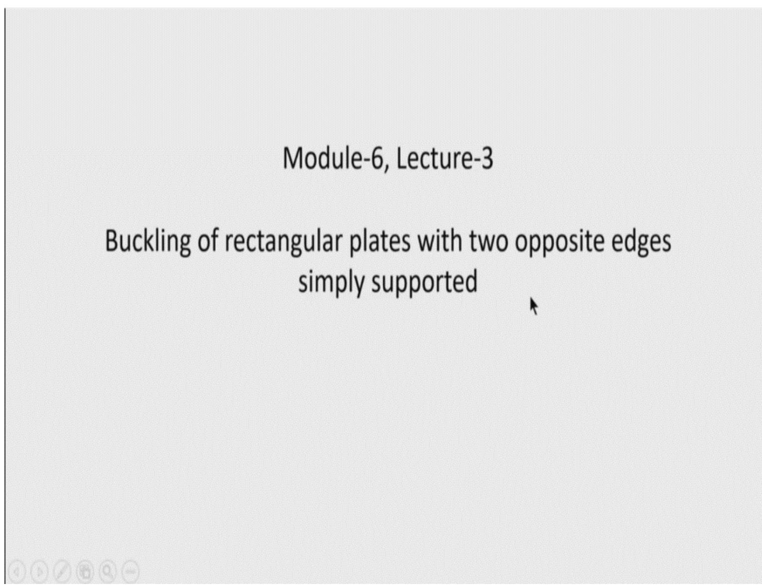
Plates and Shells
Prof. Sudip Talukdar
Department of Civil Engineering
Indian Institute of Technology, Guwahati

Lecture - 19
Buckling Load of Rectangular with Levy's Boundary Condition

Hello everybody, it is the third lecture of the module 6. We have so far discussed the buckling of thin plate subjected to uniform compression, and the solution was obtained by exact method. That is the governing differential equation was taken, and boundary conditions have been utilised to obtain the boundary value problem from which we obtain the buckling loads. And the lowest value of critical load we have obtained. Now, in this lecture will focus on two aspects.

(Refer Slide Time: 01:13)

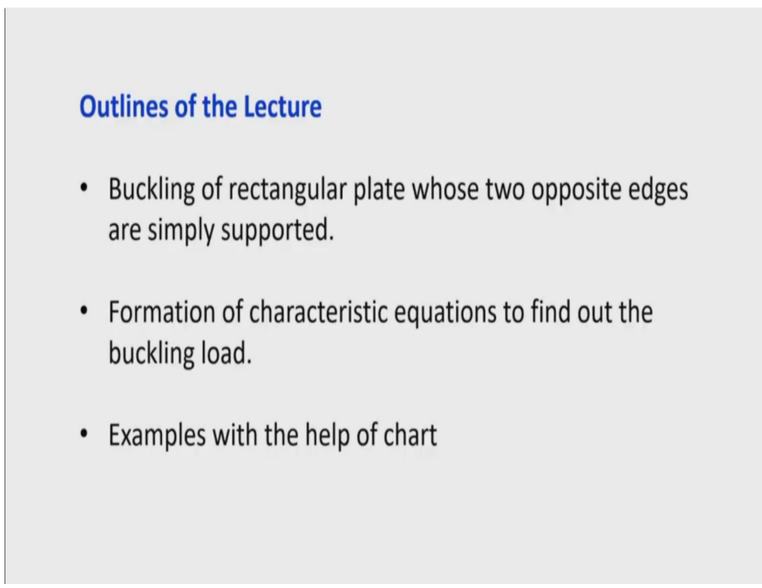


One is the buckling of rectangular plate with two opposite edges simply supported. That condition was not discussed earlier because in earlier lecture, we have considered a plate whose all edges were simply supported rectangular plate. That is, the Navier series was applied in the earlier method. But here, the buckling of rectangular plates with two opposite edges simply supported, and other two opposite edges may have any other condition.

So, that topic will be our point of discussion. And then, we have seen in earlier cases that the axial compression or in-plane load that was acting on the edges per unit length was applied in the direction of simply supported edges. That mean the loading was in the in one direction, which was simply supported. But we will see here that loading direction if now it is reversed, then how the boundary condition will change and what will be the change in this solution.

So, the today I will discuss the buckling of rectangular plate whose two opposite edges are simply supported, then we will form the characteristic equation to find out the buckling load. And some examples will solve or will try to evaluate the critical stress of the plate using the chart because charts are available for calculating the critical states of the plate in terms of non-dimensional parameter.

