Plates and Shells

Prof. Sudip Talukdar
Department of Civil Engineering
Indian Institute of Technology-Guwahati

Module-07
Lecture-21
Finite Difference Method in Buckling of Plate

Hello everybody, today I am giving the second lecture of the module 7 as you remember that module 7 was on the approximate method for solving the buckling of thin plates. Previously we have discussed the buckling of thin plates using the exact analysis. And 2 boundary conditions we have seen the general solution one is this Navier's boundary condition that all edges were simply supported.

And in other case is 2 edges opposite edges are simply supported other 2 edges may have any other condition. So, based on that Navier and Levy's boundary conditions, we have obtained the critical stresses of the thin plate using exact analysis. Then I have discussed the approximate method for finding the buckling load of the thin plate. And in that respect, first I discussed the Rayleigh-Ritz solution and then I discussed the Gallerkin method for finding the buckling load.

So, today I will discuss another very useful method for finding the buckling load of plate using the finite difference scheme. So, as you know that plate buckling or even say plate bending and also the vibration problems of any physical model say beam, arch, plate, string. Then these can be solved by exact method and sometimes the exact solution is not available, then we take the help of approximate method.

So, finite difference is one of the approximate methods for solution of the physical problem like your bending, buckling and then vibration in our structural mechanics or solid mechanics. And the most important thing in finite difference method is the control of error. That can be done by suitably adjusting the mesh sizes in this finite difference method. So, the most essential requirement of the finite difference method is first to express the differential equation, that is, the governing differential equation related to the problem in finite difference form.

Then we use this finite difference form of the equation at the particular note of a structure that was divided by different grid lines. And then, we get a n number of simultaneous equations, linear simultaneous algebraic equation, which required to be solved to find out the unknown deflection of the plate. Here problem we are discussing is related to the plate, so we will be talking about the deflection of the plate.

And the deflection is the only the unknown variables in the domain of the plate that we have to find out, and from this deflection, the other quantities can be found out. But as you know, this plate buckling or any buckling problem is a homogeneous boundary value problem. So, for non-trivial solution, we will get this characteristic equation to find out the critical value of the load or in discrete form; we actually convert this to a standard Eigenvalue problem and then we solve for the Eigenvalues.

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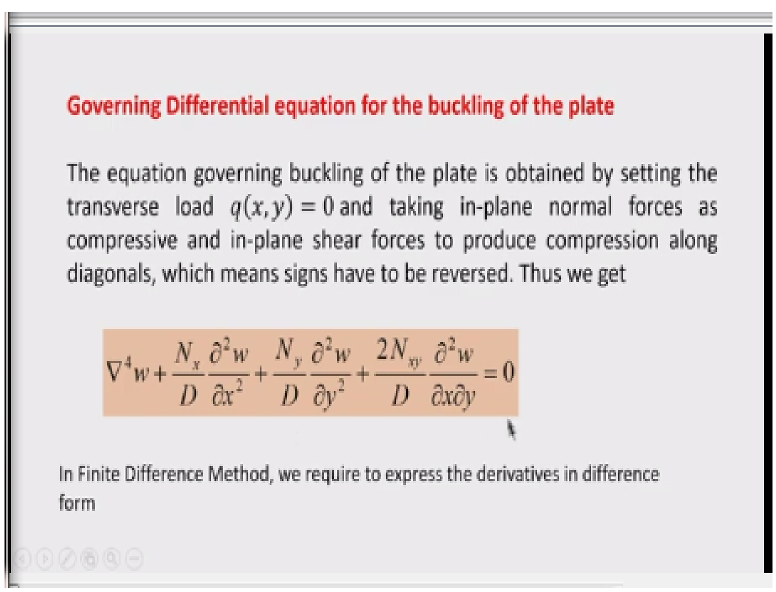

Now, let us see the differential equation. As you know that differential equation of the plate subjected to the axial compression N_x and N_y. And the in plane shear N_{xy} given by this equation that is $\nabla^4 w + \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2} + \frac{N_y}{D}\frac{\partial^2 w}{\partial y^2} + \frac{2N_{xy}}{D}\frac{\partial^2 w}{\partial x \partial y} = 0$. And there is no transverse load because the buckling is a homogeneous problem.

So, we take this transverse load that is acting on the beam is to be 0 for finding the critical load. Just like your free vibration analysis, that we will find out the natural frequency of the system, whether it is being plate or cable, then we put the external load to be 0, and we have solved this homogeneous problem. So, similar is the situation in case of buckling, we take the transverse loading on the beam to be 0, and we find the Eigenvalues or this characteristic values of the problem.

So, now to apply this differential equation in finite difference scheme, we first require to obtain the derivatives appearing in this equation in difference form. So, first, we have to obtain the first derivative in the difference form and then 2nd derivate, 3rd derivative up to 4th derivative and then we also need the mixed derivative, that is involving the twist curvature in this case of membrane shear. So, let us see.

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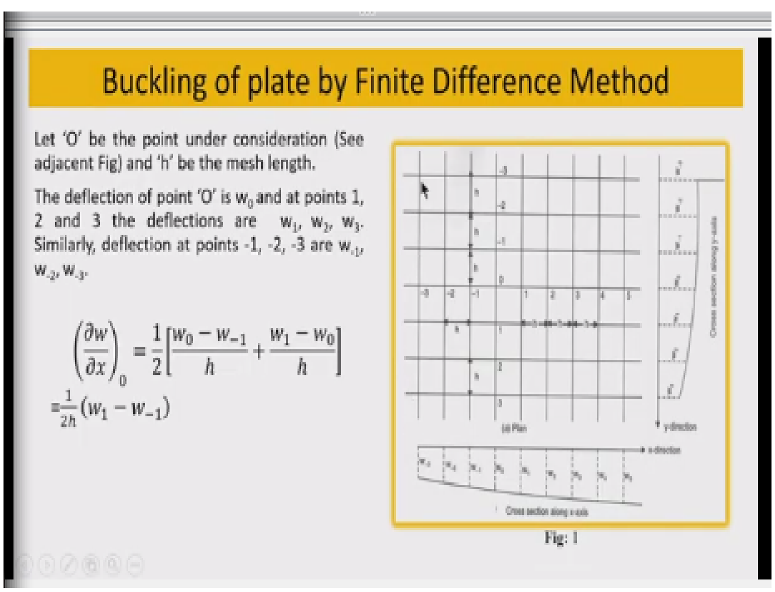

Say a domain of the plate, here it is shown, and this is divided into several divisions, and for simplicity, I am taking the equal size of the mesh in both direction x and y, but it may be different also. For example, here in this x-direction, I am taking h, y-direction is also h, but it may be say in the x-direction it may be h, in y-direction it may be say h_1. But equal mesh size is suitable for a plate which is square in shape. In rectangular plate, of course, we can vary the mesh sizes.

So, the plate is divided into number of divisions, small divisions. And you can see that say it is a node which reference to which we will discuss the difference form of the derivative. So, let us consider that 0 is a node and 1, 2, 3, 4, 5, etcetera, the node number along the x-direction. And in the positive direction of the x-axis, the positive sign that is it is +1, +2 like that and in the negative direction, we have just indicated this by -1, -2, -3.

But this is not mandatory; one can take any other convention to express the node also. So, node number should be first given and then with reference to the particular node will find the 1st derivative, 2nd derivative, 3rd derivative etcetera, in difference from. So, now let us consider the deflection of point O; there is O or 0. Whatever you call is w 0. And at points 1, 2, 3 the deflections are w_1, w_2, w_3.

And in the backward direction that is towards the negative x-axis it is -1, -2, -3, so node numbers are given. And in the y-direction, since downward direction is taken as the positive direction of the y axis, so I am taking 1, 2, 3, etcetera, is the node number in the downward direction of the y axis and -1, -2, -3 in the upward direction because here the downward direction is taken as the positive direction.

So, if I consider the slope of the deflected surface along the x-direction at node 0. So, the slope you can see that the deflection magnitude is here w_0, and here it is w_1. So, w_0 to w_1, there is a variation of displacement, and therefore the slope exist. And similarly, on the negative direction that is w_{-1} is the deflection at node -1. So, there is also the slope that is the variation of the displacement.

So, now, if I compute the slope between the node, say w_1 and w_0, then slope will be $\frac{w_1 - w_0}{h}$. So, I have calculated the slope taking the forward values of the deflection. Now, if I take the backward deflection value, then I will get the slope at w_0 at node 0 or O as $\frac{w_0 - w_{-1}}{h}$. So, 2 slopes when added and taken average, then we get this slope at node 0, so that means it represents $\frac{\partial w}{\partial x}$ at node 0 or O represents the average slope between this 1 and -1.

So, you can see here 1 by 2h is coming because the distance between this 1 and -1 is 2h. And after simplification, the expression becomes 1 by 2h bracket w 1 - w - 1. So, this is the expression for slope, a basic expression for the derivative, first derivative of w in x-direction. Similarly, the derivative in the y-direction can be written, so if I take the y-direction, then here you can see that 0, 1, -1 these are the nodes. And here, similarly, it can be written the del w by del x at node O is like that w 1 - w - 1 by 2h, but this will be in the y-direction; y-direction deflection curve is shown here.

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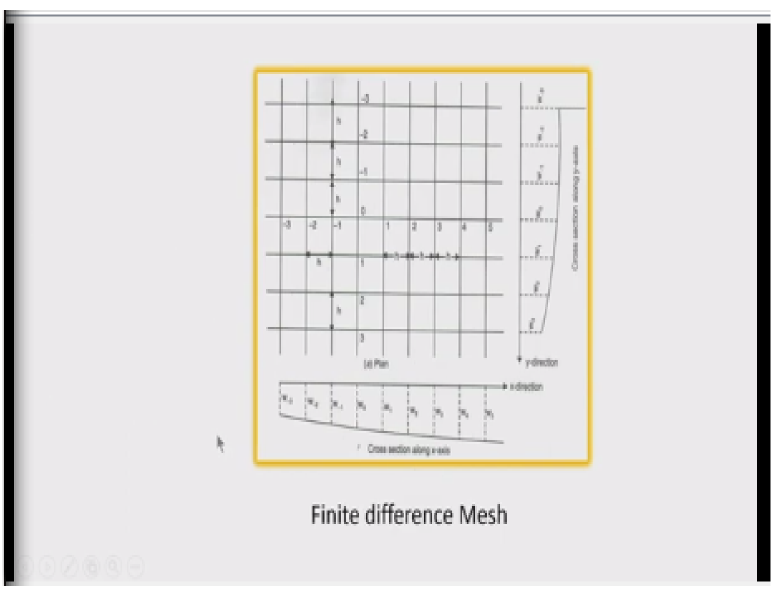

Now let us come to the 2nd derivative.

