Plates and Shells

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Module-1

Lecture-02

Theory of Thin Plate Bending

Hello, everybody, welcome to all in the MOOC course on plate and shell. Today, I will deliver the second lecture of the module 1. (refer time: 00:39) As you remember or recall that in the earlier class, last lecture that I concluded there I discussed the classification of plate theories, fundamental of plane elasticity, then deflections, slope and curvature of the plate, displacement strain relations, stress strain relations.

So, these are the important background required to proceed on other topics of the plates. Classification of plate theories - you have seen that we have classified the plate into 3 categories thin plate with small deflection, thin plate with large deflection, where nonlinearity is produced due to large deflection which is geometric nonlinearity. As a result, membrane stress is significant.

Then we have discussed the thick plate theory, in the thick plate theory I have mentioned that, if the plate thickness is less than $(1/10)^{th}$ of the larger dimension, then plate is classified as a thin plate. Otherwise, if the thickness exceeds $(1/10)^{th}$ of the larger dimension, then it is called a thick plate. And thick plate theory has to be modified based upon the shear deformation criteria.

Then we have discussed the fundamentals of plane elasticity, how the stresses are denoted? Two subscripts are used, what are their significance? What is shear stress and what is the normal stress and what are the concept of principal planes? That I have covered. Then in a plate, I have discussed the deflection and slope, how the slope is measured in 2 directions and what is the interrelation between the slope in any two mutually perpendicular directions and what is their criteria?

That the direction of maximum slope and minimum slope can be related. Then I have told you about the curvature of the plate. Curvature is obtained by secondary derivative of these deflection function and very important relationship have been obtained for curvature in any 2 orthogonal directions. That means curvature we have seen that it is not dependent on the orientation of the axis.

So, whatever way you are in the axis, the sum of the curvatures in two orthogonal directions remain constant. Then we have discussed the displacement strain relations and from that we have arrived this stress stain relation for the plate. (refer time: 03:26) Now let us go to today's topic that I want to cover. In today's lecture I want to cover the theory of bending in thin plate and related assumptions.

Expression of strain in terms of vertical deflection, effect of middle surface stretching, because the middle surface stretching is neglected in thin plate theory. However, if stretching is considered, then the effect of this stretching will influence, will alter the strain expressions, so I will discuss here. Then expression of moments in terms of vertical displacement, once the stress strain relationship is obtained, we will go for the calculation of stress resultants in the plate that is the bending moment and twisting moment in a thin plate.

And that relationship will be derived in terms of vertical displacement and of course involving their derivatives. Principal moments on any inclined plane and principal directions - I will cover and then equilibrium equations for the plate will be derived. (refer time: 04:36) So, now, let us see the theory of thin plate, how it is developed? Theory of thin plate has been developed by Love in 1888 using the assumption proposed by Kirchhoff and hence it is popularly known as Kirchhoff-Love plate theory.

You can say that Love has proposed the plate theory just extending the Euler Bernoulli beam theory. And Kirchhoff has covered this plate theory in the form of equations. So, some assumptions are made in developing the plate theory of small thickness and small deflection, the following assumptions are made ok. The material is homogeneous and isotropic. So, homogeneous material you mean that in a space variable, it does not alter, the property of the material does not alter with the variation of space variable, it remains invariant of the space index.

Then isotropic, in any direction the material properties will be same, the deflection is small that is the basic assumption in thin plate theory and these deflection is seen to be within one fifth of the plate thickness. Otherwise this theory may not give you sufficiently accurate results. Then the third assumption which is very important the normal to the middle surface of the plate before bending and after bending remains same.

This shows that the strain in z direction is 0 and the length of the normal does not change. So, that is very important assumption based on which we can say that σ_z that is the normal stress in the z direction upward direction is 0. And the shear stress components τ_{xz}, τ_{zx}, τ_{zy}, τ_{yz} are 0. We have earlier mentioned that in a thin plate 3 stresses are of importance one is σ_x and σ_y, τ_{xy}.

