Plates and Shells
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Module-07
Lecture-20
Rayleigh-Ritz and Gallerkin Method in Buckling of Plate

Hello everybody, today I am starting module 7 and this is the lecture number 1 of this module. I want to discuss today the plate buckling using approximate method.

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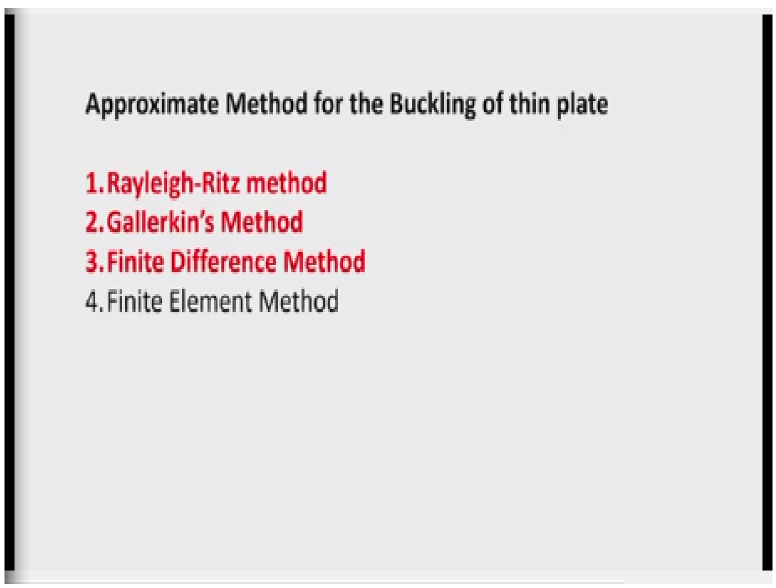

There are different approximate methods available for the solution of the plate problem. In fact, any physical problem or mechanics problem can be handled by this approximate method. These are based on numerical techniques and the efficiency of the method can be improved by suitably selecting the method of integration or by taking sufficient number of terms in the assumed deflection.

So, among the different methods I am listing here the name of the 4 methods but there may be other methods also. So, first method is Rayleigh-Ritz method, second method is Gallerkin's method, third method is finite difference method and the fourth is finite element method. The last one finite element method is not within the course prescribed here. So, therefore I will discuss the first 3 methods in studying the buckling of thin plate.

Now you have some exposure in my earlier classes on the approximate method for plate bending. That is I discussed the Rayleigh-Ritz method, Gallerkin method and finite difference method for finding the deflection of the plate due to transverse loading and also the stress resultants. So, this method will be extended to study the buckling of the thin plate. So, you have seen the Rayleigh-Ritz method and Gallerkin method these 2 methods that I am writing, 3 methods I have marked red that means 3 methods I will cover in this course.

So, first method you see Rayleigh-Ritz method and Gallerkin method. These 2 methods have their origins from energy principles. So, in Rayleigh-Ritz method we have taken the strain energy expressions and the work done by the external load on non conservative forces. And then we use the variational method of this calculus and then we found the unknown coefficients of the deflected series.

In the Gallerkin method, although the energy equation is not needed but it has it is origin from the virtual work principle. So, virtual works are done by all the forces is 0 if the body or the system is in equilibrium, so that is the principle. Or in other words that we can also say that if the system is in equilibrium, the virtual work is 0. So, if the virtual work is 0 then we say that system is in equilibrium.

But it can also be said in another way that for system is in equilibrium the virtual work is 0. So, from that condition that we have found that equilibrium equation can be used in the virtual work principle. And then we derived a Gallerkin's equation that can be used with the assumed deflection function. In both the methods the Rayleigh-Ritz method or Gallerkin method we required to use the deflected surface of the plate.

And deflected surface of the plate cannot be known or is not easily known by solving the equation or for complicated boundary condition that we encounter. So, therefore we have to select an approximately deflected function that may consist of a polynomial or it may be a combination of polynomial and trigonometric series or it may be a combination of polynomial or a hyperbolic function or it may be a combination of hyperbolic and polynomial function.

The choice of this assumed deflection function can be made in such a way that these function satisfy the geometrical and force boundary conditions at the edges of the plate. But sometimes it is difficult to satisfy both the condition geometric as well as force boundary condition. So, in that case we take a series which can at least satisfy the geometrical boundary condition that will yield the result with reasonable accuracy. That has been observed in many cases and that can be used for practical application also.

The third method is finite difference method that is another way of solving the plate problem using instead of this differential equation we take the finite difference form of the deflection equation. And then we develop a number of simultaneous linear equations that can be solved to find the unknown deflection in the plate. So, let us discuss one by one, first I will take up the Rayleigh-Ritz method, then I will take up the Gallerkin's method and then finite difference method.

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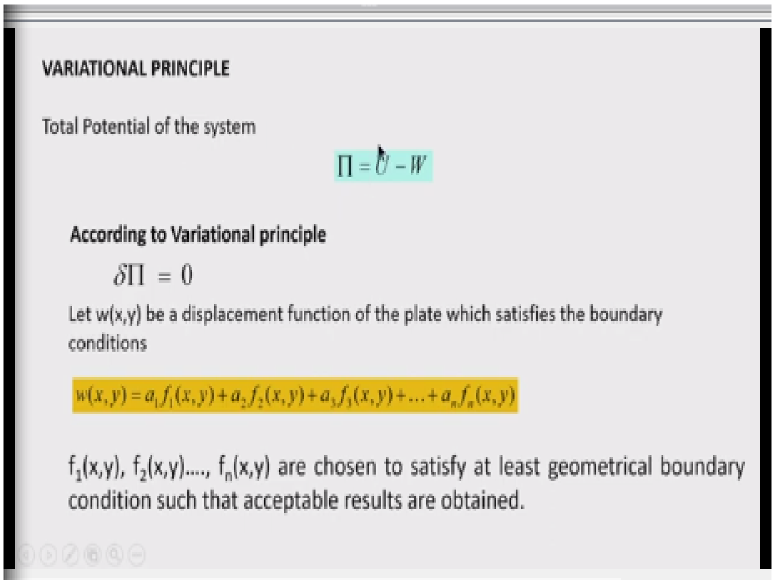

So, Rayleigh-Ritz method has it is origin on the variational principle, that you are seeing this equation that is $\Pi = U - W$ which is a very important equation and that Π is total potential. And what is this total potential U - W? U is the potential strain energy of the plate in bending and W is the work done by the forces. Now, here in the buckling problem the work done due to transverse loading will be 0, that is in buckling problem is a homogeneous problem, so we neglect the transverse loading.

Now in the work done will be evaluated based on the work that is done by the in plane forces. What are in plane forces in the plate? This N_x, N_y and N_{xy} that is applied at the edges. So, here is the difference that work done due to in plane forces have to be first found out and then it is algebraically added with the strain energy of the bending of the plate and then we can find the total potential.

Now work done due to this membrane forces that is N_x, N_y, N_{xy} that we can also term these forces as a membrane force, arises because of this in plane action. And when we calculate the work done due to membrane forces, we assume that this force does not change during the deformation of the plate. Now here work done due to membrane forces, there is involved in the expression w has to be calculated due to vertical displacement.

The in plane strain that is the direct strain arising due to in plane forces is neglected in the small deflection theory of the plate, that is the Kirchhoff's hypotheses. So, therefore we consider the work done due to membrane forces produced due to vertical displacements, so that is the difference here. Now according to variational principle, the variation of capital Π that is total potential equals to 0, so this is the equation.

That means we have to take the variation of Π, Π contains the term strain energy expression and also work done by the membrane forces. Now, you can see that this Π contains the arbitrary constant that is assumed to form a deflection function approximately. Also this W contains this arbitrary constant used to form the deflected surface. So, therefore the strain energy term or work done, strain energy is a positive quantity.

So, this strain energy term contains the square of this coefficient and therefore when we take the variation, the square term reduced to linear term. So, that is advantage of this variational principle that we have to solve finally the linear equations to find the unknown coefficients. So, we assume that $w(x,y) = a_1 f_1(x,y) + a_2 f_2(x,y) + a_3 f_3(x,y)$, in general $a_n f_n(x,y)$. So, these are the terms that is taken to form the deflected series.

Now this coefficient is arbitrary, we do not know the magnitude, sometimes when the accuracy is compromised that is just a rough approximation is required, we can only take the first term. In some cases, for example in simply supported cases if we take the series as a double Fourier series that is the Navier series double trigonometrical series, then the first term will give very accurate result for the deflected surface. And higher number of terms are required to find out or to improve the accuracy in the bending moment and shearing force expression. Now here $f_1(x,y)$ and $f_2(x,y)$ etcetera, are the arbitrary function. So, what is this function? This function maybe a trigonometrical function; maybe this algebraic or polynomial function or maybe a combination of this or maybe a combination of sine hyperbolic and this cos hyperbolic function with trignometrical function.

So, the combinations are possible for different type of function, this is taken to satisfy the boundary condition. It has also to be bear in mind, that if you choose a very complicated function even it satisfies the boundary condition completely the calculative effort that means integration that is required in the variational principle finally will be very lengthy. So, even some authors say Bishop has mentioned the Eigen functions of the beam to form the approximate deflected surface of the plate for different boundary condition.