(Refer Slide Time: 03:15)

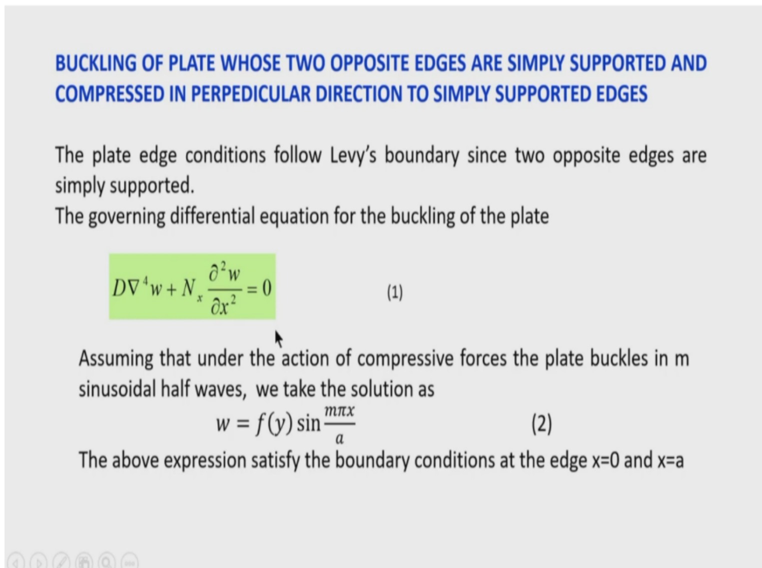


In fact, these charts were derived, or this chart was prepared based on the solution that I was discussing in earlier classes. Earlier class I discussed the solution of the buckling problem of a rectangular plate whose four edges were simply supported. Now, for that, one can get the chart that is the critical stress factor k can be found in terms of a by b ratio. And also, the graph was given in the earlier cases for any value of a by b; we can find the critical success factor.

Now, for other boundary conditions, when two opposite edges are simply supported, and other two edges may have any other condition. Means the other two opposite edges maybe have simply supported condition or maybe claimed condition or maybe in free condition or maybe any type of elastic support maybe also there. So, in that case, the characteristic equation that we will obtain will be slightly complicated, and the solution up to obtain by numerical methods.

So, solutions are obtained by various authors. And it has been represented in the form of graphs in many textbooks. So, we will take the help of this chart to illustrate some problem where the buckling load of the plate for various boundary conditions can be evaluated with the help of chart can be used for design purpose.

(Refer Slide Time: 05:05)



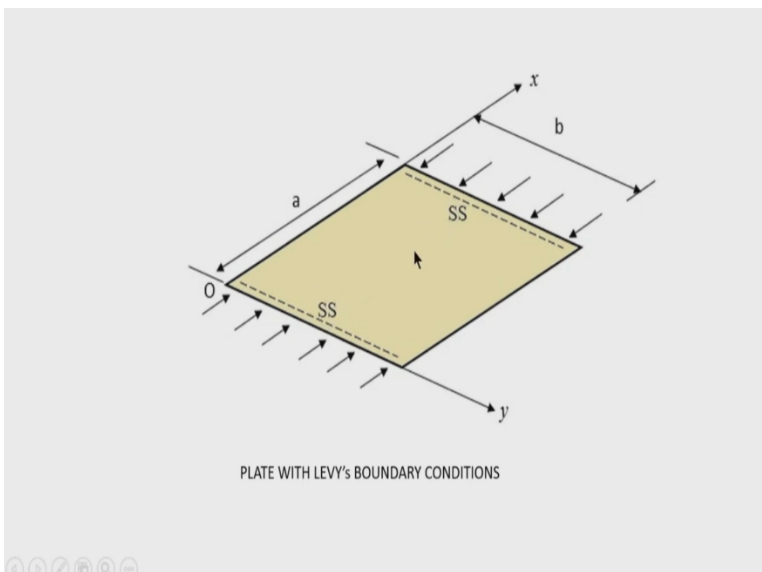
Now, the equation of buckling of the plate is given by this if the compression is in the one direction. So, uniaxial compression exists in the plate that compression is now along the x-direction. So, in that case, the buckling equation is given by $D\nabla^4 w + N_x \frac{\partial^2 w}{\partial x^2} = 0$, where N_x is the axial compression per unit length along the x-direction. So, loading direction is along the x-direction.

And if suppose in our case x-direction is replaced the length of the plate is x-direction is a and width of the plate in y-direction is b. Then a/b is the would aspect ratio of the plate. And we will be interested to find the buckling load factor or critical stress factor in terms of a/b ratio. Here D is the flexural rigidity of the plate, and we assume that plate is of uniform thickness. ∇^4, this ∇^4 is the bi-harmonic operator.

That is equal to $\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$. Now, since two edges are two opposite edges are simply supported and compressed in perpendicular direction to the simplest supported edges. If the compression is in the parallel direction of the simply supported edges, then, of course, the formulation will be boundary condition will be different.

Now, here the compression is in the perpendicular direction of the simplest supported edges that is along the x-direction. Now, in a rectangular plate the x = 0 and x = a.

(Refer Slide Time: 07:17)



If a is the length of the plate, say this is a rectangular plate whose two opposite edges are simply supported. That is x =0, x = a are simply-supported edges. Consider this as the origin so; a is the length of the plate b is the width of the plate. This uniform compression is applied at the edges along the x-direction. That means simply supported edges carries compression perpendicular to the edge line, and this edge is free from in-plane forces.