σ_x and σ_y gives you the normal stress whereas τ_{xy} or τ_{yx} because of symmetry $\tau_{xy} = \tau_{yx}$ will give you the shear stress in the plate ok. (refer time: 07:03) So, let us see the expression in terms of vertical deflection. You can see here a plate shows along the thickness ok and this is the deflected middle surface, this is the dotted line is the deflected middle surface. Because of so small deflection you are seeing this like a straight line, but actually it bends, but for small deflection and small distance it is taken as a straight line. So, θ_x is the slope in the x direction. Similarly, in the y direction which is perpendicular to the board, we will see that θ_y is the slope. Now θ_x is slope in x direction is given by $\frac{\partial w}{\partial x}$, that is the partial derivative of deflection function with respect to x. You can see that deflection function is dependent on 2 variables x and y. So, we have to write in terms of partial derivative. Similarly, θ_y slope in y direction will be $\frac{\partial w}{\partial y}$, this is the general description of the plate where the middle surface is denoted by the dotted line.

And you can see that middle surface is through the mid depth of the plate and all the stresses or stress resultant that is M_x, bending moment in x direction, bending moment in y direction and twisting moment will be referred in the middle surface ok. (refer time: 08:35) Now from that figure, we can see that the displacement in the x direction is denoted by u. You can see that z is the height and θ is the angle.

So, this can be computed based on the tangent rule and because this is small deflection, so tanθ will be θ. Therefore, we get $u = -z\ \theta_x = -z\ \frac{\partial w}{\partial x}$ that is the partial derivative of w with respect to x ok. Similarly, in y direction the strain we obtain as $\frac{\partial v}{\partial y}$ and that is nothing but z × θ_y and it will be $-z\ \frac{\partial w}{\partial y}$. So, 2 strain expressions we have obtained ϵ_x and ϵ_y, $\epsilon_x = \frac{\partial u}{\partial x}$ and $\epsilon_y = \frac{\partial v}{\partial y}$ which you can see it is related to the curvature, but it is multiplied with a variable z. That means knowing the curvature it can be seen that strain varies linearly along the depth ok.

If we measure the depth or if we measure the vertical distance referring the middle surface as the datum, we can show the strain variation along the depth of the plate or the thickness of the plate. Now, shear strain is given by the expression that expression we have developed in the first class on the fundamental of elasticity, that is $\frac{\partial u}{\partial y+\frac{\partial v}{\partial x}}$ that gives the sum of 2 angles, because of deformation of the plain surface. And because the angle that you added is nothing but is a tangent of angle but because of small deflection we have taken tanθ = θ. So, this expression gives you $2z\frac{\partial^2 w}{\partial x \partial y}$ ok. Here also I have mentioned the undeformed surface and the deformed surface, undeformed surface is the middle plane straight and deformed surface you can see here, it is the deformed surface.

So, in next page also I have shown you that this is the undeformed surface and this is the deformed surface. And normal to this deflected surface is shown which remains same, which remain again normal to the deflected surface. So, length of the normal does not change, that indicates that the strain in the vertical direction is 0 ok. So, these are the 3 important relation ϵ_x, ϵ_y and γ_{xy} that we obtain for normal strain in x direction, y direction and shear strain to be used for developing the expression for bending moment and twisting moment ok. (refer time: 11:49)

Now in thin plate theory, generally we neglect the middle plane stretching. But suppose, we want to extend this theory including the middle plane stretching then what will be the change in the expression that we have derived ok. Let $u_0(x,y,z)$ and $v_0(x,y,z)$ are displacement of middle surface by the action of in plane forces ok. So, when the main force is in-plane force then u_0 and v_0 are the displacement function in x and y direction ok.

At the middle surface z is 0, because middle surface is our datum. So, naturally at the middle surface we can refer u_0 and v_0 just as a sole function of this x and y variable, there cannot be any z, there is no significance of z in the u_0 and v_0 in the middle surface ok. Now let $\alpha(x,y)$ be the slope in x direction and $\beta(x,y)$ be the slope in y direction. I have taken α and β with a different symbol, not to confuse with the slope that we obtained when the middle surface stretching is neglected ok.