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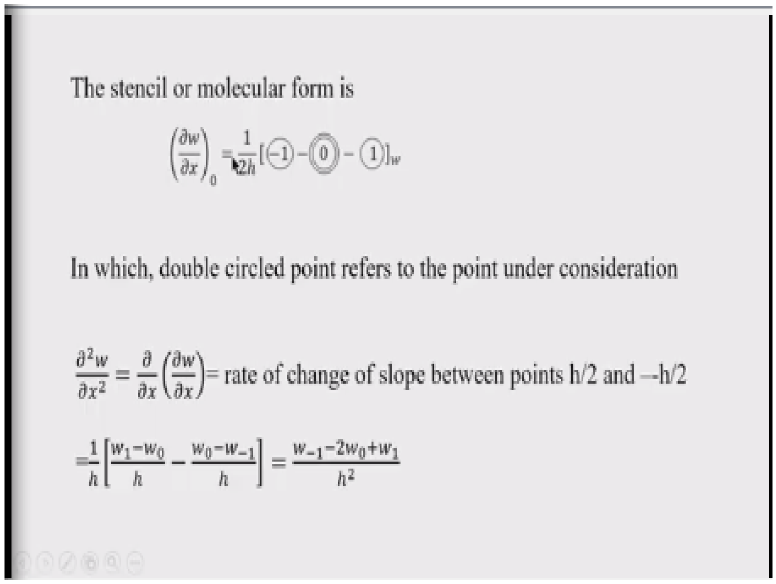

So, this is the slope which can be represented in molecular form, which is easy to remember and the convenient for use in actually solving the problem; this form is very much suitable. So, if you see the $\frac{\partial w}{\partial x}$ at node O, it can be written here like that. So, at node O, we found the slope, and it was given by $w_1 - w_{-1}$, but the value of deflection at node O is absent in this expression. So, therefore if I see the coefficient of these deflected surface, then here for w_1 it is coefficient is 1, w_{-1}, the coefficient is - 1 because the negative sign is there, and for w_0 the coefficient is 0 because no w_0 appears.

So, therefore, I can write in this molecular form like that $\frac{1}{2h}$ is the distance between node 1 and -1, and this is given say right-hand side this is w_1, the coefficient is +1. And towards the left-hand side, it is w_{-1} and coefficient is -1, and central value that is w_0 does not appear in the expression, so this value is taken at 0. So, this form is suitable because when we want to compute the slope at any point just, we place the stencil to the finite difference mesh centering this double circle point to the node where we want this slope.

So, double circle point refers to the point under consideration. So, similarly, if we go to other point, say if I want to find the slope at 3 then I will write that $\frac{w_4 - w_2}{2h}$ because the mesh size is taken same and here w_3 coefficients is 0. So, therefore naturally, there will be no value here, so 0 will be appearing there also, in the double circle, there is the centre node. Now, if I calculate the 2nd derivative of the expression, then rate of change of slope is to be evaluated, so $\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)$.

So, this quantity we have already calculated. So, now this can be calculated as $\frac{1}{h}$ because we already calculated $\frac{\partial w}{\partial x}$, and we again take the rate of change of this quantity. So, therefore $\frac{w_1 - w_0}{h}$ and then again $\frac{w_0 - w_{-1}}{h}$ because this is taken between points $h/2$ and $-\ h/2$ for curvature. So, therefore $h/2$ and $-\ h/2$ the gap is h, so therefore h appears here. And we consider the note 0 that is then towards the right-hand side the other deflected deflection value is w_1, so $\frac{w_1 - 0}{h}$.

And then change is towards the left if you see $\frac{w_0 - w_{-1}}{h}$. After simplifying this expression, it can be written as $\frac{w_{-1} - 2w_0 + w_1}{h^2}$. So, you are getting the 2nd derivative that is the curvature of the deflected surface. So, this can be written in the molecular form or stencil form. And the stencil form will be say again 3 point will be there, but here the coefficient will be h square. And there it is +1, and there it is +1, and here it will be -2 although I have not represented here but in the later expression, you will find this curvature appearing in the molecular form.

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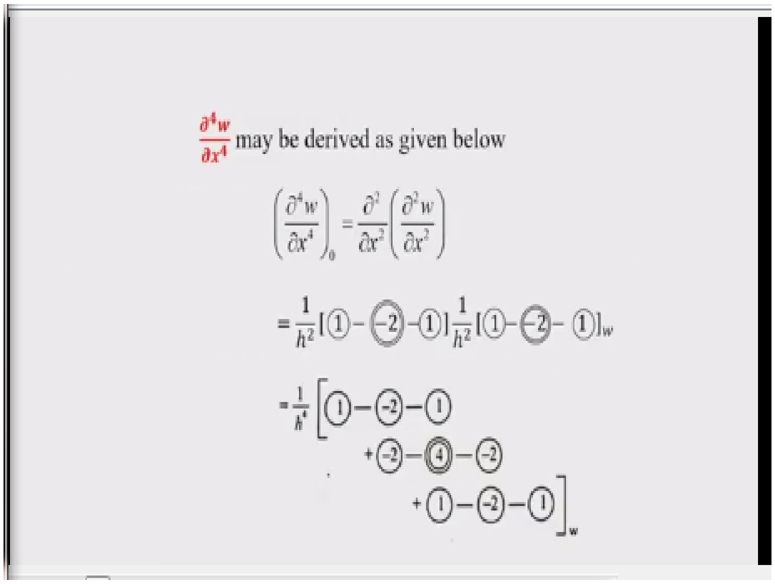

So, now the 4th derivative that is the highest derivative in the plate equation is to be evaluated. So, $\frac{\partial^4 w}{\partial x^4}$ at point node O, node O is this, we want to calculate the 4th derivative. Again the 2nd derivative is differentiated 2 times to get the 4th derivative. So, this we have already calculated, so we have written here in the molecular form that I have told you. Molecular form of these will be like that $\frac{1}{h^2}[\ (1)\ -\ ((-\ 2))\ -\ (1)\ ]$

So, this operator is this. And then again $\frac{\partial^2 w}{\partial x^2}$ here, but here w value is taken here w_1 that is w_{-1}, but coefficient is +1, then w_1 coefficient is +1, and this is w_0 that is coefficient is -2. So, term by term multiplication, you can get the range it in this form. So, if you multiply it, say 1 then -2 then 1, we are doing this first row, then 2nd row if you do the multiplication -2 to -2, 4, then -2, 1 is -2. Then the last quantity that is 1, so 1 into 1, 1, then 1 - 2 is -2, then 1 into 1 is 1, then you add this, you will get the 4th derivative.

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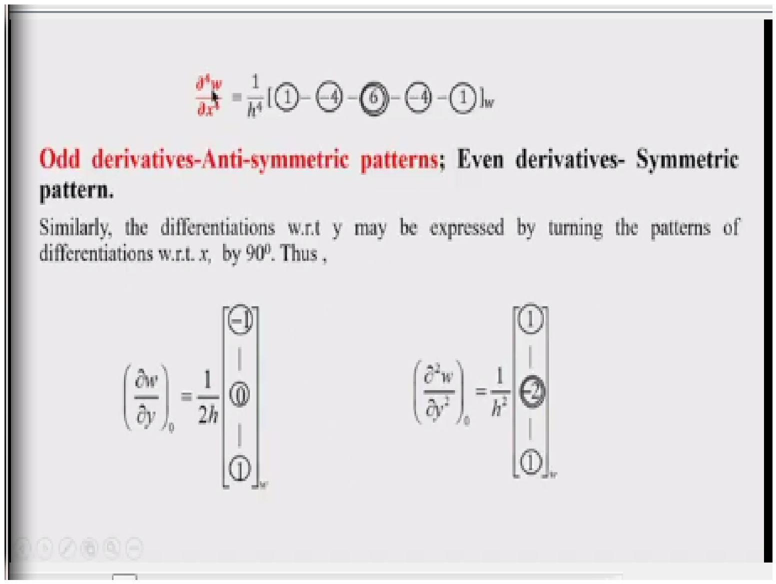

So, after adding, we get the 1st term is 1, 2nd term is -4, 3rd term is 6 then 4th term is -4 and 1. So, you can see here that this the 4th derivative in the molecular form is written, the central value is 6, coefficient of deflection in the nodal point there is the node centering which you will write the finite difference equation that coefficient is 6. And towards right and towards left, the same coefficient appears because it is even number of derivative, so the quantities are symmetry.

So, you are getting this -4, 1 towards the right side, and towards the left side, it is also -4, 1. So, interestingly you will see that odd derivative are antisymmetric, and even derivatives are symmetric. So, after getting the 4th derivative, now the pattern in the y-direction can also be written just by rotating the derivative that is found with respect to x by 90 degree. So, if I rotate this with 90° then I can get $\frac{\partial w}{\partial y}$ just rotation by 90°.

Similarly, $\frac{\partial^2 w}{\partial y^2}$ rotating this quantity is $\frac{\partial^2 w}{\partial x^2}$. So, if I rotate this quantity, this becomes -2 in upwards it is 1, and this is 1. So, horizontal direction is x-direction, and vertical direction is y-direction, so it is written like that.

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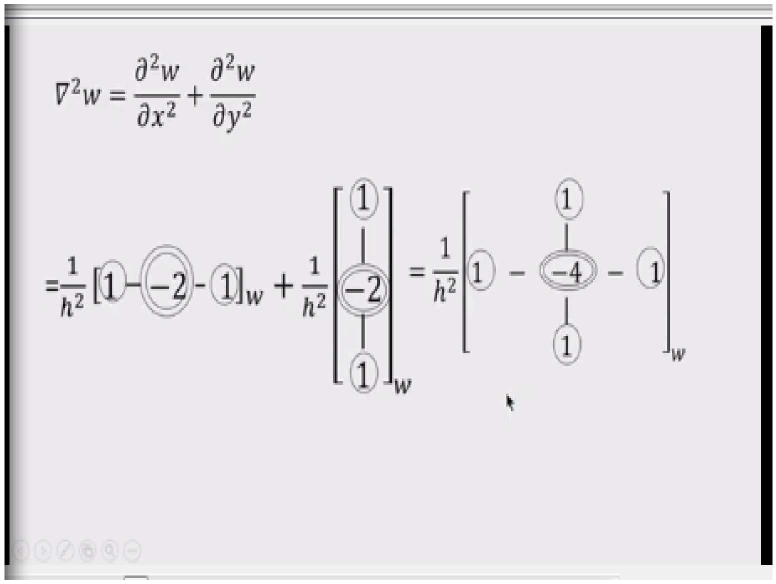

So, then Laplacian operator that is $\nabla^2 w$, we need sometimes because this if under the biaxial compression and the biaxial compression the equal amount of compression is applied in both x and y direction the plate. Then you will find that this Laplacian equation is necessary. So, let us also express this $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$. So, we have found that 2nd derivative of w with respect to x is $1/h^2[(1) - ((-2)) - (1)]_w$.

