to be 0. This is one component of the tension on the element, ab. This is another component of the tension on the other end of the element ab, like that, we have got the component of the tension on the element, which is parallel to the x axis, that is the element ad and element bc, so that will be here for element ab and element bc.

So, that component we have written and plus the external force that is T×dxdy and summation should be 0 for the equilibrium condition, because the vertical force will be resisted by the component of the tension. And you can see after the simplification that is if I do the term by term multiplication, then we get this into $Tdy\frac{\partial z}{\partial x}$ and then we get here, $T\frac{\partial^2 z}{\partial x^2}dxdy$, dx is already there.

$$-Tdy\frac{\partial z}{\partial x}+Tdy\frac{\partial z}{\partial x}+T\frac{\partial^2 z}{\partial x^2}dxdy-Tdx\frac{\partial z}{\partial y}+Tdx\frac{\partial z}{\partial y}+T\frac{\partial^2 z}{\partial x^2}dxdy+qdxdy=0$$

And double derivative will come here, $\frac{\partial^2 z}{\partial x^2}$, then here I got $-Tdx\frac{\partial z}{\partial y}$, then plus we get this $Tdx\frac{\partial z}{\partial y}$. So, that we got here. And for other term, we get $T\frac{\partial^2 z}{\partial x^2}dxdy$, and external force qdx dy = 0. You can see some terms are canceled. So, this term is cancelled, again, this term is cancelled, so then divide both sides by dxdy, you can see dx dy are common in all the terms here, whatever terms are left. And then dividing we get the final equation $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=-\frac{q}{T}$.

So, this is a very common equation in mathematical physics and it is physical meaning of this equation is that you have this stress membrane. So, when a membrane is under tension and it is subjected to air pressure q, then the phenomenon is represented by the differential equation,

$\nabla^2 z = -\frac{q}{T}$ where q is the air pressure and T is the uniform tension in the membrane. And ∇^2 here is the Laplacian operator that is $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

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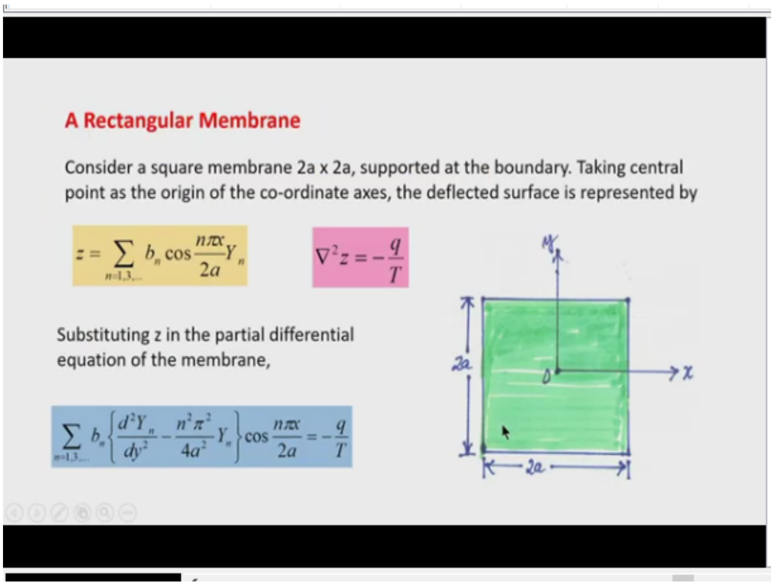

So, then, let us explain this method or differential equation that we derived now to solve a rectangular membrane. Membrane can be of any shape, it can be circular or any other shape, but let us see a solution for case of rectangular membrane that is plane is rectangular and it is supported at the boundary. So, let us take the side of the membrane. It is 2a and other direction also 2a.

So, this is a square membrane 2a by 2a, supported at the boundary, taking central point at the origin, if I take center point at the origin of the coordinate axis, the deflected surface can be represented by a series, we take the help of series solution.

$$z = \sum_{n=1,3,...} b_n \cos\frac{n\pi x}{2a} Y_n$$

While b_n is a coefficient and we take this function of x as $\cos\frac{n\pi x}{2a}$ and a function of y, that is Y_n. Now, these function $\cos\frac{n\pi x}{2a}$ is seem to satisfy the boundary condition at x = a because it a square membrane, if I take x = a or x = - a.