Now you can see here $u(x,y,z)$ will be equal to $u_0(x,y)$ that is the middle surface displacement due to stretching, then $z \times \alpha$. So, suppose this is your slope in x direction. So, z × $\alpha(x,y)$, slope is again a variable involving 2 index. So, x and y are involved in slope also, in x direction similarly the displacement in y direction that is $v(x,y,z)$ is shown as middle surface stretching plus these stretching or the extension that is developed due to bending of the plate, that is $z \times \beta$ ok.

Assumption is thin plate theory is that the Υ_{xz} is 0 or $\Upsilon_{yz} = 0$. So, taking $\Upsilon_{xz} = 0$, we get $\alpha + \frac{\partial w}{\partial x} = 0$, this shows that $\alpha = \frac{-\partial w}{\partial x}$ ok.(refer time: 14:16) Similarly we get Υ_{yz}, here Υ_{yz}

you can see $\frac{\partial v}{\partial z+\frac{\partial w}{\partial y}}$, from the plane elasticity equation. And here we get $\beta + \frac{\partial w}{\partial y} = 0$ this gives you $\beta = \frac{-\partial w}{\partial y}$ ok. Hence we can write ϵ_x and ϵ_y considering the middle surface stretching in a modified form. So, ϵ_x and ϵ_y will be now in a modified form because these extra terms are coming, ∂u_0 and ∂v_0 terms are coming.

So, therefore Υ_{xy} will be $\frac{\partial u}{\partial y+\frac{\partial v}{\partial x}}$ is also written. So, you can see the change in the stain expressions because of middle surface stretching. However, in the thin plate theory, we simplified the analysis neglecting the middle surface stretching ok. (refer time: 15:20) So, in the plate theory, we will use this expression for ϵ_x, ϵ_y, Υ_{xy}, there will be no u_0 and v_0 terms, which indicate the middle surface stretching ok. (refer time: 15:34)

Now, let us obtain the expression for stress resultant. So, very important stress resultant in the plate is the bending moment, because the plate resists the load by bending action or flexural action in both the direction, in x direction and y direction. Also because of shear stress, the twisting moment is developed. So, let us obtain the 3 quantities and their 3 expressions - M_x, M_y and M_{xy} ok. Now take 2 adjacent sides, one is parallel to x axis and another is parallel to y axis ok.

When the side is parallel to x axis ok and if I take the normal stress, it will be σ_y and total force on the element will be $\sigma_y \times dx$, this will be the total force on the element. Now first we are considering the bending moment about x axis. So, we are considering the stress normal to this face, that is normal direction is along the positive direction of the x axis ok.

So, here you can see, we have considered a small element in the plate and here the thickness is dz and the height of the element above the middle surface is z. We have taken a unit width of the plate. So, the force is $\sigma_x \times dz \times 1$, $dz \times 1$ is the area and force into distance will give you the moment and if we integrate from –h/2 to +h/2, we will get the total moment. Now substituting the σ_x that we have obtained in the previous lecture, i.e., the expression for stress in terms of curvature, they will be substituted here. And then integrating with respect to z. So, $z^2 dz$ has to be integrated that means $z^3/3$ and then we have to put the limit –h/2 to +h/2. As a result, you will get this constant $\frac{Eh^3}{12(1-\nu^2)}$. Now, you can realize that if a unit width of the plate is taken and h is the thickness, so, $1 \times h^3/12$ is the moment of inertia of the section of unit width. So, $h^3/12$ represents the moment of inertia of this section of unit width. But additional term, $1 - \nu^2$ appears here because of the 2 actions. So, now this is the expression for bending moment. You can easily recognize that this is a constant term which is generally termed as flexural rigidity of the plate, this is the curvature in x direction + ν is the Poisson ratio into curvature in y direction ok. (refer time: 18:49) Same logic is applied to the bending moment in y direction.

So, here in the y direction we have taken a strip of unit width 1 and the thickness of the element is dz at a distance z. So, similarly we have found the moment here for elemental area and then we have integrated across the thickness –h/2 to +h/2 and we get this expression. So, after simplification we get M_y. Now let us come to the twisting moment. Twisting moment is resulted due to shear stress.