But in some cases the Eigen functions of the beam are very complicated in nature. For simply supported beam it is only a sine function say $\sin(n\pi x/l)$, l is the length of the beam. Now in case of plate we take $\sin(n\pi x/a)$, that is one term. Then in the y direction also if it is simply supported $y = 0$, $y = b$ then we take $\sin(n\pi x/b)$. So, all these 2 functions actually satisfy completely the simply supported boundary conditions.

So, Eigen function for simply supported beam is a very simple in nature but it is not so if I take a cantilever beam. So, for cantilever beam Eigen functions are complicated consisting of cos hyperbolic and sine hyperbolic function as well as trignometrical function. How the Eigen functions are found out? That is another question, Eigen functions are found out from the solution of the free vibration equation.

So, free vibration of the equation can be solved for different boundary condition to find out the Eigen function and that can be chosen as a trial function in the plate, that has been suggested by Blshop. But this method of taking Eigen function in the plate problem requires a lengthy calculation but it will yield the accurate result compared to some trial function that does not completely satisfy the boundary condition.

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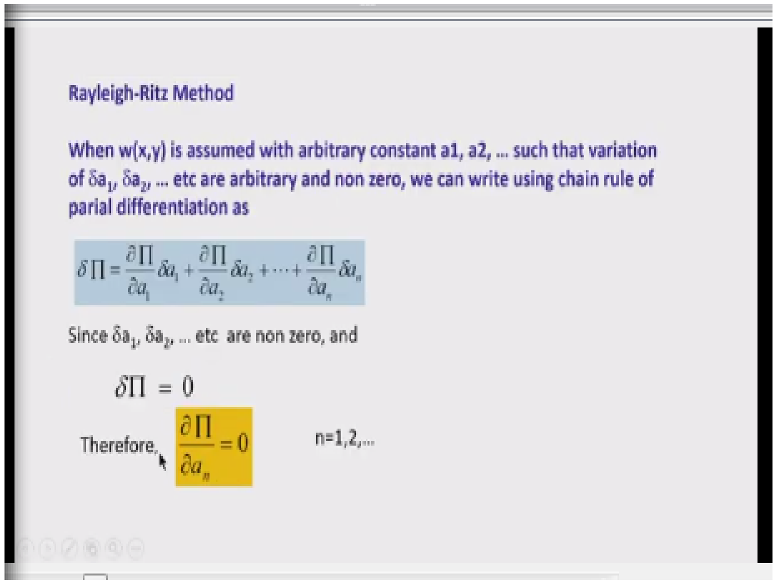

So, let us discuss what is Rayleigh-Ritz method? Suppose $w(x,y)$ is assumed with arbitrary constant a_1, a_2 etcetera such that variation of this a_1, a_2 etcetera upto a_n are arbitrary and non-zero. Then we can write this equation variation of Π that is the total potential equal to differential coefficient of Π with respect to a_1 into δa_1, δ is the variation symbol δa_1 + $\frac{\partial \Pi}{\partial a_2}\delta a_2$ + and up to n^th term if we take $\frac{\partial \Pi}{\partial a_n}\delta a_n$.

Actually in the practical computation we have to take a finite number of terms although the Navier and Levy's method you have seen that summation contains the infinite number of terms that is a infinite series but practically the number of terms have to be truncated. So, therefore you can take 3 terms, you can take 4 terms, 5 terms up to n that is the finite number of terms you have to take. So, if I write this $\partial \Pi$ using the partial differential coefficient then we can take each of the derivative that $\frac{\partial \Pi}{\partial a_1}$, $\frac{\partial \Pi}{\partial a_2}$ and $\frac{\partial \Pi}{\partial a_n}$ = 0. Because δa_1, δa_2 etcetera cannot be 0 and we have taken these are the arbitrary variation and non zero quantity. So, therefore $\delta \Pi$ that is the variation of total potential 0 indicates that $\frac{\partial \Pi}{\partial a_n} = 0$, where n varies from 1, 2 up to n, n number of terms that you have taken. So, with this method now we can find the linear simultaneous equation containing a_1, a_2 etcetera and we can solve it to know the complete deflected surface.

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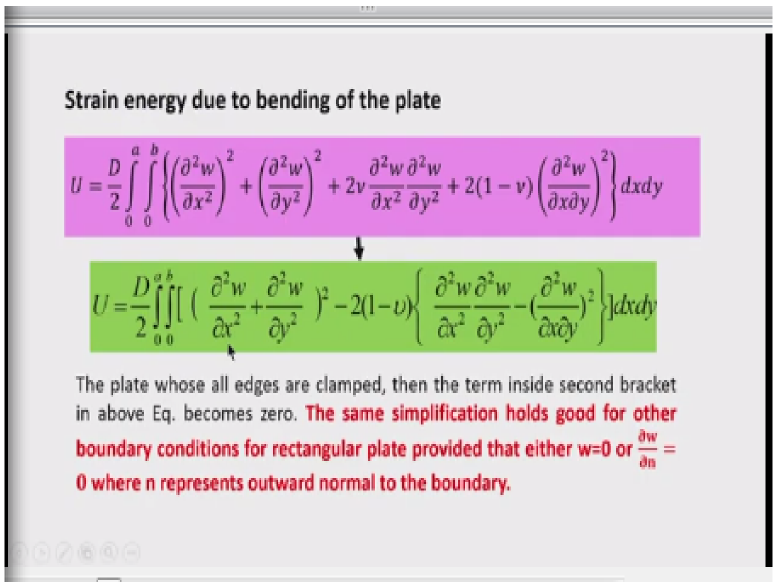

So, in applying the Rayleigh-Ritz method, we must know the strain energy expression that is U for the plate during bending. And if this is the flexural rigidity of the plate, what is the flexural rigidity of the plate? Flexural rigidity is related to the Young's modulus of elasticity, Poisson ratio and thickness of the plate. So, we can write that $D = Eh^3/12(1 - \nu^2)$, that is also possible.

So, D is the flexural rigidity of the plate and the expression for strain energy of the plate is derived earlier and it is written here

$$U = \frac{D}{2}\int_0^a\int_0^b \left\{\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\} dxdy$$

$U = \frac{D}{2}\int_0^a\int_0^b \left\{\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\} dxdy$ and integration within the domain of the plate, domain of the rectangular plate of size a and b is in the length direction from 0 to a, in the width direction 0 to b.

So, we have substituted the limit of the integral as 0 to a for x and 0 to b for integration with respect to y. Now adding a quantity $2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$. And subtracting the same quantity from this expression we can arrange this expression in this form. So, $(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})^2-2(1-\upsilon)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-(\frac{\partial^2 w}{\partial x\partial y})^2\right\}$.

Now this quantity $\frac{\partial^2 w}{\partial x^2}$ has the physical meaning, that it is represents the curvature of the plate during bending in the x direction. Similarly, curvature of the plate in bending due to y direction is $\frac{\partial^2 w}{\partial y^2}$. And what is this quantity? $\frac{\partial^2 w}{\partial x\partial y}$ that is the twist curvature. So, twist curvature is related to the twisting moment in the plate.

Now it is seen that the plate whose all edges are supported, support maybe it is a clamp support that is a fixed support or it may be a simple support. So, when it is supported along all edges, then the term inside the parentheses vanishes, that can be also observed. Now you are familiar with this plate problem with this say Navier's method for rectangular plate. So, if you take a Navier series, so for example you take on the first term w is equal to say a $\sin(\pi x/a) \times \sin(\pi y/b)$.

Then substituting their secondary derivative here and multiplying them and subtracting this twist derivative square of the twist curvature from this quantity you will get 0. So, this is also proved if the boundary conditions for rectangular plate whose all edges are supported either clamped or simply supported and has $w = 0$ and the slope along the normal direction to the boundary is 0.

So, that quantity indicates that this strain energy expression where it is applied to circular plate because circular plate boundary is curved. So, when we compute the slope for a curved edge, we have to take the normal direction and then $\frac{\partial w}{\partial n}$ will represent the slope in the direction of normal to the boundary. And if these 2 quantities are 0, w is 0 and $\frac{\partial w}{\partial n} = 0$ then strain energy expression will consist of only this term.

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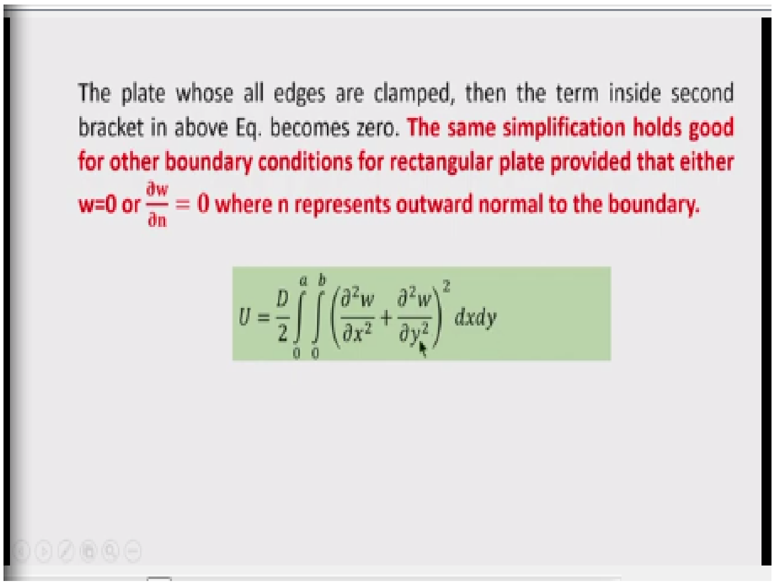

So, strain energy expression in that case is simplified just like $U = \frac{D}{2}\int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 dxdy$.