That is what the boundary, point is on the boundary at x = a, then we can see that this equation is satisfied for all odd values of n so 1, 3, 5, etc up to infinity. So, therefore, if this is the solution of the equation and b n is a constant. So, this solution can be written as a series form, that is if I expand the sum that I can tell that

$$b_1\cos\frac{\pi x}{2a}Y_1 + b_2\cos\frac{2\pi x}{2a}Y_2 + b_3\cos\frac{3\pi x}{2a}Y_3 \ldots\ldots\ldots\ldots\ldots\ldots$$

So, the summation goes up to infinity, because the infinite number of terms are there, but for vertical competition we have to truncate the terms to a finite number. Now substituting this in the differential equation $\nabla^2 z = -\frac{q}{T}$, $\nabla^2 z = -\frac{q}{T}$ was the differential equation of the membrane. So, if I substitute this in the differential equation, then we get that, you see that b_n is a constant and after substituting these we get the double derivative of Y_n, $\frac{d^2Y_n}{dy^2}$.

$$\sum_{n=1,3,\ldots} b_n\left\{\frac{d^2Y_n}{dy^2} - \frac{n^2\pi^2}{4a^2}Y_n\right\}\cos\frac{n\pi x}{2a} = -\frac{q}{T}$$

And this when the $\cos\frac{n\pi x}{2a}$ is differentiated 2 times, with respect to x, because this equation contains the second derivative of z with respect to x as well as second derivative of z with respect to y. So, when we differentiate this, this quantity with respect to x, two times, then this $\beta^2 - \left(\frac{n\pi}{2a}\right)^2$ will come outside. And the sign will be negative, then here it will be

$-\frac{n^2\pi^2}{4a^2}Y_n$. And $\cos\frac{n\pi x}{2a}$ will be common to all the terms; this is equal to $-\frac{q}{T}$. Now one after that you can experience this $\frac{q}{T}$ in terms of Fourier series or you can get it from the first principle that is applying the orthodontic conditions of the cosine function. So, let us do it from first principle that means I multiply both sides of this equation by cos n pi x by 2a. N is an integer. So, I again multiply both sides of this equation by $\cos\frac{n\pi x}{2a}$.

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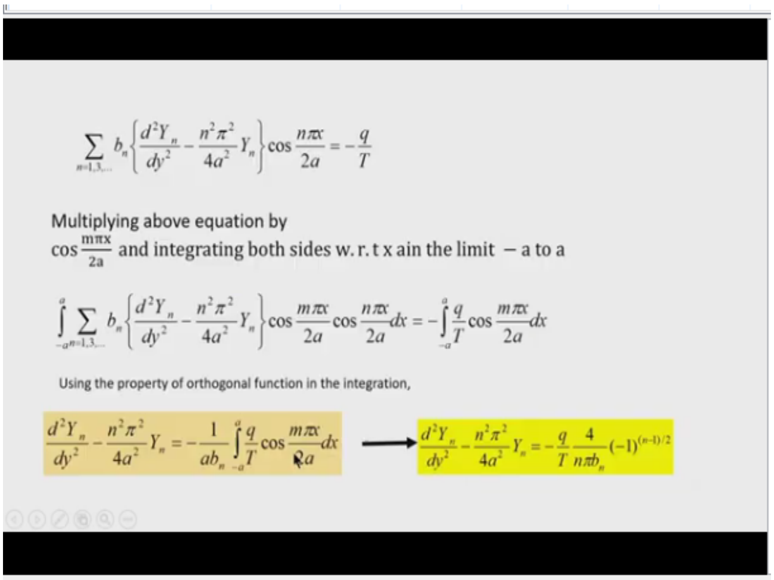

So, then multiplying this equation with $\cos\frac{n\pi x}{2a}$ both sides, I multiplied it and then use the orthogonal property of the cosine function. So, after utilizing the orthodontic property of the cosine function, we get this. This will be only a and other right hand side will be in the integral form. Then b n is there, b n will be as a constant and in doing this operation, we can see that when m = n, then only the value will be existing.

$$\int_{-a}^{a}\sum_{n=1,3,...} b_n\left\{\frac{d^2Y_n}{dy^2}-\frac{n^2\pi^2}{4a^2}Y_n\right\}\cos\frac{m\pi x}{2a}\cos\frac{n\pi x}{2a}dx=-\int_{-a}^{a}\frac{q}{T}\cos\frac{m\pi x}{2a}dx$$