So, shear stress τ_{xy} acting on an element dz will give you the shear force τ_{xy} dz × 1, this is the force and the moment about this neutral axis or middle surface will be τ_{xy} z dz × 1. And when it is integrated between the limit of thickness –h/2 to + h/2 we will get twisting moment expression as $\frac{\{Eh^3}{12(1-\nu^2)}\} \times (1 - \nu)\frac{\partial^2 w}{\partial x \partial y}$.

This expression can be simplified because $(1 - \nu^2)$ can be written as $(1 + \nu) \times (1 - \nu)$. So, after simplification and denoting the $\frac{Eh^3}{12(1-\nu^2)}$ as a flexural rigidity of the plate with a notation D, (refer time: 20:26) we can now express all the important stress resultants i.e., the bending moments, M_x, M_y and M_{xy} as this.

So, M_x is $- D\left\{\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right\}$. Then M_y - you can see the expression can be written by just interchanging the curvature. So, here this $\frac{\partial^2 w}{\partial y^2}$ is coming and with Poisson ratio you will get the multiplying factor as the curvature in the x direction. So, M_{xy} is written here which denotes the twisting moment ok. (refer time: 21:11)

So, bending moment, shear force and twisting moment in the plane which is oblique to the original x-y direction can be also obtained by resolving the vectors. The double arrow is used as a vector to understand this direction of the moment ok. So, this is the expression that we obtained for the moment along the normal, so what is this? These are the moment in the two mutually perpendicular direction i.e., *n* direction and *t* direction.

So, *t* direction is the tangential direction to the plane and *n* direction is the normal direction to the plane. So, two expressions are obtained which involve the orientation angle α. So, orientation angle if known, we can find the moment in the two orthogonal direction in any oblique plane by this expression ok. The plane in which the twisting moment M_{nt} vanishes is called as a principal plane of the moments. So, this can be obtained by equating this expression to 0 and finding the value of tan tan (α) . (refer time: 22:28)

So, tan tan (2α) is found like that and substituting this α value in the previous expression we can get M_n and M_{nt} ok. So, these are the value of M_n and M_{nt} maximum and minimum ok. (refer time: 22:44) Now let us find the equilibrium equation of the thin plate ok. The governing differential equation of equilibrium has to be known because otherwise we cannot solve for the deflection ok. To find the governing differential equation of equilibrium, let us consider a small element of the plate dx × dy ok as shown here. And the direction of the axis is shown in the adjacent figure as x, y, z ok. (refer time: 23:16)

We would draw in a magnified scale the element and show all the forces ok, like a free body diagram. So, the side of the element parallel to x axis is dx and parallel to y axis is dy and the thickness of the plate is h. This dotted line is denoting the middle surface alright. Now here you can see that on that side the angle is M_x. On the opposite side the angle is the moment about the y axis.

Actually moment along the x axis is M_x, but M_x is the moment about the y axis. About an along, 2 terms are used, you should distinguish between them. M_x when we call the bending moment along x direction we denote the bending moment by M_x but this bending moment is actually bending moment about y axis. So, similarly same meaning is applied for M_y ok. So, this side it is M_x and on the far side there will be some increment.

So, this increment is found by Taylor series expression and keeping only the first term, neglecting the high order term. So, this term is $M_x + \frac{\partial M_x}{\partial x} dx$ and twisting moment is shown here say M_{xy}. And on the opposite faces the increment is also added in the twisting moment ok. On the edge which is parallel to x axis that is perpendicular to y axis, the principal moment is M_y, here M_y and on the opposite face it will be M_y plus some increment ok.

So, this increment is found from Taylor series expansion, taking dx is very small quantity then we expanded Taylor series and we retain after the first term and get this quantity. So, M_{xy} is also given here, actually I did not distinguish between M_{xy} and M_{yx} to write the expression in figure because you know that from symmetry M_{xy} should be equal to M_{yx} ok. So, it is written M_{xy} and on that side the increment is given $M_{xy} + \frac{\partial M_{xy}}{\partial y} dy$.