Now you can use either of these expression even say you do not know say these shape function is not very clearly defined. Because in simply supported case it is clearly defined by Navier and it satisfies the boundary condition exactly.

Now in some cases when the shape function is not defined properly or not tested properly. Then we can use the original expression also, there is no harm in using the original expression. Original expression is the general expression and here you can see that $\left(\frac{\partial^2 w}{\partial x^2}\right)^2$, that means if you assume a deflected surface then these curvatures square you have to integrate.

You see the integration of each term that means if you integrate this term say this is integration part 1, I_1. Then integration of this part will be I_2 the integration of $2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$ will be third integral I_3. And then fourth integral $2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2$.

So, you can integrate it in different components I_1, I_2, I_3, I_4 then you can add it. Sometimes it becomes convenient instead of squaring this quantity because squaring of this quantity may lead you some complicated expression to integrate it or it can be integrated you will arrive at the same results but the complicacy arises because of the expansion of this quantity. So, the original expression can be taken in solving the problem or you can take this expression after taking this term to be 0 for plate which has all edges simply supported or clamped, supported edges.

So, either of these expressions you can use. So, that is the strain energy due to bending and that must be accounted for in calculating the total potential of the plate. Now question arises what will be the work done by the membrane forces? Now since there is no external load applied for buckling. So, due to vertical displacement of the plate, the work done by the membrane forces should be calculated.

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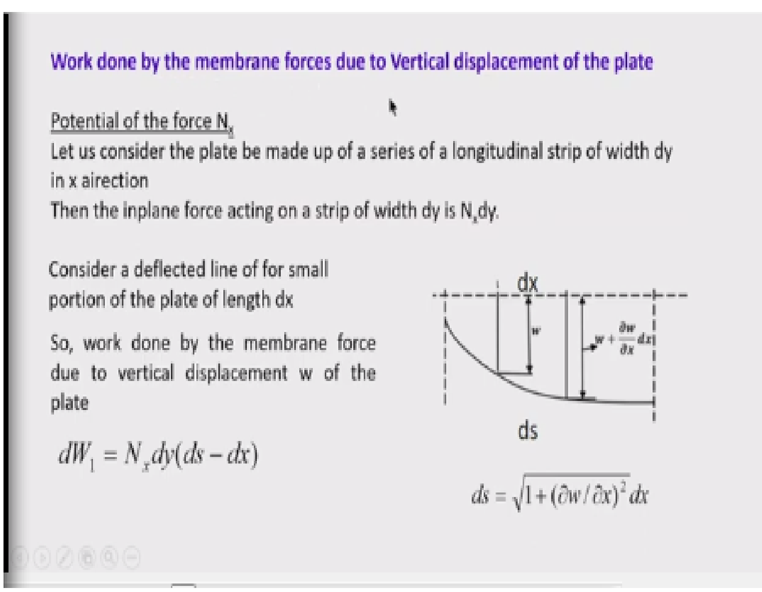

So, next task is to calculate the work done by membrane forces due to vertical displacement of the plate. So, we shall calculate the work done by each of the membrane forces N_x, N_y and N_{xy} and then we will add to find the total work done by the membrane forces due to vertical displacement of the plate. Now, let us consider the plate to be made up of a series of longitudinal strip.

So, consider a plate rectangular plate in x direction we take a strip of weight dy. So, in this strip if the N_x is the force along the x direction, then $N_x\ dy$ is the force on the strip in the x direction. Now consider the deflected surface in the x direction, so this shows the deflected surface in the x direction and the length of the element in the x direction is dx. So, in that case you can see that if w is the deflection here and at the end of the element the deflection will be w plus increment.

What is the increment? If the slope is $\partial w/\partial x$ if this angle is $\partial w/\partial x$ then for small angle, this angle multiplied by this distance that is you can call it this say $dx \tan\theta$ but $\tan\theta$ is here θ, so we can write $w+(\partial w/\partial x)dx$. So, this is the deflection at this end. So, now we have to find the work done due to membrane forces $N_x\ dy$ on the strip. So, this work done dW_1, I have denoted this here with $dW_1 = N_x\ dy$ this is the force and displacement that is the change in position of the N_x will be $ds - dx$.

Because ds is the curve length and dx is a straight length, so $ds - dx$ will indicate the deformation that is taken into account to calculate the work done by $N_x\ dy$. Now, if I take ds as a small quantity, then we can form a right angled triangle, ds is the hypotenuse and dx is the base and dy is the altitude. From the right angled triangle, we can write $ds^2 = dx^2 + dy^2$. So, naturally ds is $\sqrt{1+(\partial w/\partial x)^2}\,dx$, dx is the outside the square root, so you can take this quantity.

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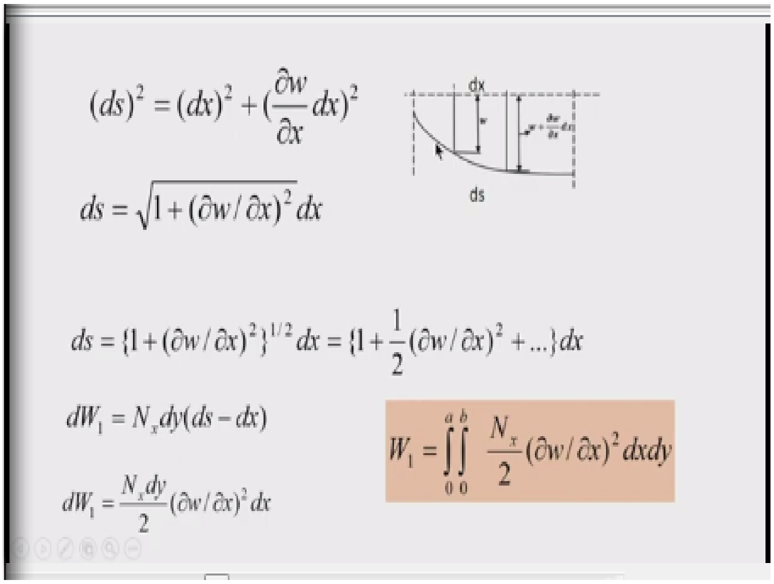

So, ds^2 is nothing but this the base of the right angled triangle we take ds consisting of this side ds, dy and dx. So, ds^2 that is the hypotenuse of the small right angular triangle $ds^2 = dx^2 + dy^2$ that is the base plus the height. So, height is $\frac{\partial w}{\partial x} dx$, this distance, so this square. So, this is giving you the ds^2, so ds is this that I have shown in my earlier slide also.

Now this expression can be expanded using the binomial series. So, this expression can be written as $\{1 + (\partial w / \partial x)^2\}^{1/2} dx$. Now expand this using the binomial series or binomial theorem, so provided the $\partial w / \partial x$ is less than 1. So, we can write it $\{1 + \frac{1}{2}(\partial w / \partial x)^2 + ...\} dx$ that is, plus other terms higher order terms will come. So, we will retain only these 2 terms because the deflection is small.

So, therefore dW_1, that is the work done by the elementary force $N_x dy$ is now $N_x dy$ and instead of ds - dx we can now write $\frac{(\partial w / \partial x)^2 dx}{2}$, that we can write. That is the force and half of this is the net displacement because after subtracting these dx from ds will get the net deformation, that must be considered for calculating the work done by $N_x dy$ is $\frac{(\partial w/\partial x)^2 dx}{2}$.

So, we write the work done by the $N_x dy$ force on the strip of width dy as $\frac{N_x dy}{2}(\partial w/\partial x)^2 dx$. Then total work done is nothing but the integration of this quantity within the domain of the plates. So, if it is a rectangular plate of side $a \times b$ then we can write the $\int_0^a\int_0^b \frac{N_x}{2}(\partial w/\partial x)^2 dxdy$. So, you have understood how the work done due to membrane force in x direction is calculated.

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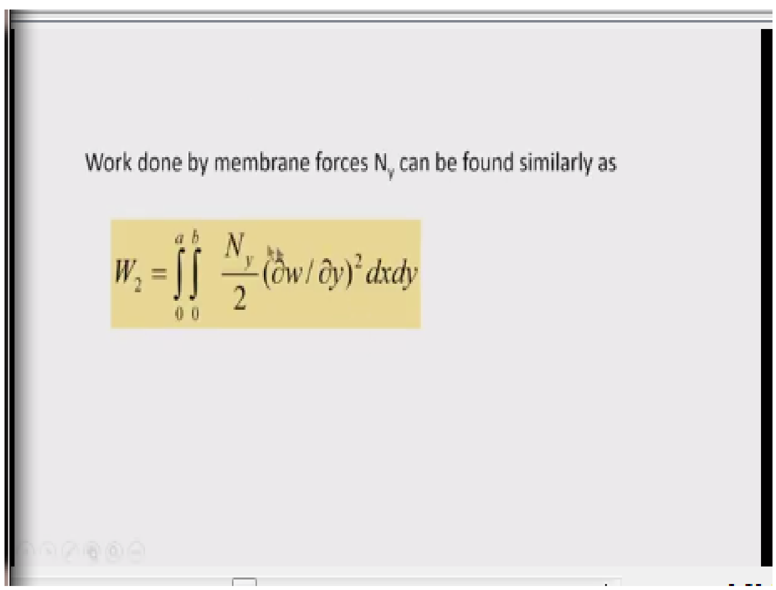

So, similarly the membrane force N_y in the y direction can be calculated and the expression for work done due to membrane force N_y acting along the direction y on the edges say $y = 0$ and $y = b$ can be calculated in this fashion. So, $\frac{N_y}{2}(\partial w/\partial y)^2 dxdy$, this is calculated in the same way as we have done. So, instead of slope in the x direction, now we will consider for calculating the work done due to N_y, we will consider this slope here this $\partial w/\partial y$.