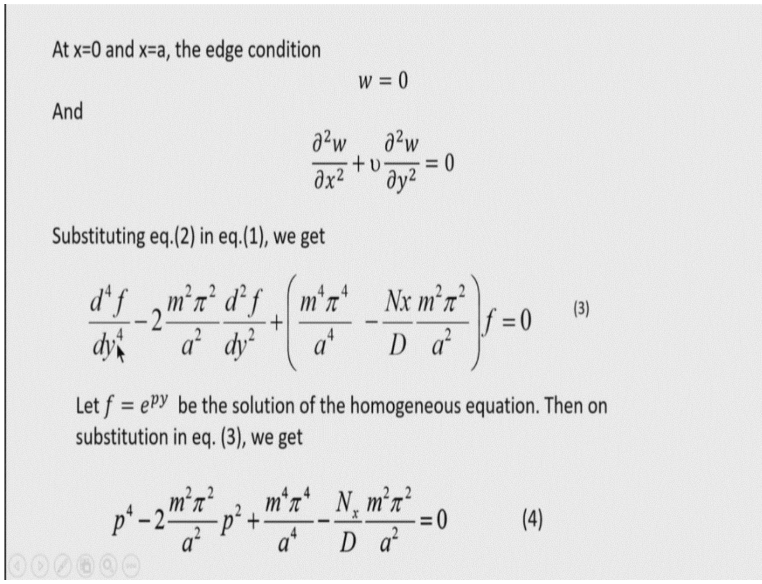
But boundary condition may be anything. Boundary condition means, here, the edge may be simply supported, or it may be claimed or maybe free. Here also different boundary condition edges for these edges. It is not necessary that boundary conditions should be symmetric in the two opposite edges. But these two edges are taken simply supported. So, that we; can express that deflection in terms of Levy's series.

Levy's proposed for such cases, when two opposite edges are simply supported that x = 0 and x = a edges are simply supported. Then Levy has proposed his series; it is a series for different values of m the deflection can be calculated and summed up. So, this a is function of y and sin $sin\frac{m\pi x}{a}$. Because the plate is supported at x = 0 and x = a. So, therefore, sin function is taken to represent the boundary condition at x=0 and x = a, $f(y)$ is unknown function there have to be found from the solution of this equation after substituting the boundary condition.

Now, if this is the solution or if this is the deflected curve, then it must satisfy the differential equation. So, substituting this equation number 2 in equation 1 we need to evaluate the derivative of w because here $\nabla^4 w$ contains the 3 derivative terms. That is $\partial^4 w/\partial x^4$ that means, we have to differentiate this sin $sin\frac{m\pi x}{a}$ 4 times with respect to x. So, naturally, the factor $\frac{m^4\pi^4}{a^4}$ will be appearing.

And then when we differentiate with respect to x, then $f(y)$ will remain as it is. Then again, when we differentiate two times with respect to x, then $-\frac{m^2\pi^2}{a^2}$ term will come. And when we differentiate four times with respect to y, then we got this $\frac{\partial^4 f}{\partial y^4}$. So, therefore, the differential equation now can be written after substituting the derivative in this form.

(Refer Slide Time: 10:49)



That this differential equation now becomes like that $\frac{d^4 f}{dy^4}$. Because after differentiation, when we separate the variables, this will be a ordinary differential equation $\frac{d^4 f}{dy^4} - 2\frac{m^2\pi^2}{a^2}\frac{d^2 f}{dy^2} + \left(\frac{m^4\pi^4}{a^4} - \frac{Nx}{D}\frac{m^2\pi^2}{a^2}\right)f = 0.$

Again, we will get this $\frac{m^2\pi^2}{a^2}$. So, this is the differential equation after taking this derivative in the plate equation. Plate equation for buckling is this, and after taking the derivative, then we get this differential equation in this form. So, this is a homogeneous differential equation, and its solution can be obtained by assuming you have $f = e^{py}$, where p has to be found from the after substituting this function trial.

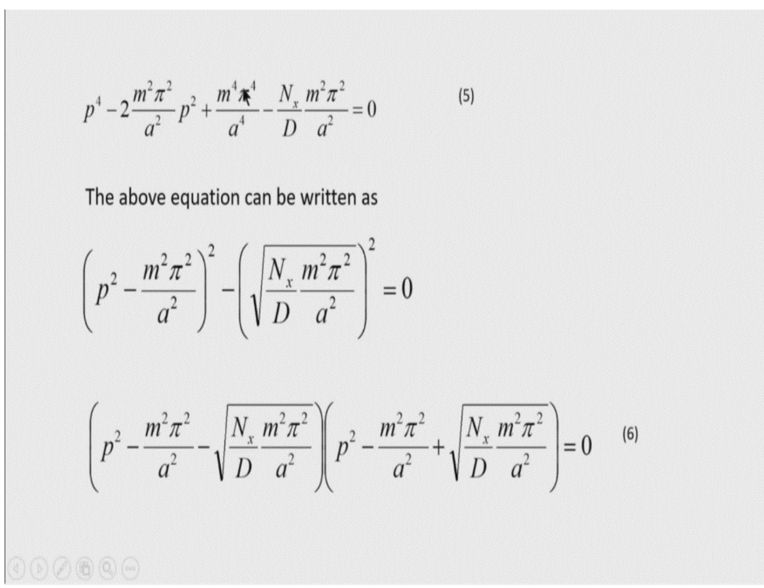
This is the assumed solution in the differential equation. So, substituting $f = e^{py}$, here in this equation. We now get say after differentiating 4 times we will get p^4 then differentiating two times will get $\frac{m^2\pi^2}{a^2}p^2$will be there $- 2 + \frac{m^4\pi^4}{a^4}$ this term is coming. And then again, this is there. So, when we substitute $f = e^{py}$.