Whatever integral is there, integral will be here is a, because the limit is - a to a. So, integral value will be this a only. Now here you can see that when m is not equal to n, then all the integrals will be vanished. So, therefore, we get only one term, saying, general we get this

$$b_n\left\{\frac{d^2Y_n}{dy^2}-\frac{n^2\pi^2}{4a^2}Y_n\right\}\times a$$ equal to $$-\int_{-a}^{a}\frac{q}{T}\cos\frac{m\pi x}{2a}dx$$

So, then taking this only the second bracket term

$$\frac{d^2Y_n}{dy^2}-\frac{n^2\pi^2}{4a^2}Y_n=-\frac{1}{ab_n}\int_{-a}^{a}\frac{q}{T}\cos\frac{m\pi x}{2a}dx$$

Now this equation has to be solved for the value of Y n. So, after integrating this equation, we now get the right hand side as $-\frac{q}{T}\frac{4}{n\pi b_n}(-1)^{(n-1)/2}$. Well, n is a integer, but it will exist only for odd integers.

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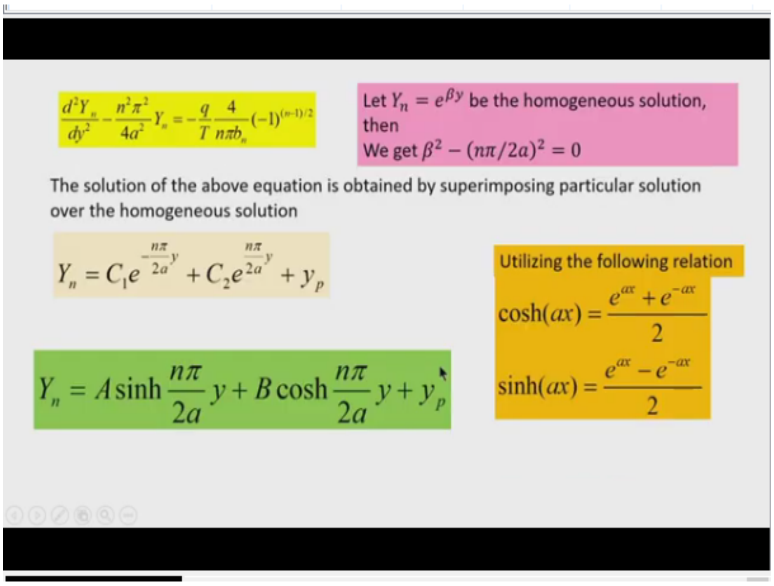

So, we get this equation, now, the right hand side now constant. Now you can see this equation is a non homogeneous equation. So, solution after we obtain in 2 parts. One is homogeneous solution and other is particular integral. So, let us first obtain the homogeneous solution. So, in case of homogeneous solution, we put right hand side to be 0. Then let us assume that the solution is $Y_n=e^{\beta y}$ $Y_n=e^{\beta y}$, where β is the roots of the characteristic equation.

Then, after substituting these, if this is a homogeneous solution, then it must satisfy the equation $\frac{d^2Y_n}{dy^2} - \frac{n^2\pi^2}{4a^2}Y_n = -\frac{1}{ab_n}\int_{-a}^{a}\frac{q}{T}\cos\frac{m\pi x}{2a}dx$

d square y and dy square - n square pi square by 4a square into Y n = 0. So, if I substitute this $Y_n = e^{\beta y}$ here. We will be getting this characteristic equation beta square $\beta^2 - \left(\frac{n\pi}{2a}\right)^2 = 0$.

Now, this characteristic equation has 2 roots, one root is $+\frac{n\pi}{2a}y$, another route is $-\frac{n\pi}{2a}y$.

So, solution now can be written as $Y_n = C_1 e^{-\frac{n\pi}{2a}y} + C_2 e^{\frac{n\pi}{2a}y} + y_p$

Now what is Y p, because this Y p is a particular solution because of the presence of nonzero term in the right hand side of the differential equation? So, that will be evaluated later on. Now let us keep it as Yp. Now, utilizing the following relation that we know from trigonometry, that $\cosh(ax) = \frac{e^{ax} + e^{-ax}}{2}$ and $\sinh(ax) = \frac{e^{ax} - e^{-ax}}{2}$.

So, these 2 relations can be used here. And then it is simplified in this form, say a constants are renamed, actually. So, $Y_n = A\sinh\frac{n\pi}{2a}y + B\cosh\frac{n\pi}{2a}y + y_p$, y_p is the particular integer that we have not evaluated till now.