So, that means -2 is the coefficient of the deflection value and the node where you want to place this stencil, then towards right it is 1, and this is 1. Similarly, for second derivative in the y-direction, y-direction is vertical, so we write in the vertical line, and we are writing say -2 central value, coefficient of the central deflection and then upward it is +1 and it is downward it is +1. Since the even derivatives are symmetry, so there is no change of sign if I consider the coefficient in the top and bottom.

Similarly, if I consider the coefficient towards left and towards right, there is no change in the sign. So, after addition of this, you can see that Laplacian operator appears in this form 1 by h square 1, then -4, 1. Because -2, -2 when added it becomes -4, then 1 in the top and bottom again 1 and this left and right also coefficient 1. So, this Laplacian operator is important when you consider the plate under biaxial compression of equal magnitude in buckling problem. So, mixed derivative also required because if I consider the membrane shear force or shear buckling of the plate, then we have to consider this term.

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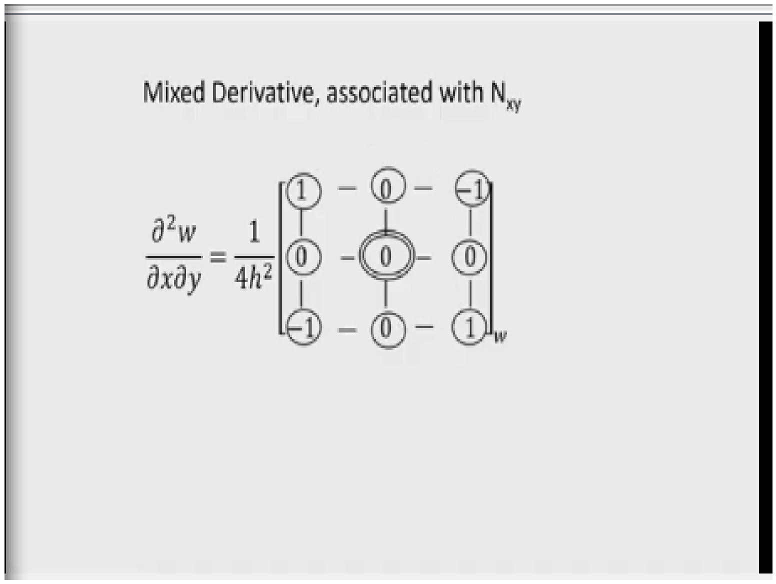

That is associated with the N_{xy} term that I have shown in this slide, that $\frac{N_{xy}}{D}\frac{\partial^2 w}{\partial x \partial y}$, so mixed derivative term is associated with N_{xy}. So, we should also know this the finite difference form of the mixed derivative. So, let us see the mixed derivative now can be written as this 1 0 that is central value is 0 and towards right and towards this left, it is also 0 coefficients. And you will get the corner values are towards the left-hand side it is 1 top and the bottom in the left-hand side it is -1, in the right-hand side you will find that in the top it is -1 and in the bottom, it is 1. And you can see the pattern that it appears as a antisymmetric quantity.

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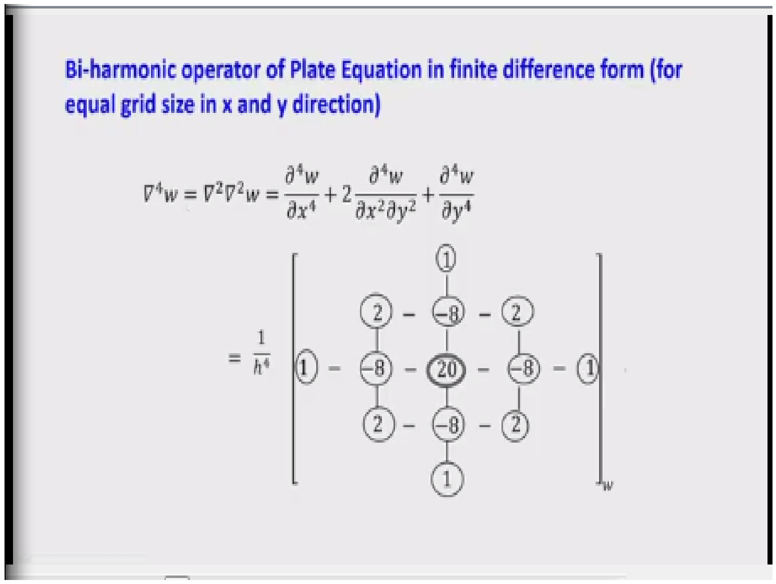

So, we have got this 4th derivative. So, now 4th derivative in x-direction I have obtained and as the y-direction also can be written just by aligning the stencil in the vertical direction. So, after arranging this, you will get the 4th derivative that is the $\nabla^4 w$ that is the biharmonic operator that is used with the deflection surface to form the plate equation. So, $\nabla^4 w$ is now is equal to $\nabla^2 \nabla^2 w$ we have got.

That is a Laplacian operator we have got here; this is the Laplacian operator. And the Laplacian operator is this, and when it operates with w, it is this; the w should be associated with this. So, therefore we can write this as this $\nabla^2 \nabla^2 w$, so 2 operators one is Laplacian operator, and this is $\nabla^2 w$ that we have already obtained. So, now after expansion that can also be written as $\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}$

So, after collecting all the terms now, the appearance of this $\nabla^4 w$ is like that, you can see in the centre line, horizontal line the stencil is to cover the 5 points. If this is the central point that is central point, I mean where you are centering the stencil. Suppose there are say 10, 12 nodes and at particular node say 5 you are centering the stencil. Then this value has to be placed in the node 5, and then automatically, other molecule will get their position. So, towards the left, you are finding that -8 1 in this central line, and this is -8 towards the right also -8 1, it is symmetrical because it is the $\nabla^4 w$ is your this even derivative.

So, therefore in the vertical line also you are getting symmetric, and there other lines are also have symmetrical distribution. So, 2, -8, 2 and here also you are seeing 2, -8, 2 and so 5 points need to be covered by this stencil in this central position as well as central, horizontal position as well as central vertical position. Central position I means that where you centre the stencil, centering may be done at the centre node, or it may be at any other node; it does not matter.

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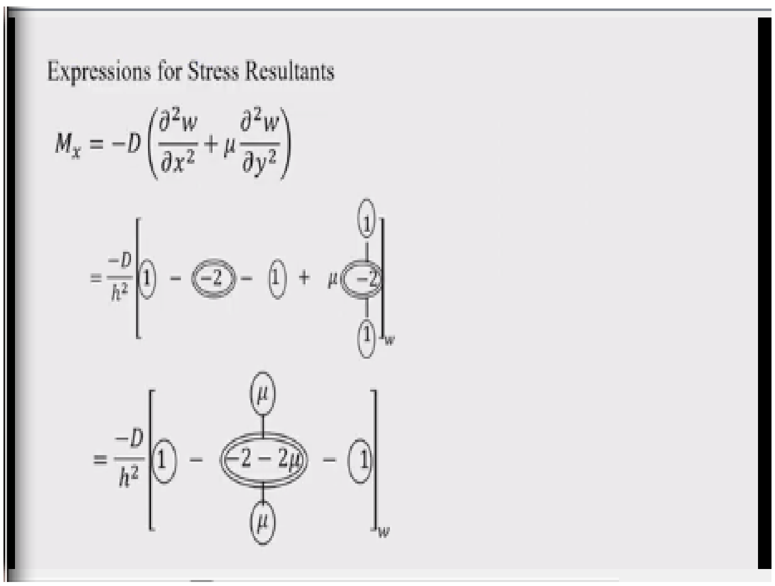

So, we should also know the bending moment expressions are not required, but to apply the boundary conditions in finite difference method, we required to know the bending moment expression in finite difference form and shear force expression also. So, bending moment expression is nothing but combination of this $-D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right)$.

And if you combine these 2 derivative and $\frac{\partial^2 w}{\partial y^2}$ is multiplied by μ that is Poisson ratio, we can write it 1, -2, 1, and this is + μ, -2, 1 and 1. So, after adding this, you will get the central node is here -2, -2 into μ and in the upward direction you are finding the μ into 1, it is there, in the downward node it is also μ into 1, and other nodes are 1 and 1. So, this is the full stencil for M_x, similarly, we can write the bending moment M_y.

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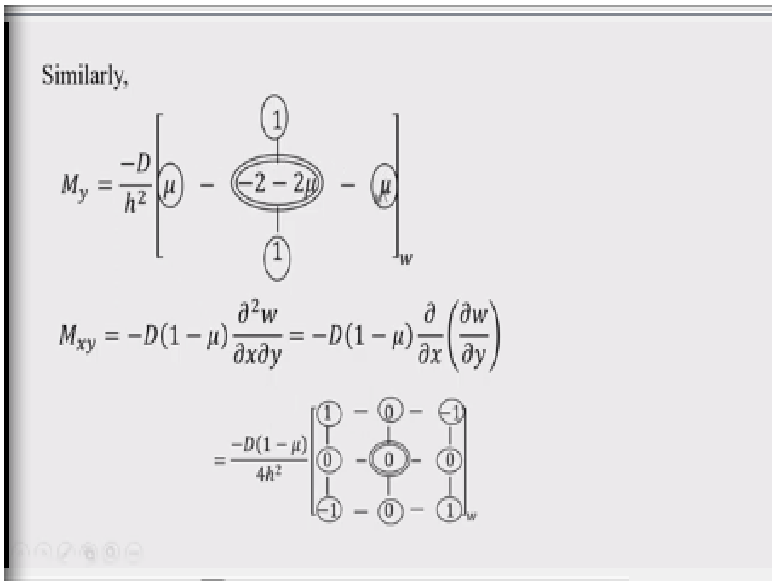

So, M_y is written like that you can see the difference, this is just placed now in the horizontal position, this vertical is now placed in the horizontal position, so we get this the M_y, so M_y is obtained like that. Then M_{xy} the mixed derivative is also needed because if we consider the shear buckling, how the shear buckling equation is formed? Shear buckling equation is formed with the term $\frac{2N_{xy}}{D}\frac{\partial^2 w}{\partial x \partial y}$.

And other in plane forces that is N_x and N_y are associated with the 2nd derivative of their respective directions. So, therefore this M_{xy} has to be known also for this shear problems, specially shear buckling problem. I mean that mixed derivative should be known for shear buckling problem and also the M_{xy} and the shear force the Q_x or Q_y has to be combined to give a Kirchhoff edge shear that is very important for free edge condition.

Because free edge condition you know that there are 3 conditions that have to be equated to 0, what are these 3 conditions? One condition is that bending moment is 0, second condition is this twisting moment that is M_{xy} or M_{yx} is 0, and third condition is Q_x or Q_y whatever maybe as per their edge location is to be 0. So, the 3 equation, 3 quantities to be equated to 0, later on that Kirchhoff were modified this condition and have given 2 expressions instead of there.