So, that quantity has to be considered and then proceeding in the same way we can arrive the work done due to membrane forces N_y is nothing but 0 to a, 0 to b N_y, this is the membrane force acting in the y direction per unit length divided by 2 into $(\partial w/\partial y)^2 dxdy$. So, this represents the work done by the membrane forces N_y. So, so far we have obtained the work done due to membrane forces W_1 and work done due to membrane forces W_2 . Now we want to calculate the work done due to membrane shear force N_{xy}. Then we can add these 3 quantities of work done and then can find the total potential completely.

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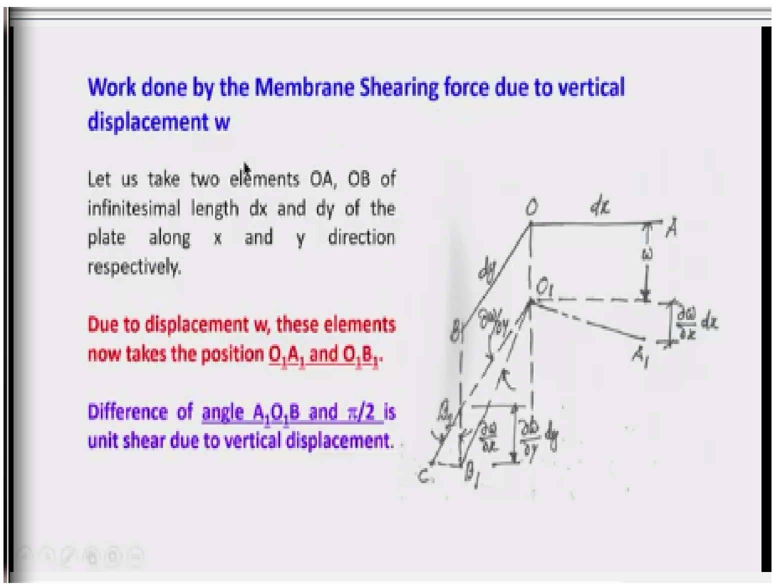

So, work done; by membrane shearing force due to vertical displacement w. So, let us see how the shearing deformation takes place in a deflected plate and how this deformation can be taken to consider the work done by the membrane forces. So, let us take 2 elements OA and OB, here you were seeing they are OA and OB these are the straight elements of the plate of small length infinitesimal length dx, dy.

dx is the length of the element in the x direction, dy is the length of the element in y direction. And then let the plate undergoes the deformation just at the buckling point w, that is not the deformation due to external load or the transverse load. So, this is a small deflection of course and let the line OA on OB takes the new position O_1A_1 and O_1B_1. So, these 2 positions I have marked here, O_1B_1, here O_1B_1 and O_1A_1.

These 2 positions are the new position after undergoing the vertical displacement *w*. Now the slope here along the *x* direction is $\partial w/\partial x$. So, naturally for small angle we can write this increment of deflection and this end is $(\partial w/\partial x)dx$. Increment of deflection along the *y* direction if $\partial w/\partial y$ is the slope along the *y* direction. Then we can write this the increment of deflection that $B_1B_2 = (\partial w/\partial y)dy$.

We want to find the shear deformation, so that this can be accounted for in calculating the work done. So, difference of angle $A_1O_1B_1$ and the right angle that is $\pi/2$ is unit shear due to vertical displacement that we know from our elementary strength of material. So, let us now proceed to calculate this; what will be this angle?

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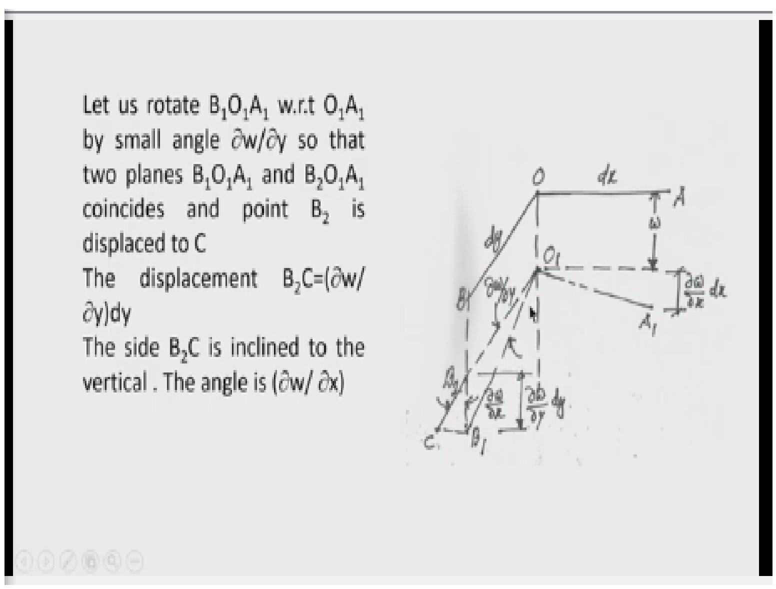

Let us rotate $B_1O_1A_1$ this angle, let us rotate this plane this plane is formed after the deformation with respect to O_1A_1. That means taking this as the reference line, we rotate this plane. So, this plane is rotated by small angle $\partial w/\partial y$, so that 2 planes B_1A_1 and $B_2O_1A_1$ now coincides. And the point B_2 is displaced to C, so this shearing deformation will be now B_1C. So, we are now interested to find what will be the displacement B_1C.

Now if you see the displacement B_2C this B_2C for this angle that B_2C is inclined with the vertical line B_1B is nothing but the slope angle $\partial w/\partial x$. Now for small deflection, the length of the vertical side is approximately equal to the length of the inclined side B_2C. So, the B_1C can be easily calculated knowing the vertical side B_1, B_2 and the angle between the vertical side and inclined side. So, then the side B_2C is inclined to the vertical, so that angle is $\partial w/\partial x$ is the slope in the *x* direction.

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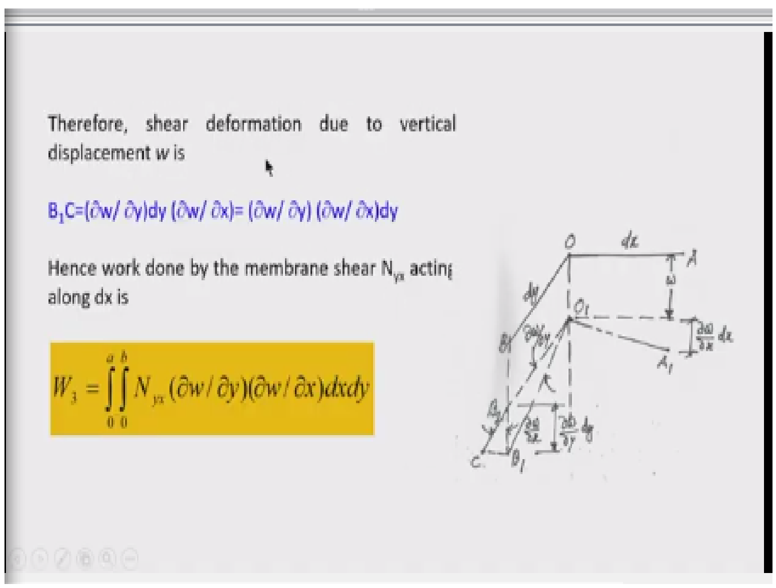

Therefore, shear deformation due to vertical displacement *w* is now B_1C, so what is B_1C? Because B_2C is now $(\partial w/\partial y)dy$, so this vertical side is approximately equal to B_2C. So, B_1C is nothing but your say this $B_2C \times \sin\theta$, θ is the small angle here, θ there is a small angle that is $\partial w/\partial x$. So, actually this B_1C now becomes this is the length of the side B_2C and the angle $\sin\theta = \theta$, so this is the angle $\partial w/\partial x$.

So, ultimately the shear deformation now becomes *(∂w/ ∂y) (∂w/ ∂x)dy*. So, hence the work done by the membrane shear force N_y, N_{yx} acting along *dx* is nothing but $N_{yx}(\partial w/\partial y)(\partial w/\partial x)dxdy$ and after integration over the domain we get the total work done by the membrane shear force.