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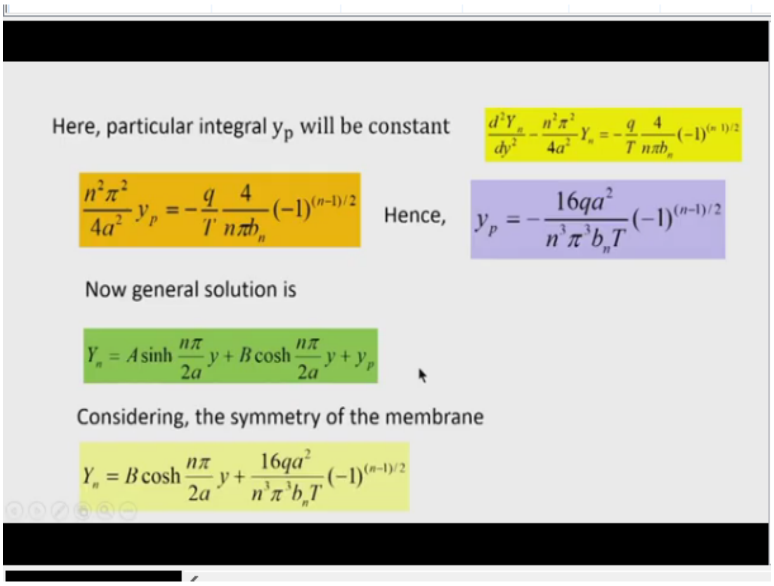

Now here particular integer will be constant, because if we observe this equation, we can see the right hand side is a constant quantity because q is a constant term, T is a constant term because this tension in the membrane is assumed to be constant, n is integer, so it is a constant term, and b_n is also a coefficient which is constant. So, therefore this particular integer can be treated as a constant.

So, if I take the particular integral as y_p and then substituting these in the differential equation, we get because the differentiation of this constant term will be 0, so this term will not be considered in the particular solution when you substitute y_p here because of the differentiation and y_p is constant, so it will be 0, only this term will be coming here. So,

$$\frac{n^2\pi^2}{4a^2}y_p=-\frac{q}{T}\frac{4}{n\pi b_n}(-1)^{(n-1)/2}$$ Now here. $$y_p=-\frac{16qa^2}{n^3\pi^3b_nT}(-1)^{(n-1)/2}$$

And this index is there, because when n is put as an odd number, this function only exists. For even number of n this solution does not exist that we have illustrated earlier also, because when we create function and the result of this integration is this and it is valid only for odd integers.

So, particular integral is obtained and this solution is now written the homogeneous solution plus the particular integral. So, if this solution is written here, then the general solution now expressed as $Y_n = A\sinh\frac{n\pi}{2a}y + B\cosh\frac{n\pi}{2a}y + y_p$

So, you can see in this general solution of y_n, we have the sine hyperbolic term, that is $\sinh\frac{n\pi}{2a}$ one term, plus $\cosh\frac{n\pi}{2a}$ term is also coming here because of the two exponential terms in the solution, $e^{\frac{n\pi}{2a}y}$ and also $e^{-\frac{n\pi}{2a}y}$ and utilizing these properties, we now get this hyperbolic function in this solution. So, solution is written in terms of hyperbolic function for convenience. Now you can see here, the membrane that we have taken. For example, here it is a symmetrical problem. That is the air pressure or whatever pressure is applied on the membrane is uniform. And the origin is taken in such a way that it is symmetrical about x and y axis. So, therefore, it because of symmetrical nature of the problem, the deflected surface also is symmetrical.

So, therefore, there should not be any odd functions in the deflected surface. So, if you see this function, the odd function is $\sinh\frac{n\pi}{2a}y$. So, this we can omit now in the deflection equation. So, deflection equation is now written as $Y_n = B\cosh\frac{n\pi}{2a}y + \frac{16qa^2}{n^3\pi^3 b_n T}(-1)^{(n-1)/2}$

So, that is the deflection in the direction of y that is the function y_n.