So, that is M_{yx} is combined with shear force Q_x or Q_y to give the edge shear, 1 edge shear there is V_x or V_y. So, therefore the M_{yx} expression is also needed for calculating the edge shear value.

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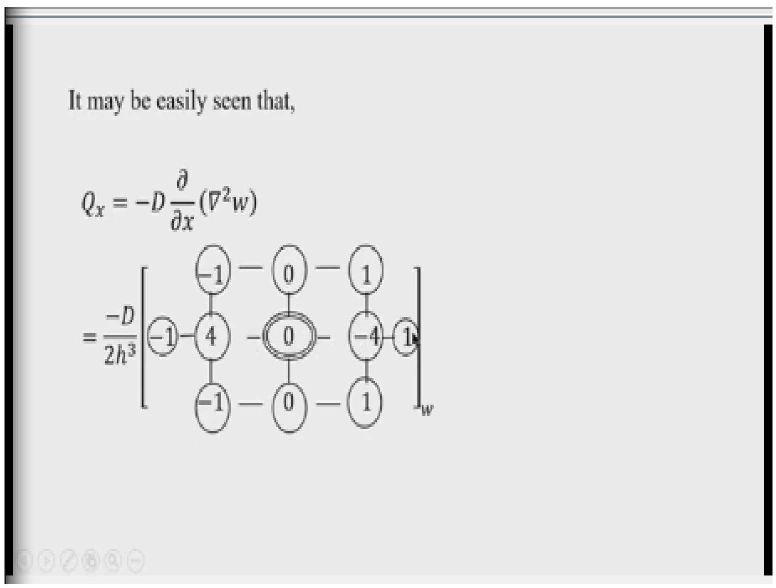

So, Q_x that is the shear force in the plate along the edge where the x coordinate is specified and y is varying. Then it is given like that $-\ D\frac{\partial}{\partial x}\left(\nabla^2 w\right)$, $\nabla^2 w$ we have already evaluated, now we operate with this, $\partial/\partial x$ operator is already known that it is $\frac{1}{2h}\,[(-\ 1) - ((0)) -\ (1)]$. So, and this operator is also known, so when we operate this $\partial/\partial x$ with this Laplacian operator ∇^2 Laplacian equation $\nabla^2 w$, then we get this stencil for Q_x.

So, Q_x is now $-\frac{D}{2h^3}$, and in the horizontal line centrally, you are getting -1, 4, 0, -4, 1, -1 +4, 0, -4, 1, and you can note here that the values are antisymmetry. So, towards the right, it is -4, but towards the left, in the corresponding location, it is +4. Similarly, in other also -1 and +1, so in the vertical direction also you can note this, and in the central location in the vertical direction, this is 0. So, you are getting Q_x.

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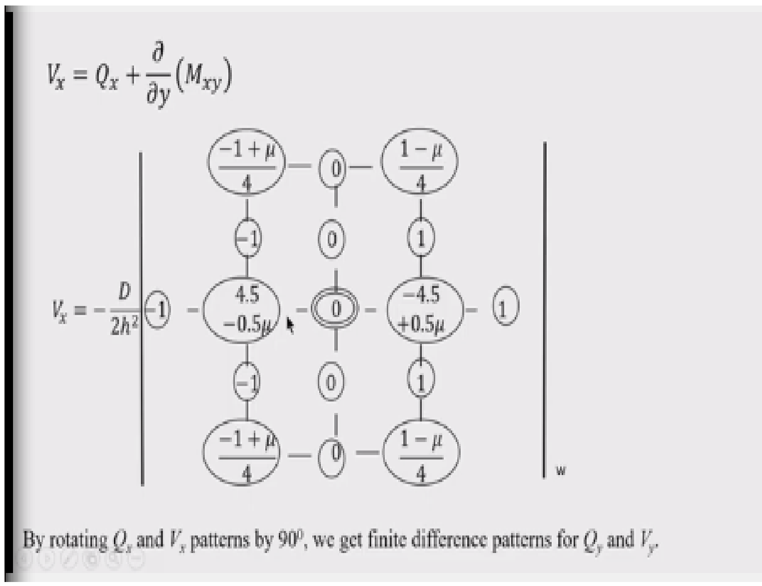

So, therefore V_x that is along the x = a or some edge at the boundary, you want to apply the edge shear condition as a boundary condition. Another boundary condition at the edge is say bending moment, specially for free edge the boundary condition becomes if x = a edge is free, then the boundary condition at x = a will be$M_x = 0$ and $V_x = 0$ So, instead of 3 equations that we earlier told that $M_x = 0$, $Q_x = 0$ and V_x and $M_{xy} = 0$.

We write now 2 condition that is the Q_x and M_{xy} are combined. So, after combining these 2 term, we get the edge shear force V_x. So, edge shear force is also containing the 3rd derivatives, the odd derivatives, and then you get that this is also appearing anti symmetrically. So, antisymmetric functions and antisymmetric values are noted in the quantity V_x. So, if we want this V_y, then just rotating the pattern by 90°, the finite difference for V_x, Q_x, V_x, etcetera can be found. But $M_{xy} = M_{yx}$, so the expression for M_{xy} will remain valid for M_{yx} also.

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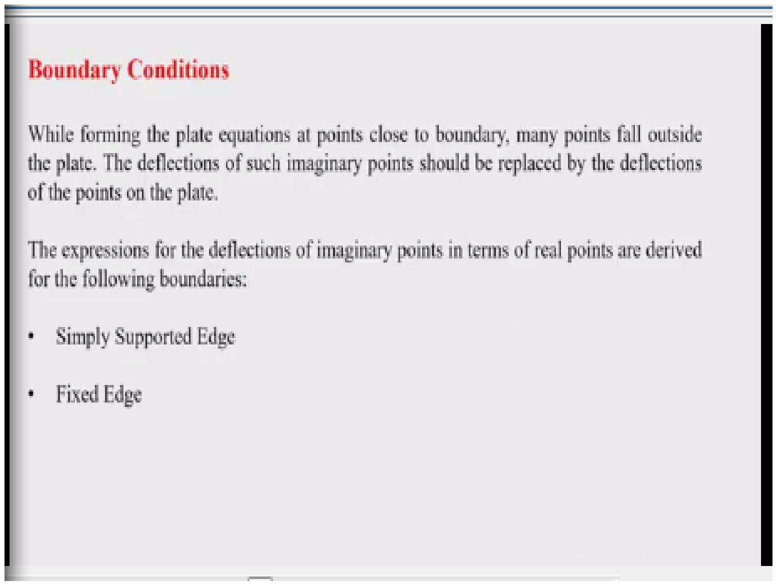

Now boundary condition that we encounter in the plate buckling problem, this may be simply supported edge or fixed edge or maybe free edge. Now free edge condition I have not mentioned here because the free edge condition is slightly lengthy. But one can apply the free edge condition using this V_x at the free edge and Q_x and M_x or M_y at the free edge to be 0.

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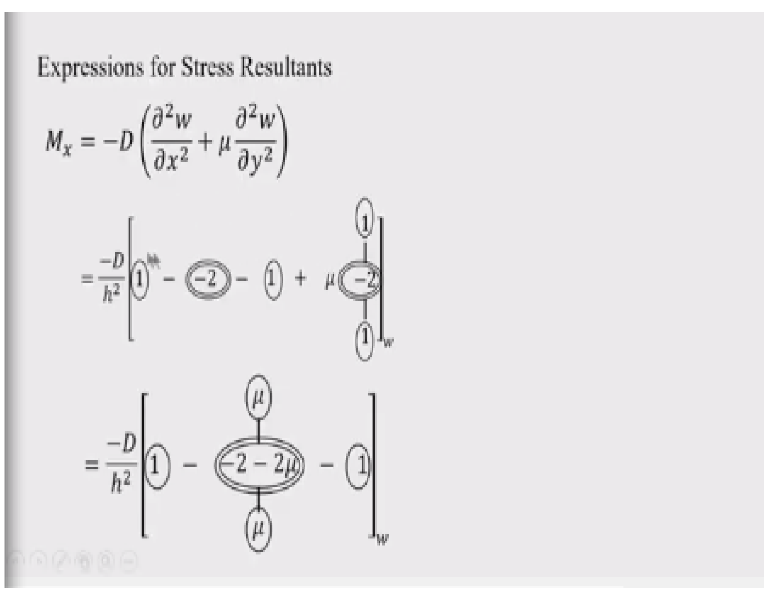

Now when we apply, suppose you apply this conditions say M_x is this 1, -2, -2μ, -1. So, if this M_x is to be applied at the boundary, that boundary value if the node is falling in the boundary here, but you can see that one point is outside the boundary. So, that problem is overcome by taking a dummy point here and relating the displacement of the dummy point with the displacement of the real point, just real image point. So, you can see here that if I consider a simply supported condition, just let me discuss with a figure.

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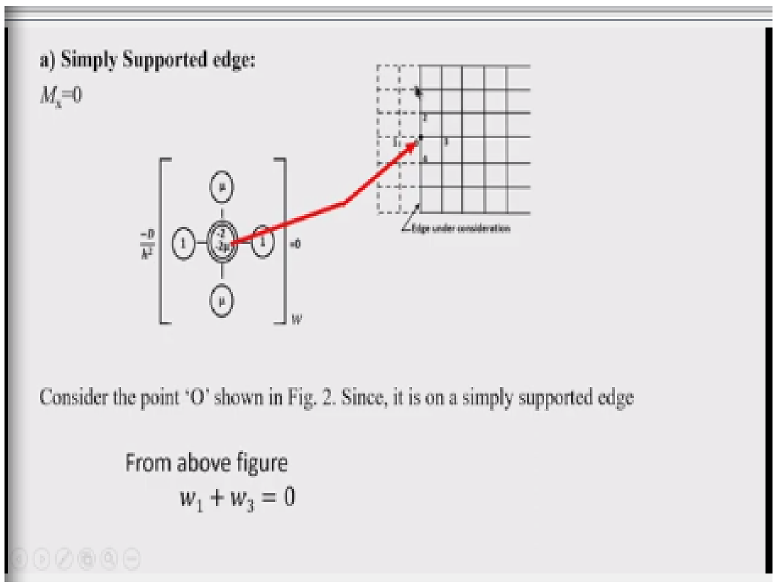

We consider a simply supported edge; this edge is simply supported. So, this is the plate domain that is shown by solid lines. But dotted lines are not really existing it is taken for this finding the expression in terms of the displacement of the plate domain. Now let us say this edge M x = 0, so our stencil was this for this bending moment. So, if I put this is the node, so if I put this here, just I take this here.

Then at the boundary, displacements are 0, so you are getting the displacement of 2 displacement of 4 or 0. And displacement of this node say 0 is also 0, because this is falling on the boundary. But displacement of 3 is someone, say w_3, and here we are getting some say imaginary node, so that is the deflection is w_1 but coefficient is +1 and +1. So, that means if I now write this equation, then I have to write $w_1 + w_3 = 0$.