And the vertical load that is applied on the plate is $q(x, y)$ which is continuously distributed. That means that $q(x, y)$ is not discontinuous for deriving the equation of the equilibrium. We take it on the element or throughout the plate is distributed, when the concentrated load comes then of course, this equation cannot be used directly, we have to see this. So, it has continuous distribution that means this function can be differentiable ok.

So, we have shown all the forces acting on the element. Then the vertical shear on this edge is Q_x and on the opposite face it will be Q_x + increment, then on the edge which is parallel to y axis it is Q_y - vertical shear, and on the opposite edge it is will be $Q_y + \frac{\partial Q_y}{\partial y}dy$. You can see the directions of forces and moments in the two opposite edges are taken of opposite side.

This is because to keep the body in equilibrium ok. Then one important thing is that all the quantities that I have mentioned here M_x, M_y, Q_x, Q_y, M_{xy} all are quantities per unit width of the plate. So, suppose the bending moment unit is say kilonewton meter. So, in the plate problem it will be described by kilonewton meter per meter. Suppose the shear force unit is kilonewton in the plate problem it will be described by kilonewton per meter. So, all quantities are per unit width of the plate ok. (refer time: 27:55)

Now in the middle plane, it is very convenient to show the moment by double arrow. The meaning of double arrow or the use of double arrow in expressing the moment can be understood by this simple right hand screw rule ok. (refer time: 28:12) So, suppose our thumb is the direction of moments. So, if I curve the fingers along the direction of rotation, then thumb will point towards the vector, so moment vector is towards the thumb. So, the moment vector is expressed or written by double arrow ok. (refer time: 28:40)

So, double arrow is used here in the moment vector and that we have seen in the middle surface, the existence of moments and the shear forces. Then taking the equilibrium equation, ok what are the equilibrium conditions that have to be written? One is bending moment about the x axis $M_x = 0$. Then bending moment along x axis not about x axis is summation of all moments about y axis, that is bending moment along x axis is 0.

So, by that equation we get these quantities ok, and we have substituted all the terms from free body diagram and equated to 0. So, neglecting the product of $(dx)^2$, ok, $(dx)^2$ has to be neglected. Because it is a small quantity, so square of small quantity again it will be square ok, so naturally we will neglect this in the derivation. After neglecting, and division by 'dxdy', we ultimately get an equation of equilibrium in terms of moment $\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x$.

Similarly, by summing up all the moments and equating to 0 about the x axis that means moment along the y axis, we get this equation $\frac{\partial M_y}{\partial y} + \frac{\partial M_{yx}}{\partial x} = Q_y$. So, these two equations are very important which relates the bending moment, twisting moment with the shear force. So, bending moment and twisting moment related to the shear force, you can understand the significance of each of the term the rate of change of bending moment along the x axis ok.

We will give shear force in the x direction, ok vertical shear along the x axis ok. But here if you see that $\frac{\partial M_{xy}}{\partial y}$, this is the rate of change of twisting moment along y has to be added to obtain the vertical shear along the x direction. Similarly, you can see here with the rate of change of bending moment along y axis with respect to y that is equal to Q_y but the added term is there because of twisting effect.

And the rate of change of twisting moment is added along the x direction, so you are getting Q_y. So, these 2 equations should be noted very carefully only for this term which is slightly non-conventional compared to beam. So, in beam theory only this term was there, this term was there, but here we are getting additional two terms for twisting effect. (refer time: 31:47)

So, summing up all the forces in z direction, now we get the vertical equilibrium of forces $\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q(x, y) = 0$. Substituting above equations, in the previous slide we have obtained the Q_x and Q_y ok. Now in the next slide instead of Q_x and Q_y, we will substitute the expression that we have obtained earlier. So, by substituting this and with differentiation will ultimately arrive an equation of this type which is the equilibrium equation of the plate in terms of moment ok.