So, e^{py} will be common for all the terms, and then it will get cancelled. So, therefore, ultimately, we are getting this equation $p^4 - 2\frac{m^2\pi^2}{a^2}p^2 + \frac{m^4\pi^4}{a^4} - \frac{N_x}{D}\frac{m^2\pi^2}{a^2} = 0$. So, you are getting this as the algebraic equation that is a polynomial equation of fourth-order polynomial.

That we have to find the four roots, and then we can find the solution of the differential equation. Now, this characteristic equation this is called the characteristic equation, is obtained for the plate under consideration which has two opposite edges simply supported. And other two edges, the boundary condition is not specified. So, for any boundary conditions on the other two edges, the function that is to $w = f(y) \sin sin \frac{m\pi x}{a}$ will remain valid.

So, the boundary condition at other edges was equal to 0 and y = b must be utilised to get the proper or complete solution of the homogeneous equation. Now, let us see how we can solve this polynomial in a meaningful way.

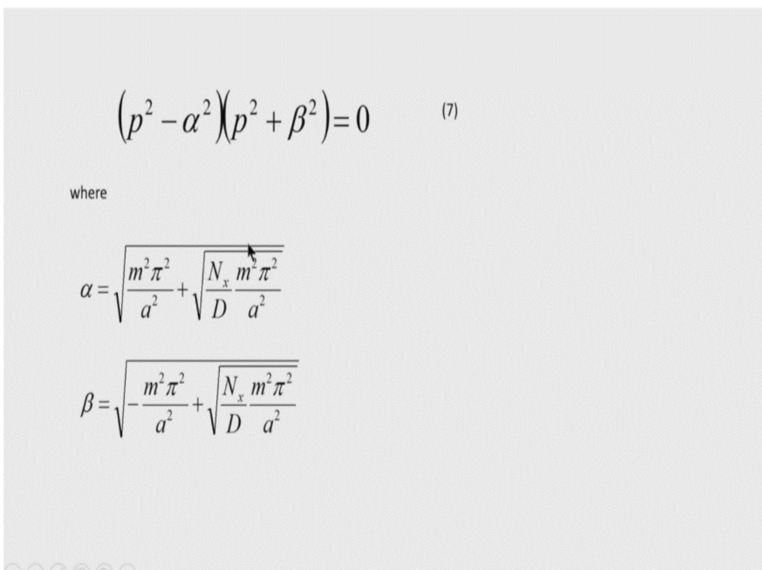
(Refer Slide Time: 14:38)



Now, this is the polynomial or characteristic equation fourth-order polynomial. The above equation can be written in this form. Suppose if I take say p^2 and then this is another factor $\frac{m^2\pi^2}{a^2}$, whole square that we can take, and you see this term can be now written $\left(p^2 - \frac{m^2\pi^2}{a^2}\right)^2$. Then this term that $\frac{N_x}{D}\frac{m^2\pi^2}{a^2}$ can be written in this form root of bar $\left(\sqrt{\frac{N_x}{D}\frac{m^2\pi^2}{a^2}}\right)^2$.

So, now this polynomial is written in this form that it can be factorised. So, main intention is to factorise, and from this factorised expression, we can get the roots p for roots of p. Now, this is in the form of $(a^2 - b^2) = 0$. So, $(a + b)(a - b)$ we can utilise equal to 0. So, if I take this quantity is a, and this is b, $(a - b)$ and $(a + b)$. So, this is written in this fashion. Now, to simplify calculation or to write in a way that one should not confuse with so many terms.

(Refer Slide Time: 16:13)