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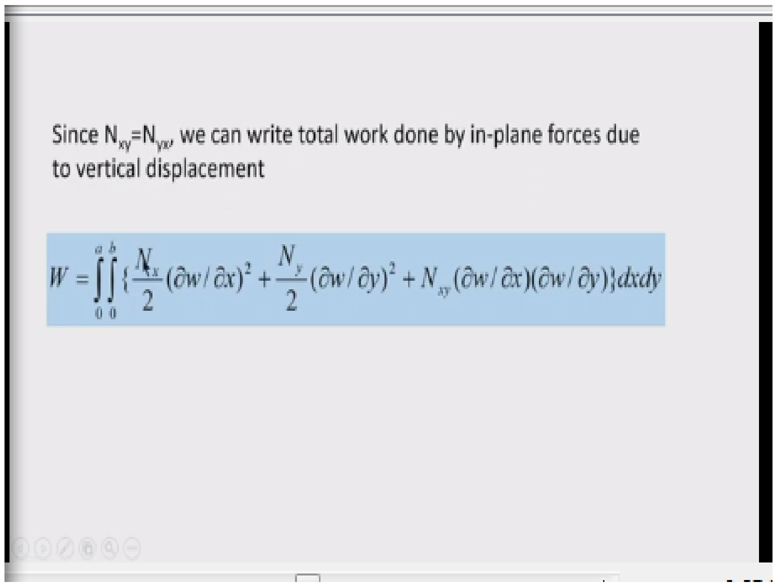

So, since $N_{xy} = N_{yx}$, so there will be no difference when we write N_{yx} or N_{xy}, total work done by the in plane forces or membrane forces now can be expressed in this form. $\frac{N_x}{2}(\partial w/\partial x)^2+\frac{N_y}{2}(\partial w/\partial y)^2+N_{xy}(\partial w/\partial x)(\partial w/\partial y)$ and integration. You can note here this component is the work done due to membrane force or in plane force in the *x* direction due to vertical displacement. And this component is the work done due to membrane force in *y* direction due to vertical displacement of the plate. And this is the work done by the shearing force membrane shearing force N_{xy} or N_{yx} due to vertical deformations. So, the vertical deformation introduces additional shear that is $(\partial w/\partial x)(\partial w/\partial y)$. So, this has been taken into account to find the work done. And it is also to be noted that during deformation this N_x, N_y, N_{xy} does not change.

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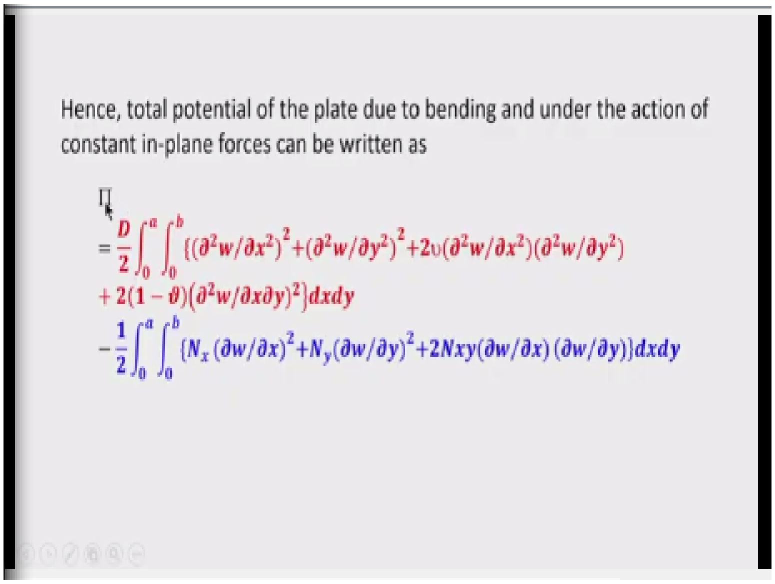

So, now with the help of this we can now write the total potential capital Π. So, total potential capital Π for the buckling problem is now given as *D*/2 integration 0 to *a*, 0 to *b* that is the curvature in the *x* direction square + curvature in the *y* direction square + 2 × *v*, *v* is the Poisson ratio $(\frac{\partial^2 w}{\partial x^2})(\frac{\partial^2 w}{\partial y^2})$. So, 2 curvature product and then + 2 × 1 - *v* this twist curvature $(\frac{\partial^2 w}{\partial x \partial y})^2$ and integration is done over the domain of the plate.

So, red colour term is the strain energy during bending of the plate and blue colour term is the work done due to membrane forces due to vertical displacement. So, work done; of membrane forces when the plate is undergoing the vertical displacement, then this quantity is written as the word done and we take this total potential Π as *U* - *W*. So, total potential has to be computed first in case of Rayleigh-Ritz method then we can apply the method taking a suitable shape function.

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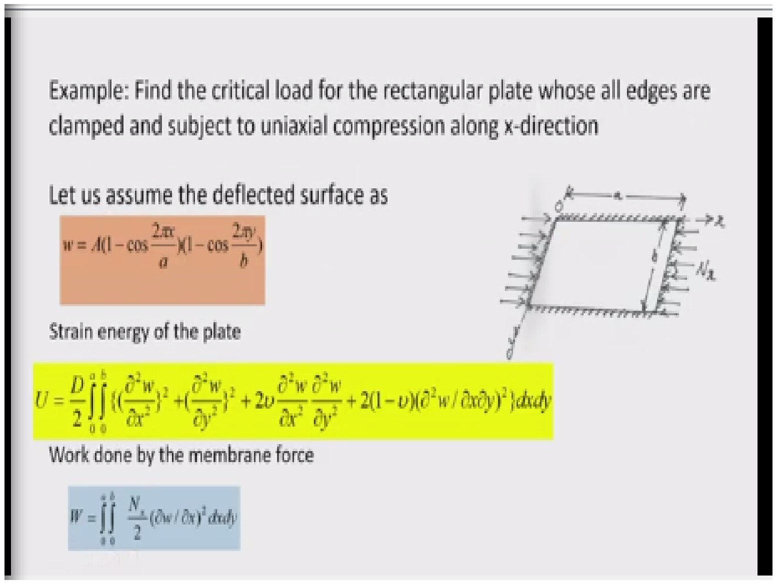

Let us illustrate this method with an example. So, I have taken an example of this plate which has this fixed edges, so all the edges are fixed. So, this type of problem cannot be solved by using the Navier's method or the Levy's method. So, therefore we take the help of approximate method. Now here you can see the assumed deflection function, I have taken only the first term other terms can also be written taking origin at O.

I have taken this function, the function of x that is $(1-\cos\frac{2\pi x}{a})$ and this function as a function of y $(1-\cos\frac{2\pi y}{b})$. It is easily verified these 2 functions satisfy the condition of clamped edges. What are the conditions of clamped edges? 2 conditions at the clamped edges should be present. One condition is deflection is 0 at the clamped edge; another condition is slope along the normal direction is 0 at the clamped edge.

So, if I calculate the deflection at x = 0 edge, x = 0 edge is this edge, you can see that w is 0. Because when you substitute x = 0, this factor becomes $\cos\frac{2\pi x}{a}$ becomes 1 and 1 - 1 is 0, so naturally w is 0. If I calculate the slope at x = 0 then $\partial w/\partial x$ will be this sine function will come, so a will be there and here you will get $\sin\frac{2\pi x}{a}$ and another factor will be there because of differentiation $\frac{2\pi}{a}$ will be there, so this will be constant.

So, the slope that is the first derivative again at $x = 0$ because sin0 is 0, so therefore the slope and deflection both are satisfied. This is also verified at $x = a$ edge, if you substitute here $x = a$ then $\cos 2\pi$ is again 1, so deflection is 0. And when again you differentiate it for finding the slope then $\sin\frac{2\pi x}{a}$ term is appearing and when $x = a$ again $\sin 2\pi$ that is sine of any integral value of π is 0. So, therefore slope 0 condition is satisfied at $x = a$ edge.

Similarly, $y = 0$ and $y = b$, this 2 condition $w = 0$ and $\partial w/\partial y = 0$ satisfy. So, this function adequately represents the deflected surface of the plate which has clamped edges along all the edges. So, strain energy of the plate can be written now $U = D/2$, D is the flexural rigidity of the plate 0 to a 0 to b integration, then the curvature along the x direction square + curvature along the y direction square + 2 × v $(\frac{\partial^2 w}{\partial x^2})(\frac{\partial^2 w}{\partial y^2})$ + 2 × 1 - v and then $(\frac{\partial^2 w}{\partial x \partial y})^2$ $dxdy$.

The plate is subjected to uniaxial compression. So, to calculate the work done by the membrane forces only this component is pertinent, other component of work done N_y and N_{xy} will be absent here because these forces are not acting in the plate. So, plate is acted upon by only the uniaxial compression and that compression is along the x direction. So, compression along the edge $x = 0$ and $x = a$.

So, work done due to this compressive force is now written like that $W = \int_0^a \int_0^b \frac{N_x}{2} (\partial w / \partial x)^2 \, dxdy$. This expression has been derived earlier from the basic principle, so you can use it.

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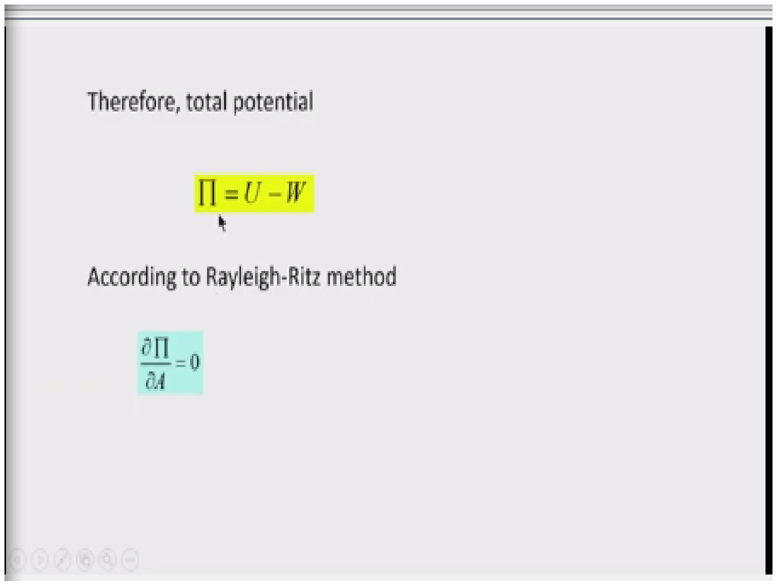

So, total potential is $\Pi = U - W$ and that has to be now applied or that has to be differentiated with respect to the constant that is used to find the deflected series. Now, since δA are the non zero parameters. So, we differentiate this to find this, actually we want to find the arbitrary variation of PI. So, we can write $\delta \Pi = \frac{\partial \Pi}{\partial A} \delta A$. Now since the deflection is a function of only one variable, we can write it as an ordinary differential coefficient also instead of partial differential coefficient.