Now substitute the expression for M_x, M_y and M_{xy} that we have obtained in terms of displacement and their derivatives ok. So, if we substitute M_x, M_y and M_{xy} in this expression, (refer time: 32:48) then we will obtain a very well-known plate equation in this form. So, rewriting equation 36, involving curvature expression, ok. Then we will get this expression and after doing some algebra, we ultimately arrive at the equation of plate, flexural rigidity

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = q(x, y)$$

So, if I denote ∇^4 as the biharmonic operator then in a compact form, I can write the plate equation as $D\nabla^4 w = q(x, y)$ where ∇^4 is the biharmonic operator. So, this operator is very well known operator in our mechanics problem and we should use this equation whenever we write in a compact form we can write $D\nabla^4 w = q(x, y)$. Now there are other possibilities of expressing the plate equations, ok what are these possibilities? Let us see. (refer time: 34:08)

An alternate form of plate equation in Cartesian coordinate system can also be written ok. So, in that case, we obtain a second order equation. So, second order equation can be conveniently expressed using the Laplacian operator and which represent the equation of the stress membrane. So, in this slide, let us again write the expression for M_x and M_y and add these two quantities, after adding M_x and M_y we get $-D(1+\upsilon)\left\{\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right\}$.

So, now letting this $\{(M_x + M_y)/(1 + \upsilon)\} \times D = M$ or D will keep in the equation. So, we will tell that $\frac{(M_x+M_y)}{(1+\upsilon)} = M$, a capital M, which consists of the summation of bending moment in x direction and y direction. So, you can see this equation can be written, if it is an equation with Laplacian operator, Laplacian operator is ∇^2 ok, ∇^2 is the Laplacian operator which is $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

So, in this way we can write this $\nabla^2 w$ and this D term I keep here, so that this is equal to M. So, $\nabla^2 w = -\frac{M}{D}$ is one form of the equation representing the deflection with a second order differential equation in relating to the sum of the moment in M_x and M_y direction. So, you can see that the membrane equation, stress membrane equation is also a second order differential equation.

So, here we are getting the equation similar to the membrane equation. So, whatever the equation for the stress membrane is there and solution method is there can be applied here with change of suitable change of variables. Now here original equation in terms of w is $\nabla^4 w = \frac{q}{D}$, where q is the distributed load per unit area of the plate, D is the flexural rigidity.

Now this can be broken up as $\nabla^2\nabla^2 w = \frac{q}{D}$. So, $\nabla^2 w$ again, we will take it from this equation. So, if I take $\nabla^2 w$ here, we can write $\nabla^2\left(-\frac{M}{D}\right) = \frac{q}{D}$. So, that gives you another second order equation but involving the variable M, M is the summation of bending moments in x direction and summation of bending moment in y direction divided by $1 + \upsilon$, where υ is the Poisson ratio.

So, conveniently the fourth order equation is used. For plate problem finding the deflection and thereafter the bending moment, shear force, we use the plate equation of 4th order. But, if anybody is interested to interpret the results, in analogy with the membrane equation, then also the second order equation be used. So, in the next class I will demonstrate the different conditions or different methods of solution.

And specially the equations I have derived here, one is fourth order equation, one is second order equation. Fourth order equation is relating the deflection to the distributed load and second order equation is relating deflection to the moment quantity. A moment quantity is the summation of 2 moments. Now you can see that any differential equation when you solve an exact solution is obtained, we need to satisfy the boundary condition ok.

Because if the boundary condition is not satisfied, this will not be an exact solution. Therefore, different boundary condition in the plate has to be known and based on the different boundary conditions; different methods of solution have been evolved. In theory, generally in exact solution, no definite method is used. As I have told in the introductory class, there are 2 popular methods, one is Navier method and another is Levy's method.

But Navier method is used for a plate which has all the edges, all 4 edges simply supported, that is it is used for rectangular plate only. So, rectangular plate has 4 edges, so all 4 edges are simply supported. Similarly, another method I have told you that is the Levy's method, which has two opposite edges simply supported and other two edges may have any other condition.

So, these two popular analytical methods are still existing and results from these two popular analytical method forms a benchmark result. Whatever you use finite difference, finite element or any other software to evaluate the stress resultant, these methods, Navier method and Levy's method give you the exact values. And you can compare the efficiency of the software or the numerical techniques, ok. So, thank you for today's lecture, I will conclude now. And next class we will see, how to go into deep of the plate equation and its solution, thank you.