Because this quantity, if I write this, will be 0 because these nodes are these molecules are falling on the boundaries. So, therefore we can write $w_1 + w_3 = 0$, that means the w_1 that is falling outside the plate boundary, that is now expressed in terms of the displacement of the image point, image point is 3, which is real point in the plate domain and therefore $w_1 =- w_3$. So, in this way, simply supported boundary condition is expressed.

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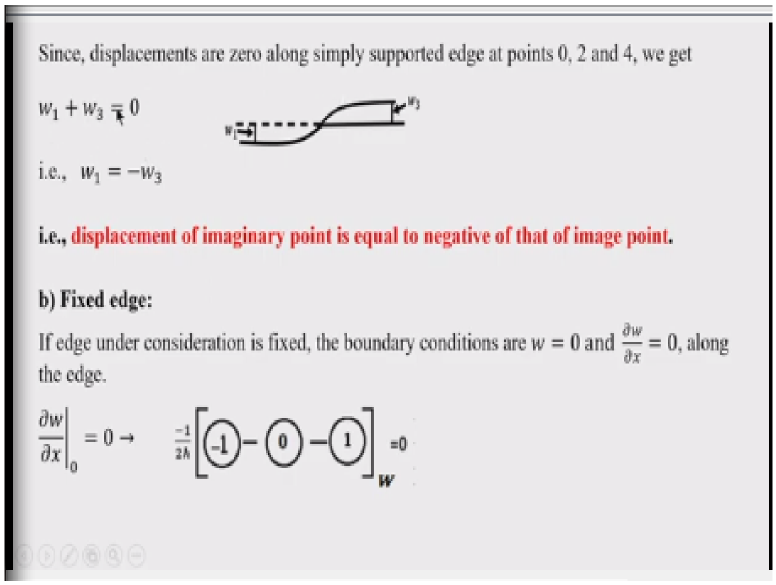

So, you can see this $w_1 + w_3 = 0$, so w_1 will be minus of w 3. So, deflected shape for purpose of expressing the deflection value of the imaginary point in terms of deflection value of the real point is taken like that. Now displacement of imaginary points that we call the imaginary point w 1 is now equal to the negative of the displacement of the image point. So, now we come at the fixed edge condition.

In the fixed edge condition, the 2 conditions are generally have to be satisfied; one is deflection is 0, another is slope along the normal to the edges should be 0. So, if it is the edge is x is equal to suppose x = a, this edge is fixed. Then w is 0 at this edge and also $\frac{\partial w}{\partial x} = 0$. Now here the $\frac{\partial w}{\partial x} = 0$, that means if I write this in the stencil form, I can express this -1, 0, -1, and it is w, that stencil we have already got it earlier.

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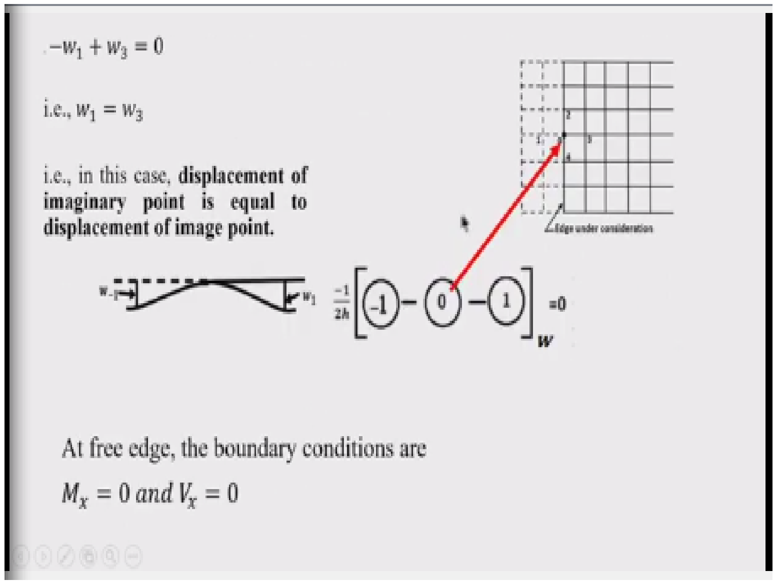

So, now this stencil is now put here; 0 is the node where you want to apply the boundary conditions. So, this fixed that is the slope this stencil is first derivative stencil is put here at 0. And then we get that 2 points; this point is on the node on the boundary. This point is on the boundary, so therefore automatically, it vanished because the deflection at the boundary is 0. But this node is outside the boundary and with coefficient -1.

So, say suppose this is the deflection w_1, and this is that the deflection w_3. So, when we expand this stencil, we have to write now $-w_1 + w_3$. Because the coefficient is -1, so we have written -1, $-w_1 + w_3 = 0$ therefore $w_1 = w_3$. So, in this case, the displacement of the imaginary point, this is the imaginary point, this point 1 is equal to the displacement of image point. So, that conclusion we have got after applying the boundary condition.

So, we have applied the boundary condition at the edges with the simply supported condition that is the bending moment is 0 and also the slope 0 at the fixed edge condition. So, 2 equations we have got $w_1 = w_3$ for the fixed edge condition, where w_1 is the imaginary point deflection and w_3 is the corresponding image point. So, whenever it appears the imaginary point, we can now replace it by the displacement of the image point.

So, similar was the case in case of simply supported edges, but you can see the difference here. Here the $w_1 = w_3$ but there, you will get $w_1 = - w_3$, there means in simply supported case you are getting $w_1 =- w_3$. But in fixed edge condition, you are getting $w_1 = w_3$. At the free edge of course, the boundary conditions are bending moment = 0 and edge shear force = 0.

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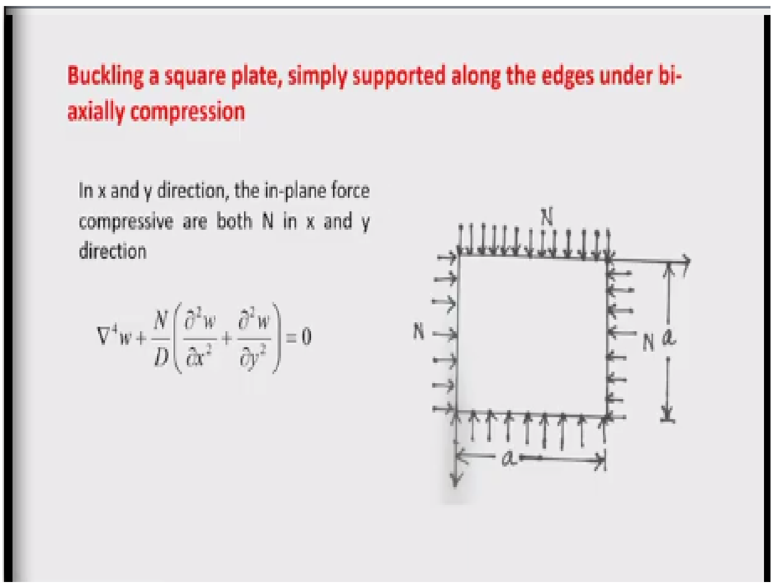

Now let us consider a buckling of square plate simply supported along the edges under biaxial compression. So, we take a square plate just to illustrate the application of this method; the compression is along the x-direction and along the y-direction both are same amount, and there is no membrane shear, there is no shear load, so shear buckling is not considered here. So, therefore the differential equation for buckling is now $\nabla^4 w + \frac{N}{D}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0$.

Because the N_x and N_y is equal here, so I have taken the N common from both the curvature term $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$. So, the equation now in this form, the differential equation is now can be seen that it is like that $\nabla^4 w + \frac{N}{D}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0$, where the meaning of D is the flexural rigidity of the plate and N is the axial compression and equal in both the direction. Let us solve this problem by finite difference from.

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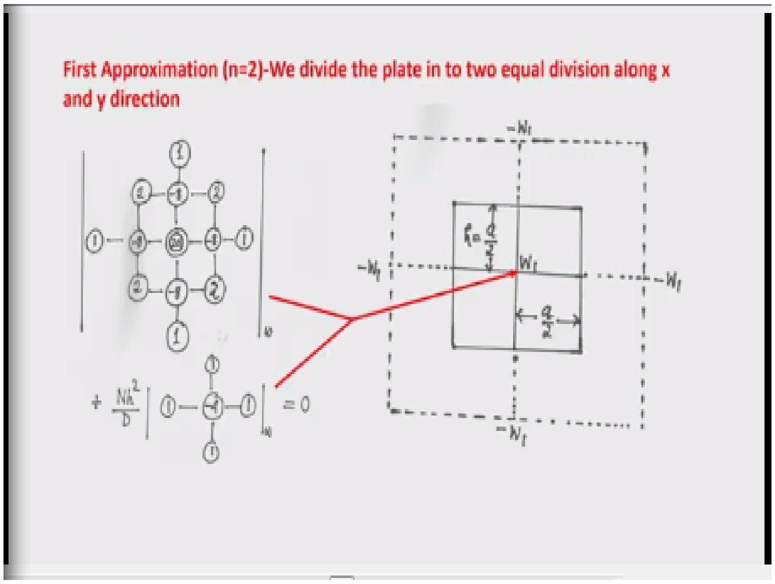

Now first, we approximate the approximation is that n = 2, that means the plate is divided into 2 equal divisions along x and along y-direction. So, a square plate whose side was a, now divided into 2 divisions $\frac{a}{2}$ and $\frac{a}{2}$. So, there are 4 divisions and this node here because the other node, if you consider all, are falling in the boundary. So, therefore deflection value will be 0 but central node, let us see this deflection is w_1 that is on the one node has to be considered here.

And you can see that the other node because if you see the molecular form of the finite difference equation of the plate, you can see that 5 nodes have to covered in the horizontal direction centrally and vertical direction centrally also 5 nodes. So, therefore other 2 nodes are falling outside the boundary, and we have given it is w_1, w_1. But coefficient -1 I have given, because I have shown you that for simply supported case if the deflection of the imaginary point is w_1 and corresponding image point is w_3 then $w_1 =- w_3$.

So, here the image point of this node is this central node w_1, so for example, 1, node number is 1. So, therefore the deflection here will be $-w_1$, if the deflection here is w_1. Similarly, in other direction, vertical direction, here deflection is $-w_1$ and here it is $-w_1$ and towards the right it is also $-w_1$. So, this is the stencil of the buckling equation and because this $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$, these 2 terms are there.

So, we have written these 2 quantities in finite difference from 1, -4, 1, 1 and 1. So, now placing this finite difference term, this is for plate bending, and this is coming for this axial compression terms. So, then if you place this stencil here, this node 20 should be placed here, this double circle, double circle should be placed here. So, if you place double circle here, this -8, the coefficient here will go here and naturally, there is this node is falling on the boundary, and therefore deflection is 0.