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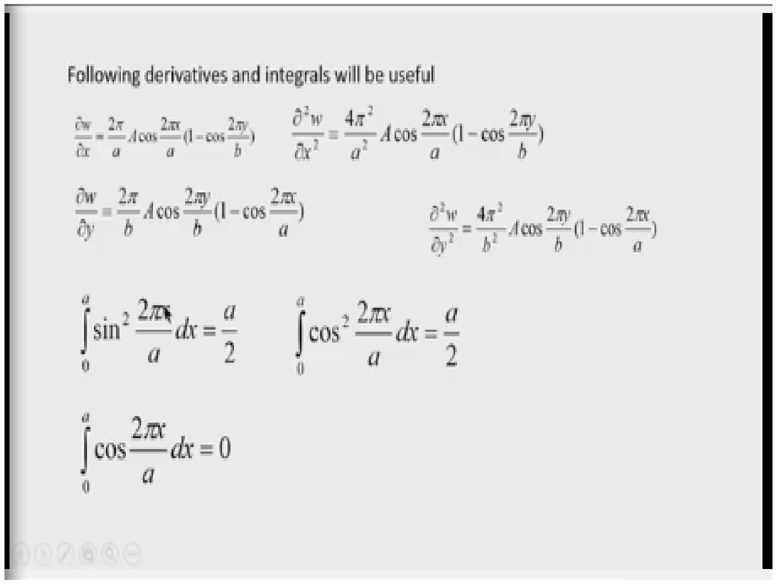

So, now we can see that the strain energy expression that has to be computed first and it contains the second derivative of the w, w is this, the assumed deflection function. As well as the cross derivative of this function, that is the mixed derivative. So, this derivative is calculated and this you can see $\frac{\partial w}{\partial x}$. You can easily see it $\frac{2\pi}{a}$ will be there and $A\cos\frac{2\pi x}{a}(1-\cos\frac{2\pi y}{b})$.

Then your this function $\frac{\partial^2 w}{\partial x^2}$ will be this type. Again this differentiating this, similarly $\frac{\partial w}{\partial y}$ will be like that and $\frac{\partial^2 w}{\partial y^2}$ will also be written like that. And these integral are also useful, these integral are useful $\int_0^a \sin^2\frac{2\pi x}{a}dx = \frac{a}{2}$ due to orthogonality condition of the function. And also the $\int_0^a \cos^2\frac{2\pi x}{a}dx = \frac{a}{2}$ due to orthogonality condition of the shape function. And when this function $\cos\frac{2\pi x}{a}$ is integrated between the limit 0 to a, gives you 0 value.

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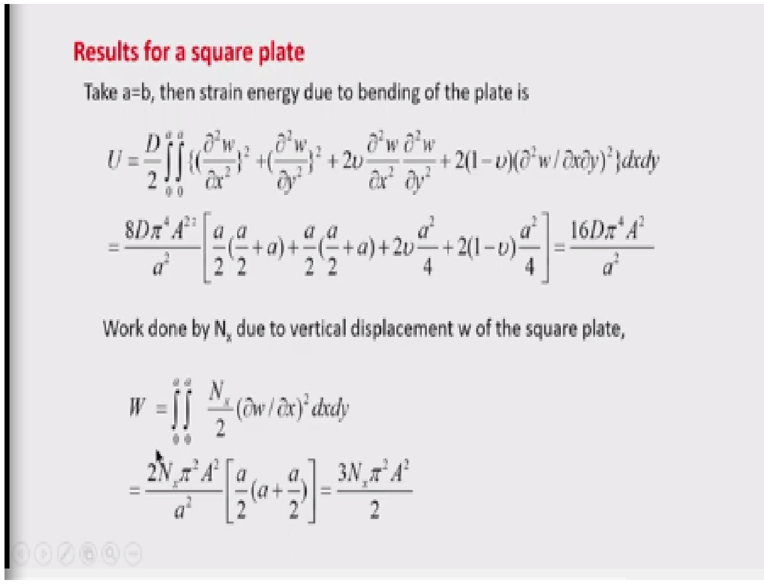

So, results for a square plate. Now, let us use this result for a square plate, taking $a = b$. So, strain energy expression now can be found out by putting the integral, individually you can see here the integral if I see this is the first part of integral, this is the second part of integral, this is the third part of integral and this is the fourth part of integral. So, all these derivatives are calculated and after calculating this then we integrate it.

Because this will be a function of sine and cosine and sometimes the orthogonality condition is used very easily. So, you will find that integration in 3, 4 parts are calculated, 4 components are calculated, this integral represents this first component $(\frac{\partial^2 w}{\partial x^2})^2$. And this is also same because it is a square plate, so curvatures in both the directions are equal.

So, therefore you will get this again for the square plate, this integral. And then product of these $2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$, when this is integrated product of the curvature is integrated. And here the square of the curvature is integrated, note the difference. Here you are integrating the square of the curvature, this is also the square of the curvature integrated yields you this result but here product of the curvature is integrated.

So, it is result will be different and this result is $2\upsilon\frac{a^2}{4}$. Then the mixed derivative that is the twist curvature is squared up and integrated. This is the fourth component of the strain energy and this gives you this quantity. So, after simplifying you are getting that strain energy $U=\frac{16D\pi^4A^2}{a^2}$, A is the unknown coefficient of the deflected series that we have to find out, divided by a^2.

a is the size of the plate, because we have taken a square plate whose side is $a \times a$ instead of $a \times b$ for a rectangular plate. Now let us calculate the work done due to the membrane force, in plane force or direct force you can call it, direct compression that is N_x, so N_x is applied along the x direction. And we know that expression for the work done due to membrane force, this N_x is double integration 0 to a, 0 to a because this is a square plate, $\frac{N_x}{2}(\partial w/\partial x)^2\,dxdy$ because this represents the work done by membrane forces due to vertical displacement of the plate. So, calculating these derivative and then integrating, we can now see that integration becomes this. And after simplification, we get this W is $\frac{3N_x\pi^2A^2}{2}$. So, 2 component of potential total potential we have calculated, now we can form the total potential.

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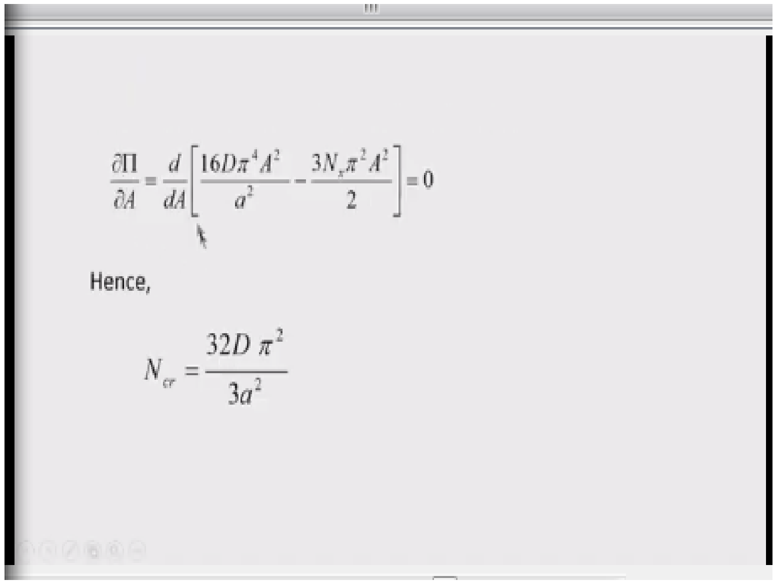

So, total potential is formed $\Pi = U - W$. So, you can write Π is the total potential = U - W, when you form the total potential, then apply the Rayleigh-Ritz method $\frac{\partial \Pi}{\partial A}$. Now since the functions or the strain energy and the work done due to membrane force is found to contain only one variable, unknown variable that we want to deal with the variational method, is A. So, therefore we can write it simply an ordinary differential coefficient d/dA.

And inside this we write here this strain energy expression that we have found minus this work done. So, after differentiating and then treating A should not be 0 for non trivial solution. Then we get the critical value of the load as $\frac{32D\,\pi^2}{3a^2}$. So, this is one method that by which you can calculate the strain energy of this buckling load of the plate. And here we have taken only one term in the series, but if you want to increase the accuracy of the result then you can take more number of terms.