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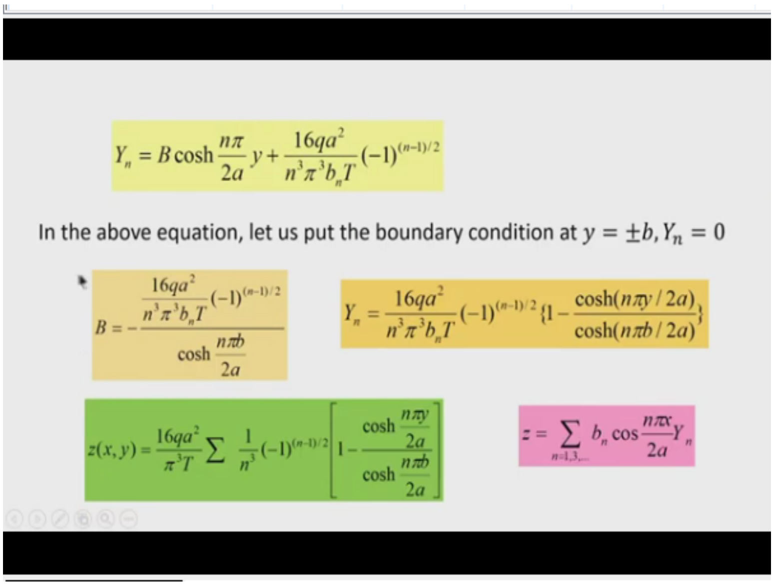

And now we have to evaluate the constant B because constant B is till now not evaluated and it is evaluated, applying the boundary condition. Since the membrane is supported along the edges. So, it is a square membrane. So, at y = a, the deflection is 0. So, therefore, by substituting this we ultimately get

$$B = -\frac{\frac{16qa^2}{n^3\pi^3 b_n T}(-1)^{(n-1)/2}}{\cosh\frac{n\pi b}{2a}}$$

So, y_n is now completely known. And it is after rearrangement, it is shown like that.

$$Y_n = \frac{16qa^2}{n^3\pi^3 b_n T}(-1)^{(n-1)/2}\{1-\frac{\cosh(n\pi y/2a)}{\cosh(n\pi b/2a)}\}$$

Actually here, this is a square moment so we can take this term as this. The deflector surface is now

$$z(x,y) = \frac{16qa^2}{\pi^3 T}\sum \frac{1}{n^3}(-1)^{(n-1)/2}\left[1-\frac{\cosh\frac{n\pi y}{2a}}{\cosh\frac{n\pi b}{2a}}\right].$$

So, that we get here, the deflector surface and z is ultimately found as this

$$z = \sum_{n=1,3,...} b_n \cos\frac{n\pi x}{2a} Y_n$$

So, therefore b_n will get cancelled from this quantity and ultimately you will be left with this term. So, this is the membrane deflection and this example I have taken to illustrate the state of stress in a membrane, which behaves like a stress skin. So, shell is also a stressed skin so we must know the membrane behaviour, then we will go to the analysis of shell.

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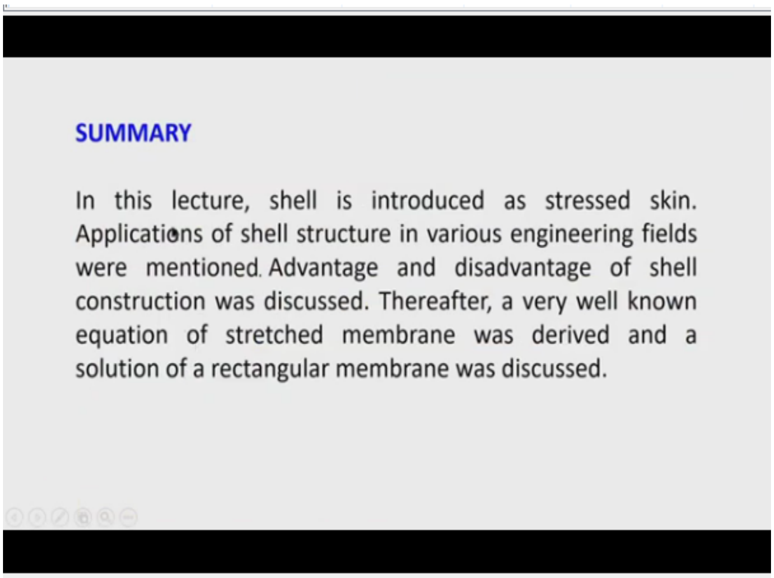

Let us see, what do we have done in today's class. In this lecture shell is introduced the stressed skin, application of shell structure in various engineering fields were mentioned advantage and disadvantage of shell construction was discussed. Thereafter, a very well known equation of stressed membrane was derived in the solution of rectangular membrane was discussed. Thank you very much.