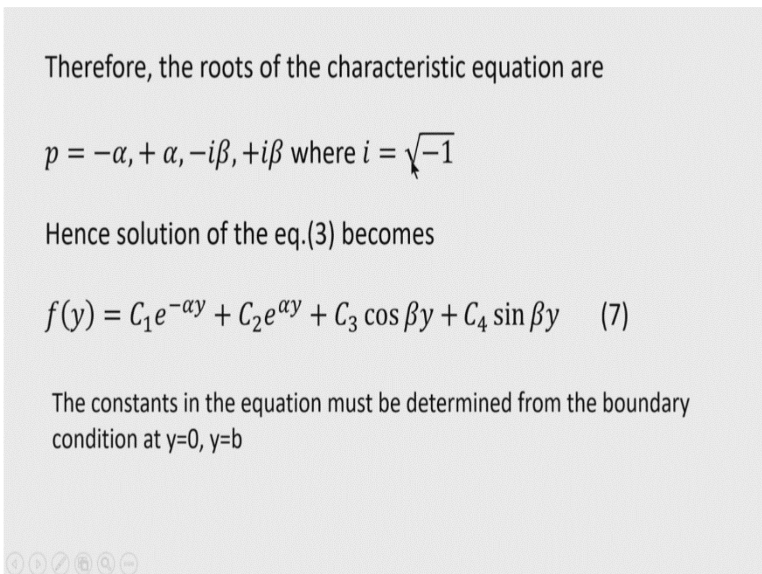


We introduced two parameters, alpha and beta. So, $\alpha = \sqrt{\frac{m^2\pi^2}{a^2} + \sqrt{\frac{N_x}{D}\frac{m^2\pi^2}{a^2}}}$. And then $\beta = \sqrt{-\frac{m^2\pi^2}{a^2} + \sqrt{\frac{N_x}{D}\frac{m^2\pi^2}{a^2}}}$. This is taken in this way because we can observe that this quantity is your say alpha square, and this is β^2.

So, now, we can write the equation in this form $\left(p^2 - \alpha^2\right)\left(p^2 + \beta^2\right)$. So, this is the characteristic equation. Now, one can easily see that the roots are obtained in a very convenient way. So, from this equation, the product of two functions one is $\left(p^2 - \alpha^2\right)\left(p^2 + \beta^2\right)$. One can write that either, $\left(p^2 - \alpha^2\right) = 0$ or $\left(p^2 + \beta^2\right) = 0$.

So, $\left(p^2 - \alpha^2\right) = 0$ will give you two real roots because alpha and beta are real quantity. And $\left(p^2 + \beta^2\right) = 0$ will give you two complex roots.

(Refer Slide Time: 17:45)



So, let us see the roots of the equation is now $p = -\alpha, +\alpha$ this is the real roots. And another two roots are $-i\beta, +i\beta$ where i is the imaginary unit which is equal to $\sqrt{-1}$. So, now solution can be written in this form e to the power, C_1 a constant of integration $f(y) = C_1 e^{-\alpha y} + C_2 e^{\alpha y} + ae^{-i\beta} + be^{+i\beta}$, a, b, are any arbitrary constant.

So, these are all arbitrary constants. They can be manipulated and changed also; symbol can be changed; the meaning is only these are arbitrary constants. Now, you can see here that this equation contains exponential term and also trigonometrical term. Trigonometrical term is coming. How it is coming? Because we have written one root, I have not written here. One root was ae^{-ib} I have considered.

So, ae^{-ib} can be expressed as $a\cos cos\,\beta y - i\sin sin\,\beta y$. Here is minus; it will $-\sin sin\,\beta y$. So, another root is $e^{i\beta y}$. So, in the form of trigonometric function that is the D Mobius theorem, it is called integral symmetry; one can write that e^{ib} as $\cos cos\,iby$ can be written as cos beta y + i sin beta y. So, when we take the exponential of $-\ i\beta$, we get cos and $\sin sin\,\beta y$.

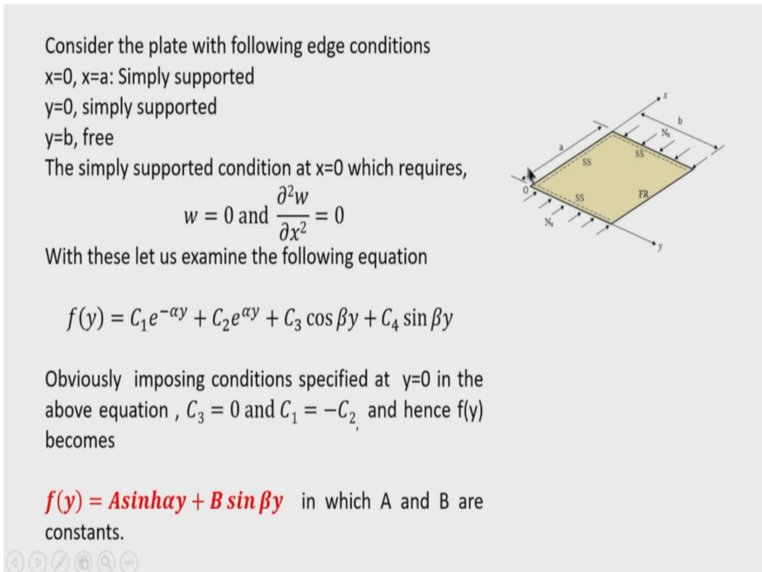
And then when we take again this exponential of $+\ i\beta y$, we are getting $\cos cos\,\beta y\ +\ i\sin sin\,\beta y$. So, then the plugging the constants of the common term these common terms in both the cases are $\cos cos\,\beta y$ and $\sin sin\,\beta y$. We can now write the complete solution in this form involving four constants of integration. So, complete solution is $f(y) = C_1e^{-\alpha y} + C_2e^{\alpha y} + C_3\cos cos\,\beta y +\ C_4\sin sin\,\beta y$

Now, in this function that we obtain the one part of the solution $f(y)$ other part is completely known because this is simply supported edges or x = 0 and x = a. So, there is no difficulty to write it by $\sin sin\,\frac{m\pi x}{a}$ only unknown thing is $f(y)$. Now, we have arrived the solution for in this form $C_1e^{-\alpha y} + C_2e^{\alpha y} + C_3\cos cos\,\beta y +\ C_4\sin sin\,\beta y$. So, this function can be taken for forming the boundary value problem from which we can find the characteristic routes.