So, similarly, towards the left-hand side, it is also on the boundary right-hand top it is also on the boundary, and here on the bottom, it is also on the boundary. Similarly, this coefficient, that is this node this circle that you are seeing that 2, 2, 2, 2 that are also falling on the corners. And this is because of simply supported edge, this plate is simply supported edge, so deflection value at all points along the boundary is 0.

So, that we are getting only say $20w_1$ by applying this equation, no other quantity can appear here. Then let us see this term Nh^2/D 1, **-,** -4, 1, 1, 1, 1 1 is vertically and 1, -4, 1 is horizontally. That means -4 coefficient is to be placed circle with -4 coefficient has to be placed at the nodes. So, if I place this circle that -4 containing the coefficient -4 here, other point say 1 is falling on the boundary, this is falling on the boundary, this is falling on the boundary, and this is also falling on the boundary, so naturally, the deflection is 0.

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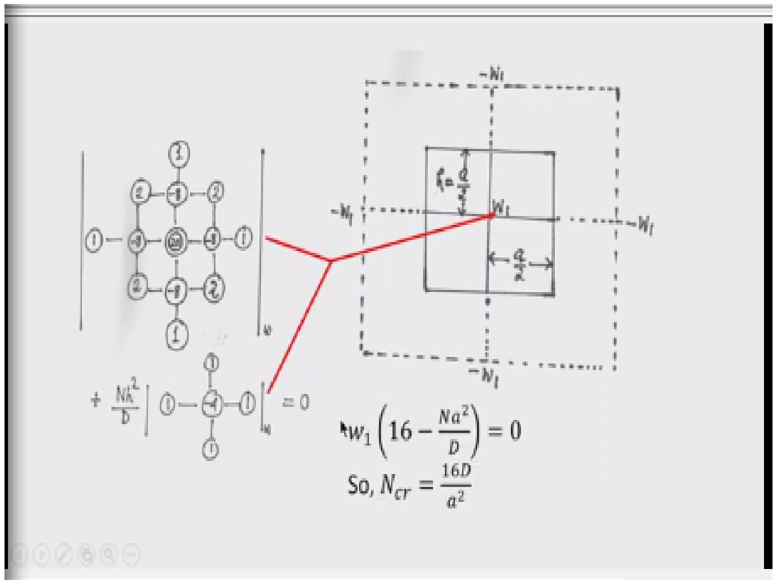

So, hence finite difference equation now can be written 16; how 16 is coming? 20 - 4, because 20 is there at the centre and no other values are important there because it is they are falling on the boundary. And this -4 because this is also again this -4, so $\frac{Nh^2}{D}$, $-\frac{4Nh^2}{D}$, a^2 is coming because $h=\frac{a}{2}$. So when you write $\frac{Nh^2}{D}$, $-\frac{4Nh^2}{D}$ and then put $h=\frac{a}{4}$, then this quantity will be $\frac{Na^2}{D}$.

So, for non-trivial solution, this quantity inside the bracket should be 0. So, the critical load for this plate is now is $\frac{16D}{a^2}$. So, by first approximation, that is, by dividing the plate into 2 divisions, we now get this value. That critical load of the plate is given by $\frac{16D}{a^2}$. So, let us see whether we can improve the accuracy of this method by increasing the number of divisions.

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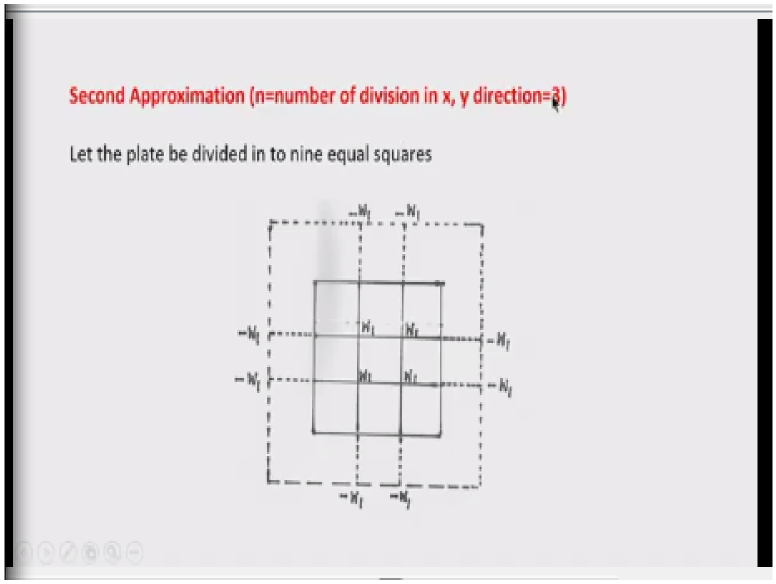

So, in the next step, we take the second approximation, we take number of division as 3. So, we divided the plate into 3 equal divisions. So, each division is now $\frac{a}{3} \times \frac{a}{3}$. So, side of a small square is now $\frac{a}{3} \times \frac{a}{3}$, that is a square small square and full plate was of the size a by a. So, 9 divisions are there in this plate total, so now 9 equal squares are there. So, we now number the node.

Now because of symmetry, the deflection here, say, is the same deflection as that deflection. So, therefore $w_1 = w_1$ and $w_1 = w_1$ because of symmetry, so these value we have assign now. And this is boundary value wherever this plate stencils, the circles of the plate stencils fall on the boundary; then the deflection value is 0. And when we consider the boundary condition that is bending moment is 0 at the simply supported edge, we encounter 1 imaginary node.

So, that imaginary node for simply supported condition has a coefficient -1, and therefore deflection value of the imaginary point that has to be taken to replace this is $-w_1$ as compared to it is deflection of the imaginary point. So, similarly $-w_1$ is written here and here also $-w_1$, here it is $-w_1$, $-w_1$ all are written accordingly.

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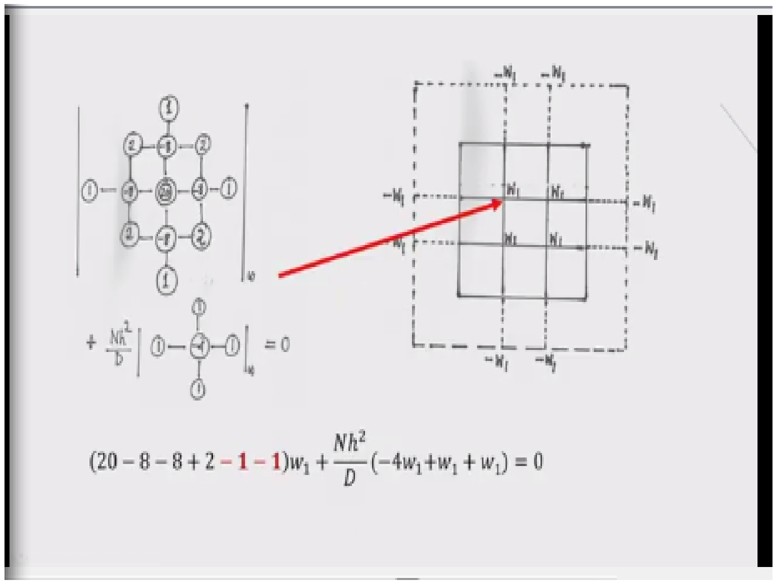

So, now place this stencil. So, this is the stencil for the buckling equation of the plate; that is, you can tell it the molecular form. So, if we now place the stencil here, there are 2 parts one is for plate bending that you can see, $\nabla^4 w$, and this is for buckling part, that is $\frac{Nh^2}{D}$. So, now let us first place this, if I place this here, then $20w_1$, so $20w_1$ is coming. Then here this circle is falling here, so -8.

Then this circle is falling here, so -8, this 1 is falling on the boundary, so 0 value. This circle 1 that is falling on the boundary, so, therefore, it is value will not be considered because the boundary deflection is 0. Then, on the other hand, this value, this circle 2 is falling here, so this value is taken 2. Now with red colour 2 numbers are written -1, -1 that is just to indicate this are the node that are outside the plate domain.

But the deflection of that imaginary point is now written in terms of deflection of the actual point, image point by using the boundary condition that is the bending moment condition to be 0 at the simply supported edges. So, therefore one imaginary point we get here, if I place this stencil here, then this $-w_1$ will be here corresponding to this w_1. Then another imaginary point we get here, so $-w_1$, so -1 is there.

So, therefore I have indicated in the red colour this 2 imaginary point deflection that is expressed in terms of the deflection of the corresponding point by using the bending moment 0 condition at this simply supported edges. So, this is for plate and plate bending and for buckling the axial compression part. Now, if you place this stencil here, you can see the coefficient is -4, so $-4w_1 + w1 + w1$ is there.

Then +w 1 is there; other nodes say these nodes and these nodes are falling in the boundary, so therefore they do not appear here. So, the equation for this $\frac{Nh^2}{D}$ that is the buckling part, is written like that.

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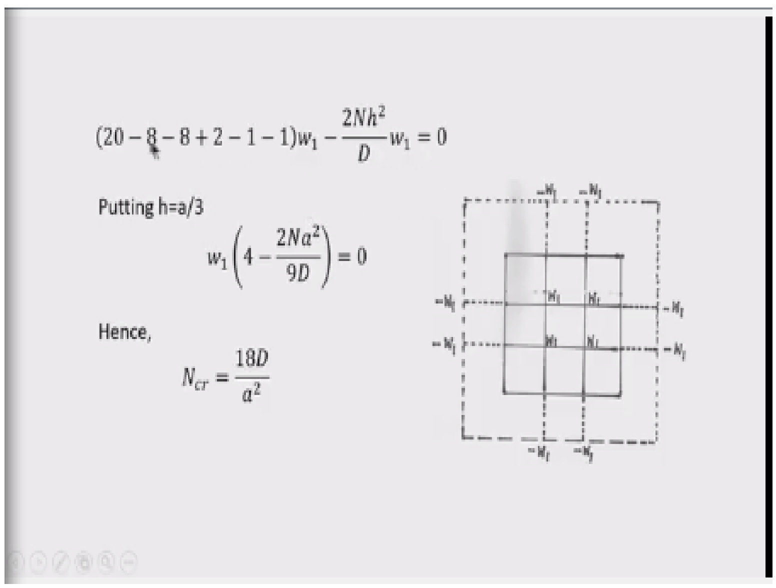

So, after simplification, now we can write this and putting 20 - 8 - 8 + 2 - 1 - 1. And this is, if you simplify it, it will be $-2w_1$, so, therefore, it is$-\frac{2w_1Nh^2}{D}$ by putting $h = a/3$ because we have divided the plate into 3 equal parts. We now write $w_1\left(4 - \frac{2Na^2}{9D}\right) = 0$

. So, therefore the critical load for buckling now becomes $\frac{18D}{a^2}$. So, let us compare the result that is obtained by finite difference method with the exact results. Now exact result for this problem is available.