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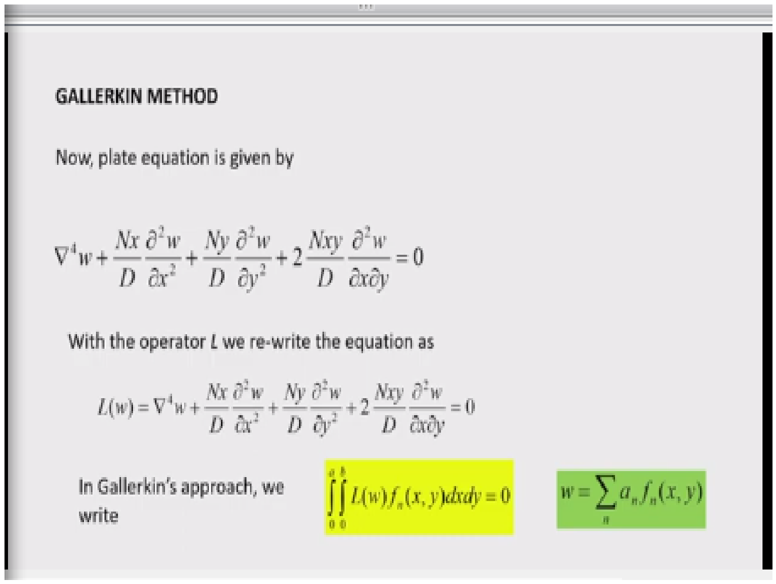

Let us see what is Gallerkin method in plate problem? Plate problem we have already used the Gallerkin method which is again approximate method. But difference of Gallerkin method with Rayleigh-Ritz method is that. Gallerkin method requires the differential equation of the plate. Whether it is under bending or it is in buckling or even in vibration problem you require the governing differential equation instead of the strain energy expression.

Although both the methods are derived from the work energy principle but one method is directed in such a way that the first method that I discussed Rayleigh-Ritz method has to be started only after knowing the strain energy of the system and work done of the system. But if we see the Gallerkin method, then instead of strain energy we start the calculation by knowing the differential equation of the system.

So, here the system is the plate and it is under buckling. So, therefore the equation for buckling of the thin plate is taken in presence of all the membrane forces which are treated as constant. But if these are not constant then this N_x, N_y, N_{xy} will be a function of x, y. Now define operator L, such that L is $\nabla^4 + \frac{Nx}{D}\frac{\partial^2}{\partial x^2} + \frac{Ny}{D}\frac{\partial^2}{\partial y^2} + 2\frac{Nxy}{D}\frac{\partial^2}{\partial x \partial y}$.

So, introducing this operator, we can now represent the differential equation in this form. So, $L(w)$ is 0, so L is a operator I have introduced, you can isolate w from this equation and then you will be able to get the operator. Now in the Gallerkin approach, again we require to choose the shape function. So, shape function is chosen in the same way that we have carried out in case of Rayleigh-Ritz method.

So, shape function is chosen such that is satisfies the geometrical and force boundary condition, that is the most important part and the choice of function in a judicious way will reduce the calculative effort and also improve the accuracy of the result. So, therefore this is the very important step in carrying out or solving the problem by Rayleigh-Ritz method or Gallerkin method to choose a suitable shape function.

Now here again we write the same function in the form of a series, series means $f(x, y)$ maybe a function say $f_1(x, y)$ maybe some trigonometric function. And then $f_2(x, y)$ maybe another trigonometrical function. So, adding all these things we get this the total deflected shape. Now here you can see that individually f_1, f_2, f_3 etcetera should satisfy the boundary condition of the plate.

Now boundary condition here I mean that geometrical boundary condition or the force boundary condition. Sometimes geometrical boundary condition is very easily satisfied and force boundary condition is difficult to satisfy at all, sometimes partly satisfied. But in any way when the geometrical boundary condition is satisfied, we take this deflection function to be used in the Gallerkin approach or Rayleigh-Ritz method.

The criteria for selection of shape function in Gallerkin method and Rayleigh-Ritz method, both are same. So, after choosing the shape function, then you form the Gallerkin equation like that.

This is the differential equation into $f_n(x, y)dxdy = 0$ which is integrated in the domain of the rectangular plate. Now here you can see the shape function is approximate that we take sometimes we get it exactly choose it exact function sometimes.

Say for example in simply supported plate along all edges we will take the Navier's series then it represents the exact deflected shape, so therefore the function is exact. So, but in general the shape function is not an exact and it is approximation of the deflected surface. So, when you substitute the approximate deflected surface in the differential equation, the right hand side you will not get 0. So, therefore it represents an error and non zero quantity which represents error, this may be small or this may be large depending on the choice of the shape function.

Now when this error, you can see after substituting *w* in the differential equation, this $L(w)$ represents the operator *L* into *w*, $L(w)$ represents the error introduced due to assumed deflection function. So, the error multiplied by each of the function, taken to form the deflected surface is integrated with respect to *x* and *y*, between the limits 0 to *a*, 0 to *b* and equated to 0. So, this is the Gallerkin equation that has been derived from the first principle, virtual work principle.

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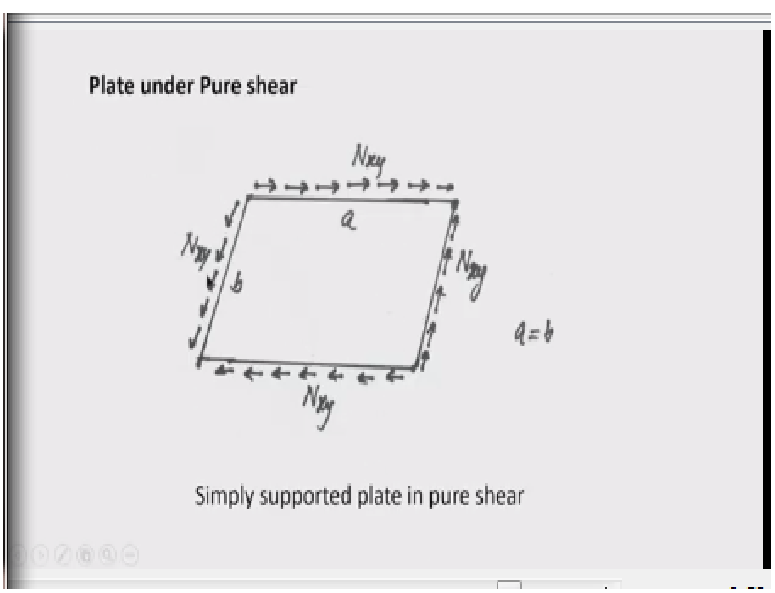

Now here we take a problem of plate under pure shear. So, this pure shear case is commonly seen in case of this plate girder say, web of plate girder. For example, this is the web of plate girder, a portion of a panel of a plate girder**,** web panel of a plate girder. Now you can see that due to the shear acting on the element panel, you can see diagonally it is compressed but in the other diagonal there is a tension. And this tension sometimes reduces or minimizes the chance of buckling failure. But due to compression here, there is a chance that shear buckling may takes place in the panel.

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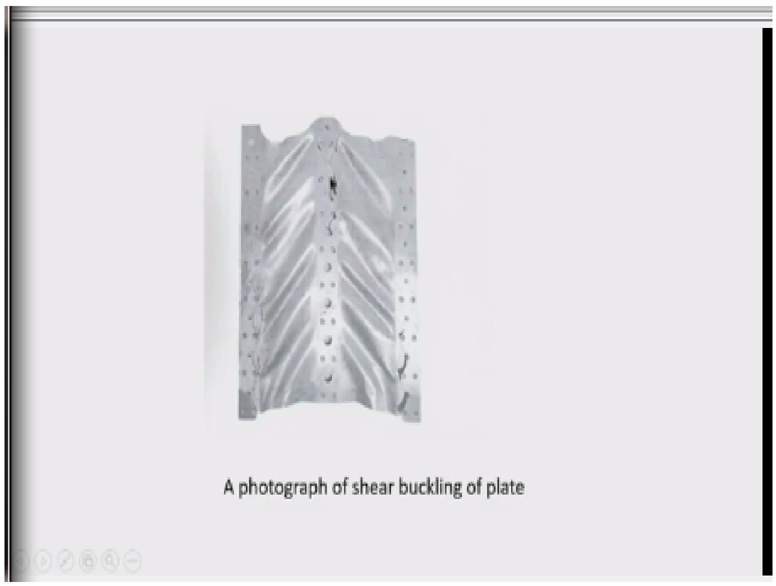

So, the shear buckling phenomenon is observed in many practical cases, this is one photograph of a plate used in the ship construction. And you can see due to diagonal compression that the buckling takes place due to compression. So, that is particularly observed, therefore the study of buckling of plate due to in plane shear is also important, significant. And here you can see that in plane forces N_{xy} is applied and we take $a = b$, that is a square plate to simplify the calculation.

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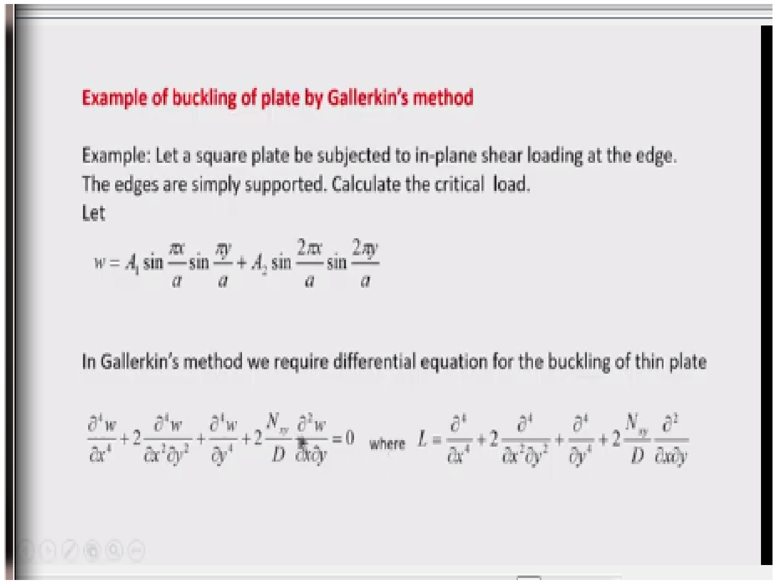

Now we take the function, the shape function as consisting of 2 terms of the Navier series. Because we have taken the support as a simply supported, so we take the support as simply supported and therefore we take this shape function as consisting of the 2 terms of the Navier series. So, $A_1 \sin\frac{\pi x}{a}\sin\frac{\pi y}{a} + A_2 \sin\frac{2\pi x}{a}\sin\frac{2\pi y}{a}$. Now in Gallerkin method, we take the differential equation that I have introduced earlier. And this differential equation now has to be satisfied by this approximate function. So, therefore naturally an error will be introduced.