That is, which contains the axial load and x from which we can get the critical load. Now, constants in the equations are four numbers because it is a fourth-order differential equation. So, solution which naturally yield four constants of integration. So, four constants of integrations should be found by imposing the four boundary conditions in the equation number 7. The constants in the equation should be determined from the boundary condition at $y\ =\ 0,\ y\ =\ b$.

Now, see the plate again; this plate at x = 0 boundary conditions is known. And x = a boundary condition is again known, x = 0 is the set and x = a is this edge, but y = 0 this edge I have not specified any boundary condition. At this edge also I have not specified any boundary condition. So, for this condition, that is the Levy's condition; if I tell it for this plate, we get the solution $f(y)$ as this. And this is valid for any boundary condition on the edges y = 0, y = b.

(Refer Slide Time: 23:19)



Now, let us consider the plate with the following S condition on the other opposite side edges. On the two opposite edges, x = 0, x = a boundary condition and simply supported. So, therefore, we have taken the function in the deflection series as sin $sin\ \frac{m\pi x}{a}$. That function satisfies the boundary condition at x = 0 and x = a. But y = 0 we take now, for this problem y =0 simply supported and y = b as a free condition.

It is not necessary that y = 0 simply supported and y = b free. It may be vice versa also, and again the both the condition both the boundary condition may be free also and when these two conditions are simply supported again. Then we can go back to the Navier solution that we have already used. So, this problem is more general compared to your Navier series that we have taken earlier. In which case, all the edges of the rectangular plate were simply supportive.

The simply supported condition and x =0 said this is the function. If this y = 0, x = 0 simply supported condition gives y = 0, $\frac{\partial^2 w}{\partial x^2} = 0$ that we know. Now, let us impose the boundary condition at y = 0, y = 0 we have taken again simply supported. So, at y = 0, if we take the simply supported condition, then put y = 0 here, you will get $C_1 + C_2$ then you will get C_3, and this will be 0.

Now, for 0 deflection, deflection is 0; that is, w should be 0 in for what condition it will be 0. If $C_3 = 0$ that is one condition because we are getting $C_1 + C_2 + C_3 = 0$. So, that is possible if $C_3 = 0$ and another condition is $C_1 =- C_2$. So, two constants, one constant completely known that $C_3 = 0$ and another constant is known in terms of other constant. And C_4 is not coming here because when we put 0, then C_4 is meaningless.

So, we ultimately get the solution of the equation because here, we get this $C_3 = 0$. So, that means, we can get now solution in this form four constants are now transformed into two constant by using the first condition of boundary that is y = 0 simply supported, y = 0 simply about the first condition is deflection is 0 at this simply-supported edges. So, for that condition, if $C_3 = 0$ this term will not be appearing in the deflection equation, so, $\sin sin\, \beta y$ term will be coming.

And this term $C_1 e^{-\alpha y}$ and $C_2 e^{\alpha y}$ will be there, but instead of C_2 I can write $- C_1$. So, one constant C_1 and another constant C_4 that means, now, I renamed the constant as A and B. Now, you can see that $e^{-\alpha y}$ can be expressed in terms of $cosh$ and $sinh$. So, $e^{-\alpha y}$ can be written as $\cosh cosh\, \alpha y - \sinh sinh\, \alpha y$.

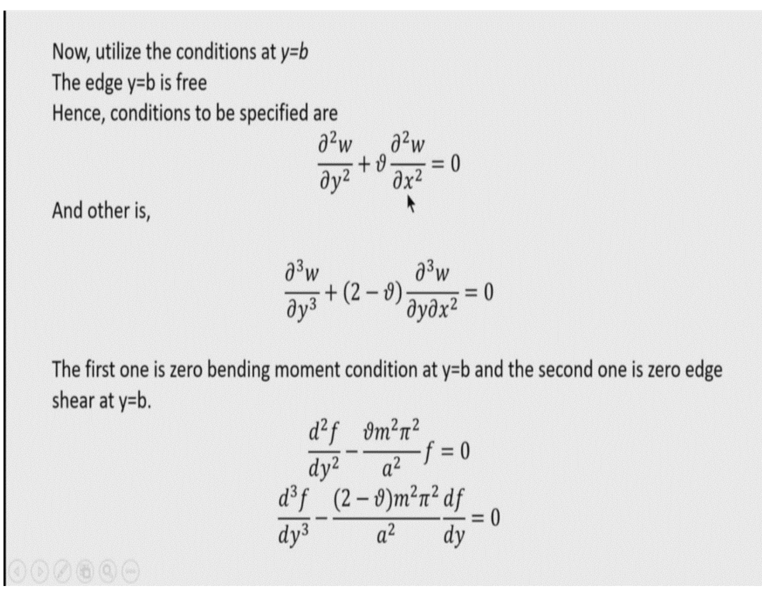
Similarly, that $e^{\alpha y}$ can be written as $\cosh cosh\, \alpha y + \sinh sinh\, \alpha y$. How will you get it? Because the $sinh$ function says $\sinh sinh\, \alpha y$ function is $\frac{e^{\alpha y} - e^{-\alpha y}}{2}$. What is $cosh$ function?