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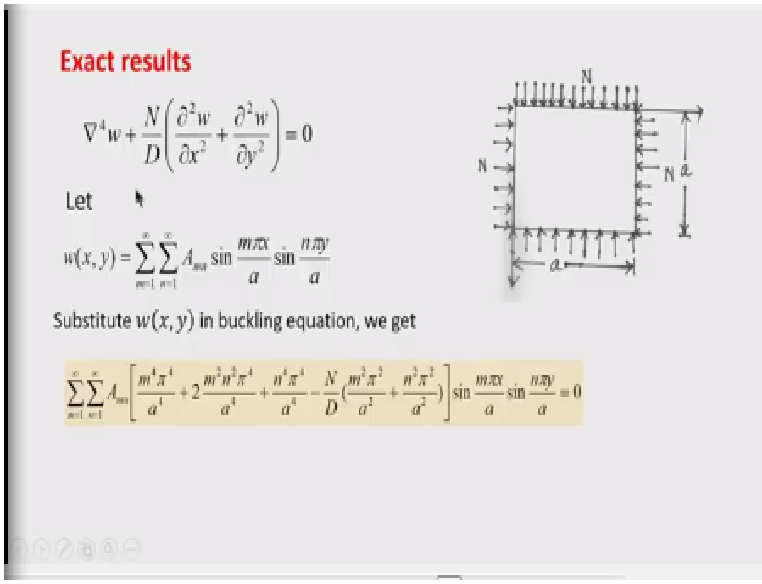

So, if I see the differential equation is this, and the edge condition is simply supported. Then I can take the double trigonometrical series that Navier series to find the solution of the deflection. That is, I want to solve it by exact method. So, therefore substituting this in the differential equation.

We now get this form that is the $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left[\frac{m^4\pi^4}{a^4} + 2\frac{m^2n^2\pi^4}{a^4} + \frac{n^4\pi^4}{a^4} - \frac{N}{D}\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{a^2}\right)\right] \sin sin\frac{m\pi x}{a} \sin sin\frac{n\pi y}{a} = 0.$

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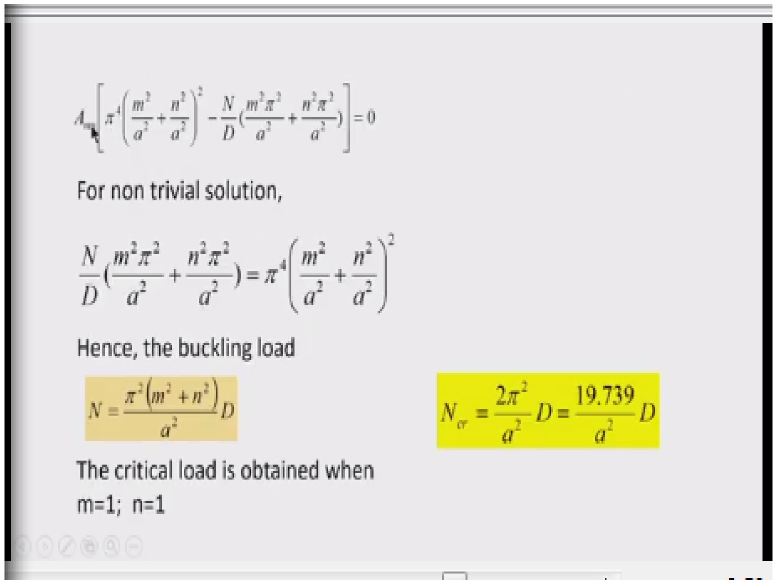

So, I can simplify this expression as like that $A_{mn}\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2-\ldots\right]$ because it is a square plate. So, $\ldots-\frac{N}{D}\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)]=0$. Because a square plate is taken, so that $a=b$ in this plate, and therefore the double summation now appears as A_{mn} sin $sin\ \frac{m\pi x}{a}$ sin $sin\ \frac{n\pi y}{a}$. But A_{mn} are 2 different wave numbers that represents the half wave number in the x-direction, m represents the half wave number in x-direction.

n represents the half wave number in y-direction. So, this equation is now written in this form, and for non-trivial solution, one can see that $\frac{N}{D}\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)=\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2$. Now since it is a square plate and after simplification, we get the critical load is $N=\frac{\pi^2(m^2+n^2)}{a^2}D$.

And the lowest value of load can be obtained, and this is easily observed that if m = 1 and n = 1, that means the plate buckles with half wave number in both x and y-direction. Then the lowest value of buckling load is obtained, and that will be the critical load. So, by putting m = 1, N = 1, we get the critical load for buckling as this. So, it is value is $\frac{19.739}{a^2}D$, so this is the exact value.

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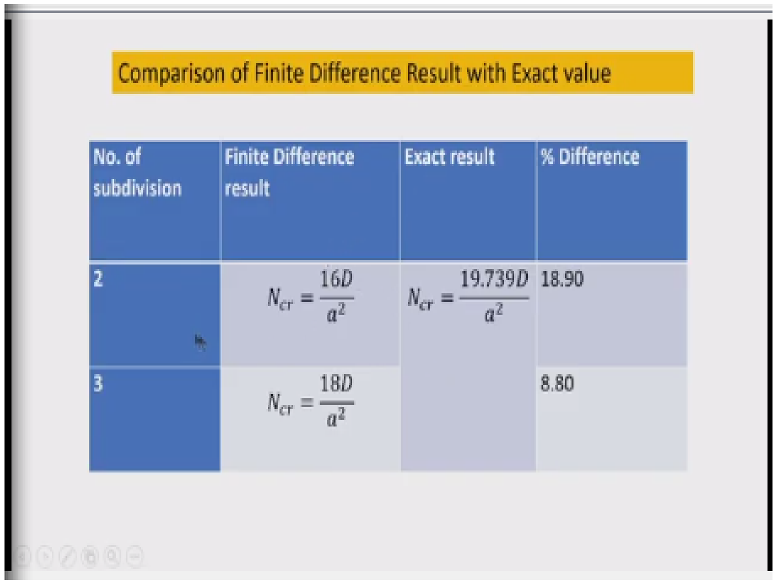

No. of subdivision	Finite Difference result	Exact result	% Difference
2	$N_{cr} = \frac{16D}{a^2}$	$N_{cr} = \frac{19.739D}{a^2}$	18.90
3	$N_{cr} = \frac{18D}{a^2}$		8.80

So, now in this table, I compare the finite difference result with the exact results. So, we get this the number of divisions if it is 3 in the plate. Then, finite difference result gives $\frac{16D}{a^2}$; exact result is $\frac{19.739}{a^2}D$. So, percentage difference of result or percentage error or you can call is 18.9. So, when we say in case the division by 3, the critical load is $\frac{18D}{a^2}$ but the exact value of the critical load is $\frac{19.739D}{a^2}$ and here percentage difference is 8.8. So, that means by increasing the number of division the accuracy is improved.

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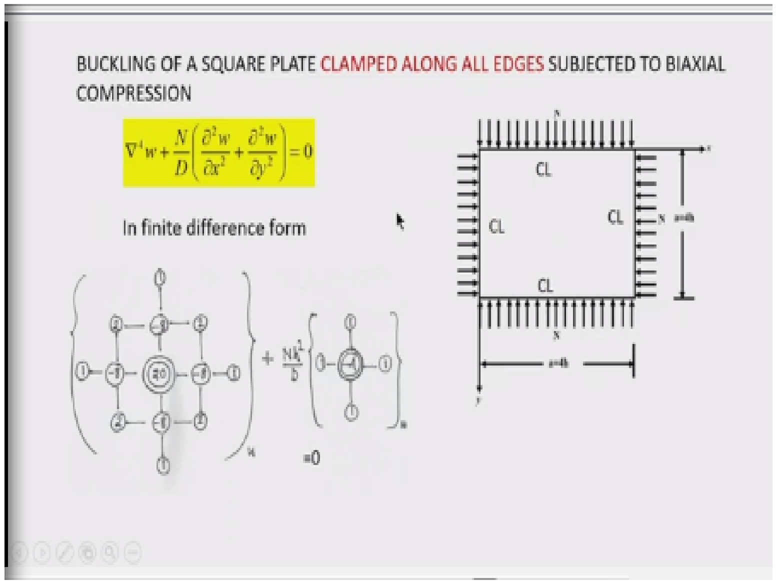

Now, here again, I give you one problem that you can see it is solved in a similar way, but the edge condition is now altered. So, edge condition is now clamped instead of simply supported, and this is the stencil for the plate buckling equation.

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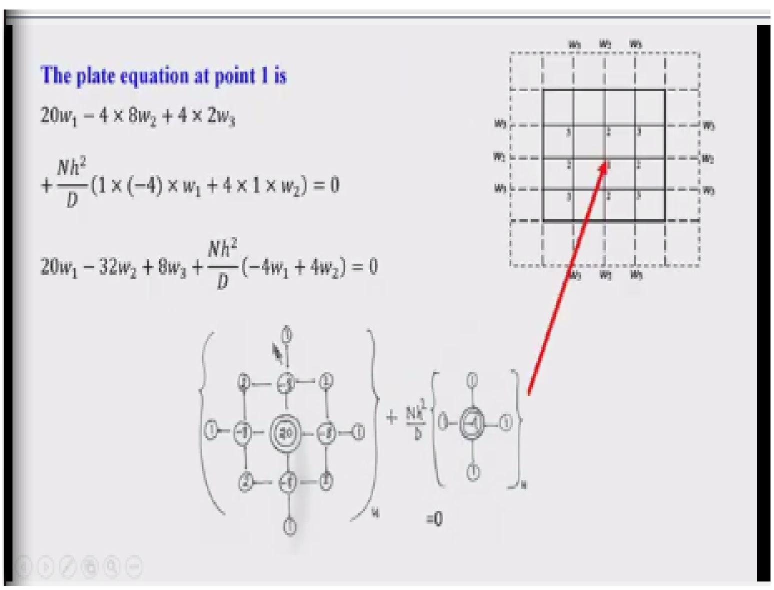

And we take this stencil, and because of symmetry, you can see if this is the central deflection w_1. Then because of symmetry, this deflection is said this value of the deflection at this node is w_2, here also w_2, w_2, w_2 and then also it is w_3, w_3, w_3 and w_3. And because of the fixed boundary condition, the deflection of the imaginary point here will be equal to the corresponding deflection of the image point.

So, if this is the deflection of the image point w_3, then this deflection will also be w_3. So, similarly, other deflection values are written. So, applying the plate equation, that is the buckling equation in point 1, we get now say this stencil if I place here. So, 20 is there and here -8, so $-\ 8w_2$. And there again -8 is here, here -8 is there, here -8 is there, so 4 – 8, so $4\times(-\ 8w_2)$, then these other value that here. These 3, 3 this point nodes are there w_3, w_3, w_3, w_3, so 4.