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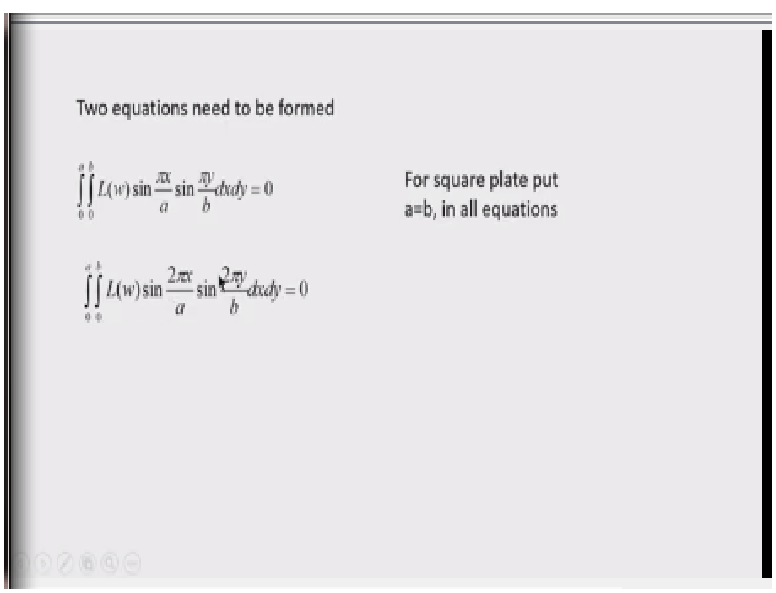

So, this error what is there will remain here and then this error multiplied by each of the functions. So, what are the functions? Functions are taken $A_1 \sin\frac{\pi x}{a}\sin\frac{\pi y}{a}$, so this is one function f_1, and f_2 is + $A_2 \sin\frac{2\pi x}{a}\sin\frac{2\pi y}{a}$. So, this is $L(w) \times f_1(x,y) = 0$ and $L(w) \times f_2(x,y) = 0$, because 2 constants are involved in forming the deflected shape, so therefore 2 Gallerkin equations are written. We take the square plate, so in all places we substitute b with a.

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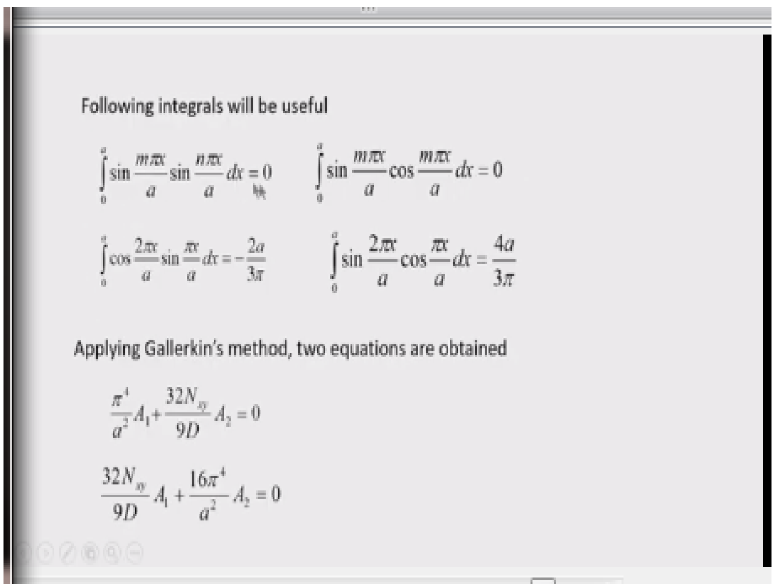

So, these integrals are important to be used here and after carrying out this integration, you will finally arrive 2 homogeneous equations in A_1, A_2. This is as homogeneous equation in A_1, A_2, is a linear equation. So, one possibility is that A_1 = 0, A_2 = 0, then these 2 equations are satisfied but this condition will not give you the result that we required for this buckling load. So, this will form a trivial solution with A_1 = A_2, A_1 and A_2 being 0, so this solution we discard. For non trivial solution, we take the determinant formed by the coefficient of this equation to be 0.

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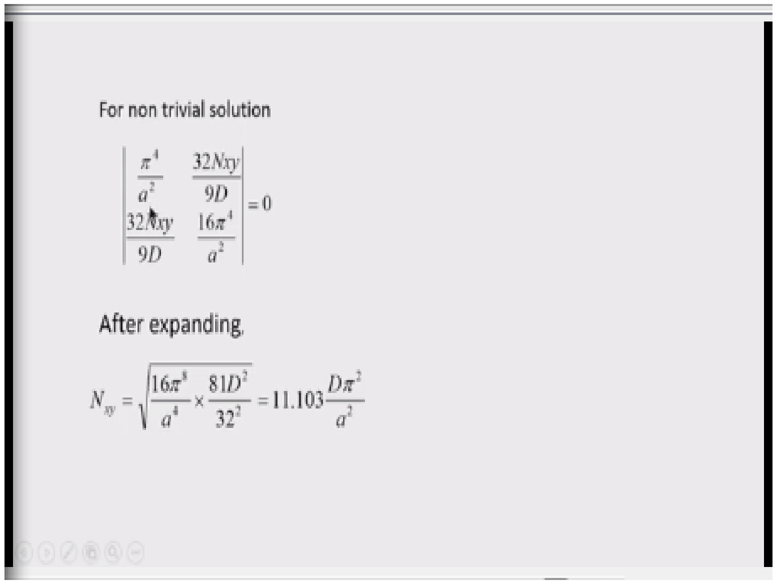

So, determinant form by the coefficient of this equation, say this coefficient with A_1 will be in the first row first column of the determinant pi to the power 4 a square, so this is $\frac{\pi^4}{a^2}$. Then second coefficient in the 1st row that is the 2nd column of first row $\frac{32Nxy}{9D}$, so this is. Then this from the second equation, we picked up this coefficient in the 1st column 2nd row and this term we pick up for a 2nd row 2nd column.

So, expanding the determinant and equating to 0 we get now the value of N_{xy}, so N_{xy} is this $\sqrt{\frac{16\pi^8}{a^4} \times \frac{81D^2}{32^2}}$. So, that is under root and after calculating we get the critical membrane shear which cause the shear buckling in the square plate is $11.103\frac{D\pi^2}{a^2}$.

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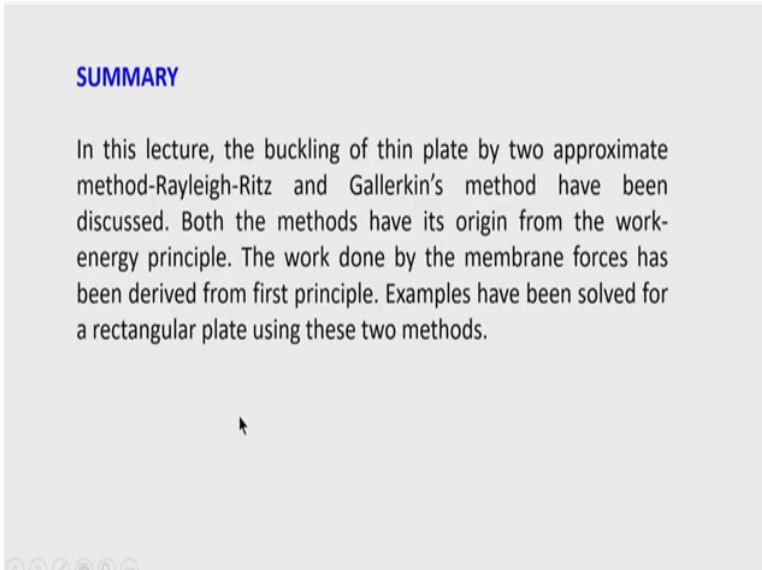

So, let me summarize the topics that I have covered in today's lecture. So, in this lecture the buckling of thin plate by 2 approximate methods Rayleigh-Ritz and Gallerkin's method have been discussed. So, formulations were carried out using the work done due to membrane forces or you can tell the potential of the membrane forces. Then Gallerkin method is also shown how it is evolved with the help of differential equations.

And both the methods have it is origin from the work-energy principle, that we have to bear in mind. The work done by the membrane forces has been derived from the first principle and work done and strain energy expressions are specifically required for Rayleigh-Ritz method. In Gallerkin method, you only need the differential equation and then of course you assume shape function, that are same in case of Rayleigh-Ritz and Gallerkin method. So, 2 examples have been solved for a rectangular plate using these 2 methods, so thank you very much.