cosh $cosh\, \alpha y$ function is $\frac{e^{\alpha y}+e^{-\alpha y}}{2}$. So, when we want to find out or express this $e^{-\alpha y}$ or $e^{+\alpha y}$ in terms of $cosh$ and $sinh$.

We should add this, and then we can also subtract it. So, after adding and subtraction, we can find this $e^{-\alpha y}$ and $e^{\alpha y}$. So, with this technique, we can express $e^{-\alpha y}$ in terms of cosh $cosh\, \alpha y$ and sinh $sinh\, \alpha y$. Similarly, $e^{\alpha y}$ in terms of cosh $cosh\, \alpha y$ and sinh $sinh\, \alpha y$. Now, since $C_1 = C_2$, so, therefore, only one constant appears.

And therefore, we can write this in this form A into sinh $sinh\, \alpha y + B$ because C_4 is taken as B, B sin $sin\, \beta y$. So, now, instead of four constants, we now get two constants in the equation f y. So, A and B are the constants which should be found out by imposing the boundary condition at y = 0 and y = b.

(Refer Slide Time: 29:49)



Utilise the condition at y = b, the edge y = b is free. That means free condition, the bending moment is 0, and edge shear is 0. So, bending moment 0 means we can write $\frac{\partial^2 w}{\partial y^2} + \vartheta \frac{\partial^2 w}{\partial x^2} = 0$

And other is. And edge shear 0 that is another condition we can write $\frac{\partial^3 w}{\partial y^3} + (2 - \vartheta)\frac{\partial^3 w}{\partial y \partial x^2} = 0$

.

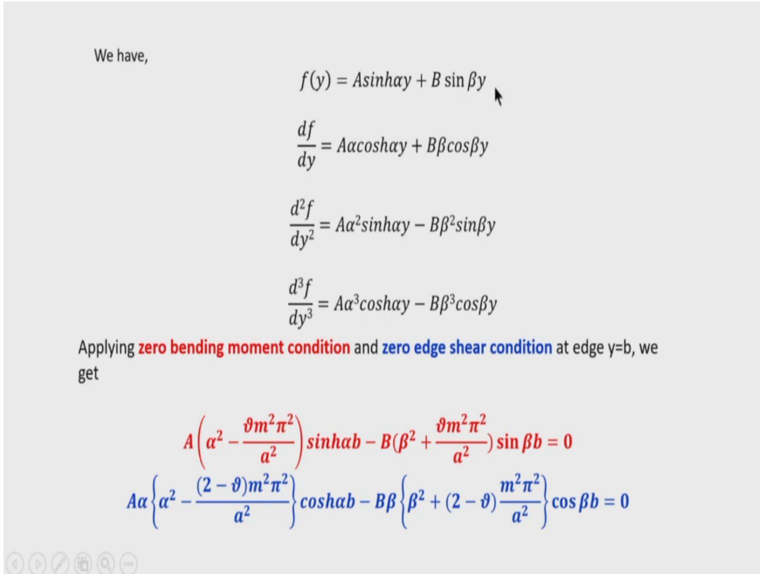
So, these two conditions are known; these two conditions have to be imposed because the condition at b is free. So, bending moment is 0, and edge shear is 0. Now, if I see this equation say $A \sinh sinh\, \alpha y\, B \sin sin\, \beta y$. How these two terms are coming? You have seen that $C_1 =- C_2$. So, we can write a constant say $C_1 - C_2 = a$. So, we can write $A\times(e^{\alpha y} - e^{-\alpha y})$.

So, $e^{\alpha y} - e^{-\alpha y}$ can be written conveniently as $sinh$ function. The factor this half or two that is coming here will be plugged in the constant A. So, it will not affect the function $sinh\alpha y$. So, one function that you are getting in the form of hyperbolic term other function because $C_3 = 0$ So, C_4 we are renaming B. So, we are getting say $B \sin sin\, \beta y$. So, two constants are now have to be found imposing the two conditions.