And coefficient of this node is 2, so therefore $4\times w_3$, we have written for plate. And other nodes, these nodes say with coefficient 1, 1, 1, 1 is falling on the boundary, so they are not appearing here. And the buckling part now can be written, say this is value, so this $-\ 4\times w_1$ plus this other say 1, this is all are same values, so w 2, so it is written 4 into 1 into w 2. So, these equations are now obtained like that, so this is say, equation 1.

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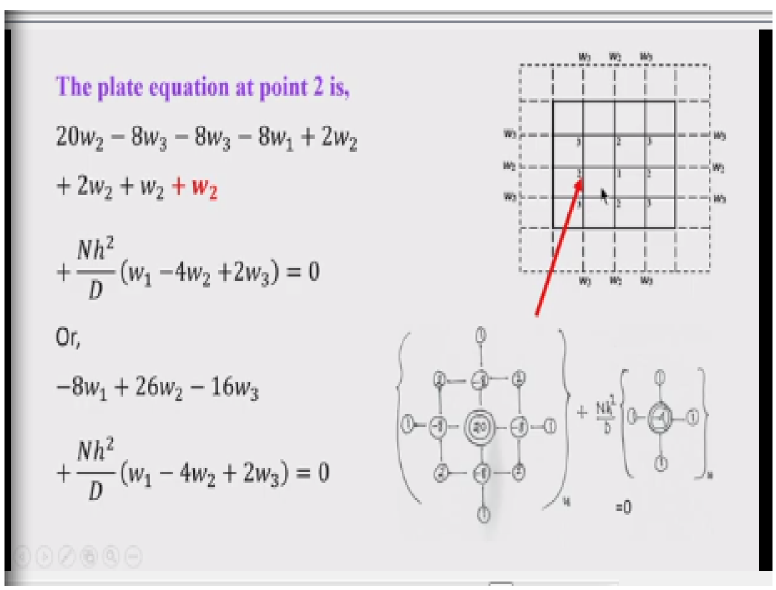

So, now at the point 2, if it is applied, then question of 1 imaginary point comes. Because if you place the stencil here, then one point that is -8 is in the boundary, but this node will come here. So, one imaginary point is there, so this value has to be expressed in terms of the corresponding image point value. And corresponding image point is 2, and this is for fixed condition, it is the deflection of the imaginary point is equal to the deflection of the image pair.

So, this term is coming here with w_2 that I have indicated with the red colour to understand that is due to the imaginary nodal deflection expressed in terms of corresponding image node deflection inside the plate. So, plate equation is now written like that $20w_2 - 8w_3 - 8w_3 - 8w_1 + 2w_2 + 2w_2 + w_2 + w_2$ and this w_2 is coming due to imaginary point. And now the buckling part is written because this node is now placed here.

So, $-4w_2$ and there you are getting the other points, say this is w_3 and this is w_3. So, $2w_3$ and this is w_1, so w_1. And this value is falling here, so naturally, it will not come. So, therefore we write this equation as this, after simplification $-8w_1 + 26w_2 - 16w_3 + \frac{Nh^2}{D}\left(w_1 - 4w_2 + 2w_3\right) = 0$.

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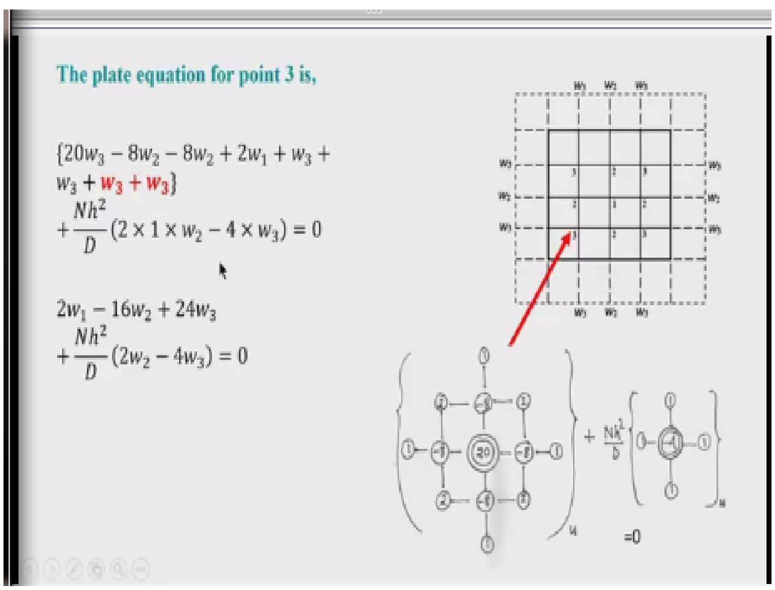

The plate equation in node 3, let us come to the node 3. If you place the stencil at the node 3, first let me see that you will encounter one imaginary node here and another imaginary node at the bottom. One imaginary node towards left and another imaginary nodes at the bottom. So, therefore 2 imaginary nodal deflection has to be now expressed in terms of corresponding image points. So, therefore corresponding image points are 3, so w 3, w 3 I have written.

And other points are written as usual, so w $20w_3 - 8w_2 - 8w_2 + 2w_1 + w_3 + w_3 + w_3 + w_3$ because these are due to points which are lying outside the boundary, and this is expressed in terms of deflection of the points inside the boundary, the corresponding image point. So, writing this for the plate and then we come for this buckling part, that is in-plane force part. So in-plane forces, the stencil is this $\frac{Nh^2}{D}$, 1, -4, 1 and vertical, it is 1 and 1.

So, if I place here this node, then you can see $-4w_3$, that is ok, and this node is coming here. So, this node is coming here, and another node is coming here, so $2\times w_2$. And other 2 nodes say 1, and this one is falling on the boundary, so naturally, these values are 0. So, after simplification, we get this equation.

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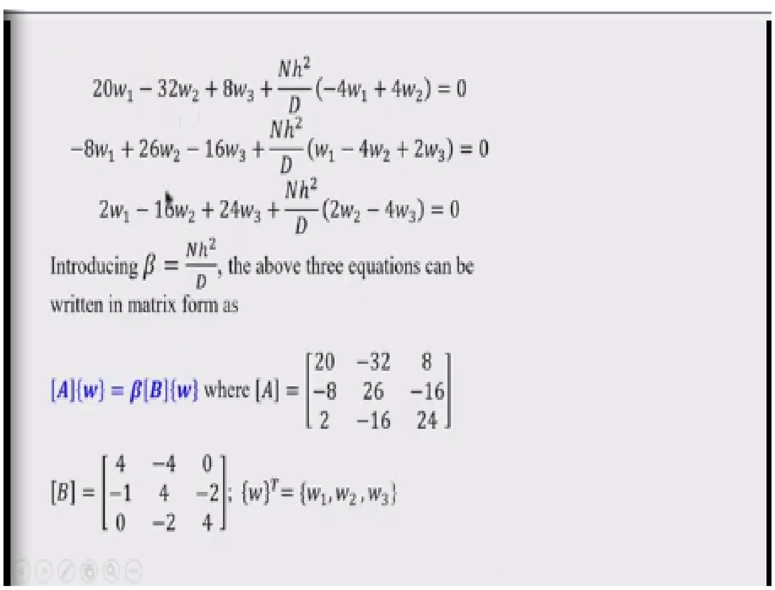

So, 3 equations now we have got, and these 3 equations now can be expressed in the matrix from. Say, A matrix, w matrix, beta is a factor which is containing the buckling load N, and it is written as $\frac{Nh^2}{D}$ and β $B\,w$, where A is the coefficient matrix that is 20 - 32, 8, the coefficients of w_1, w_2, w_3, etcetera in each equation. So, after writing this, the B equations are written as the coefficient of w_1, w_2, w_3 obtained from the in-plane force part.

And that is obtained for plate bending part. So, this coefficient you can see -4, then 4 like that it is written. So, after obtaining because these quantities are transferred in the right-hand side, so this sign will be reversed. So, therefore it is 4, and this is -4, and this is 1, so this is -1, this is -4, this is +4, and this is +2, this is -2 like that. And here, no w_1 is appearing, so w_1 coefficient is 0 and then 2, 8, if it transferred to the right-hand side then -2.

And this is transferred when transferred to the right-hand side it will become +4 and w vector is composed of w_1, w_2, w_3. So, you can see if I multiply this equation by B inverse, we can find an equation like that, if I multiply this equation by B inverse, then we can find this equation as a standard Eigenvalue equation. So, here are the Eigenvalues of the D are obtained because B inverse B, B inverse B is unit matrix or identity matrix.

And B inverse A that is a matrix D, so Eigenvalues of D are calculated, and it is found as 2.6464 and 4.747036 and 7.1500. And lowest Eigen value will give this critical value of the buckling load. So, taking $\beta = 2.6464$, we can now calculate the critical value of $N = \frac{2.6464D}{h^2}$. And putting $h = \frac{a}{4}$ because A is the plate dimension and h is the mesh size. So, we now replace the mesh size with the plate dimension. We now get these $\frac{42.34D}{a^2}$ as the critical load; this can also be written as $\frac{4.29D\pi^2}{a^2}$.

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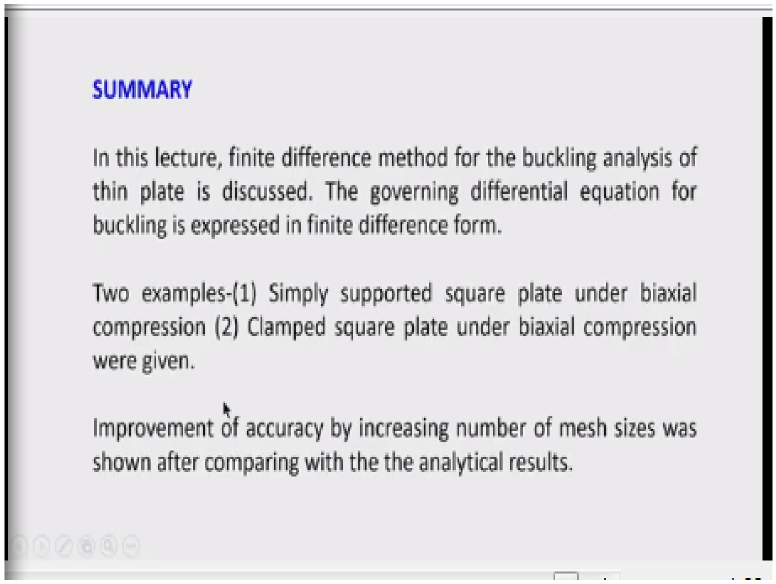

So, let us summarize what we have done in today's lecture. In this lecture, finite difference method for buckling analysis of the thin plate is discussed. The governing differential equation for buckling is expressed in finite difference form first, and then we use this. Finite difference equation in solving 2 problems, one problem is simply supported square plate under biaxial compression and second problem is clamped square plate, the plate which is also square, but the edges are clamped and also under equal biaxial compression. Improvement of accuracy by increasing the number of mesh size was shown after comparing with the analytical results. Thank you very much.