So, one condition that 0 bending moment condition when it is applied at the edges y = b, we get this equation $\frac{d^2f}{dy^2} - \frac{\vartheta m^2\pi^2}{a^2}f = 0$. And the second condition when we impose or apply at the edge y = b, we get the $\frac{d^3f}{dy^3} - \frac{(2-\vartheta)m^2\pi^2}{a^2}\frac{df}{dy} = 0$. Now, you can see the edge shear quantity the differentiation has taken twice with respect to x.

So, therefore, $\frac{m^2\pi^2}{a^2}$ term is coming. And then differentiation with respect to y that means, just we can write $\frac{df}{dy}$ because this function f is not completely known. So, these two equations are now written after substituting $y = b,\ at\ y = b$, all these function f to be evaluated.

(Refer Slide Time: 33:14)



Now, $f(y)$ we got $A \sinh sinh\, \alpha y + B \sin sin\, \beta y$, this is the function that represents the solution one part of the deflection series $f(y)$. That is found after solving a homogeneous equation in y fourth homogeneous differential equation in y. So, if I take first derivative, then I get $\frac{df}{dy} = A\alpha cosh\alpha y + B\beta cos\beta y$. Second derivative, if I want to calculate, I get A α^2.

Then $\cosh cosh\, \alpha y$, when differentiated with respect to y, it becomes again $\sinh sinh\, \alpha y$. Then $\beta \cos cos\, \beta y$ after differentiation it becomes $-B\beta^2 \sin sin\, \beta y$. Then we go up to third derivative because the edge shear contains the third derivative terms that you have seen here; the third derivative is required for edge shear force. Hence, we required to calculate the third derivative.

Now, when we calculate the third derivative, then $A\alpha^3 \cosh cosh\, \alpha y$. Now, one thing you can see that α and β these are two parameters, very important parameters for this problem. Because our targeted value that we want to find out, that is, the N_x is now inside the α and β that you have noted earlier. So, if α and β have to be known then we can find the N_x.

So, N_x is to be found out for these buckling loads and from which we can decide what will be the critical load that will give the critical safe load for buckling or the critical stress we can call if we go for design and buckling is the criteria for design of plate. Now, zero bending moment condition that if we apply here, you can see the first term is secondary derivative the second term is just function.

So, after applying this second derivative in the function, if we apply the zero moment condition, then we can get this equation $A\left(\alpha^2 - \frac{\vartheta m^2\pi^2}{a^2}\right) sinh\alpha b - B(\beta^2 + \frac{\vartheta m^2\pi^2}{a^2}) \sin sin\, \beta b = 0$ So, this equation is obtained by imposing the condition of zero moment because zero moment comes at the free edges.

At the free edges, zero moments is coming because the zero stress or zero moments is there at the free edges, zero shear and zero moments. Now, after this, this is the solution after getting the relation between the constants, and only two unknown constants are remaining to be found out. Then we get after applying this condition that is the zero bending moment condition; we get the first equation which is returning the red colour.

So, $A\left(\alpha^2 - \frac{\vartheta m^2\pi^2}{a^2}\right) sinh\alpha b - B(\beta^2 + \frac{\vartheta m^2\pi^2}{a^2}) \sin sin\, \beta b$. You can see this is the constant term; this is also constant term, sinh $sinh\, \alpha b$ because this is also constant. How it is becoming constant? Originally it was a function of y, but now, we have substituted the value y = B. So, now, it renders to be a constant term. So, these are also constant.

So, we are writing this equation said this is equation 1, but still, A and B that constants we have taken as; an integration constant still remain unknown. In this second equation, the zero-edge shear condition is applied at y = b. Now, let us see what is zero edge share condition. Zero edge shear condition is this $\frac{d^3w}{dy^3}$ that is first term. Then second term is $+ (2 - \vartheta)\frac{\partial^3 w}{\partial y \partial x^2}$.

You can see here the second derivative is taken with respect to x two times, and then again it is the result is differentiated with respect to y. So, since the w is now in terms of f and $f(y)$ and sin $sin\ \frac{m\pi x}{a}$. So, this can be written as because this is the derivative with respect to y. So, $\frac{d^3f}{dy^3}$ that is the coming here. For bending moment, the second derivative will be there, but for shear force that the second condition that we are considering.

The third derivative will becoming $\frac{d^3f}{dy^3}$. And then sin $sin\ \frac{m\pi}{a}$ will be there, but it will be there in both the terms. So, therefore, it will get cancelled. Then the second term is $(2 - \vartheta)\frac{\partial^3 w}{\partial y \partial x^2}$. That means first, let me differentiate w with two times with respect to x. What is w? $w = f \times \sin sin\ \frac{m\pi x}{a}$. Now when we differentiate w two times with respect to x, then $\frac{m^2\pi^2}{a^2}$ will be coming in the minus sign will come.

So, minus sign is there $\frac{m^2\pi^2}{a^2}$ term is there and $(2 - \nu)$ this term is a constant. So, it is remaining as it is. Then if I differentiate the result with respect to y because sin $sin\ \frac{m\pi x}{a}$ was also already there. So, therefore, we only differentiate this $\frac{df}{dy}$. Then because sin $sin\ \frac{m\pi x}{a}$ will get cancelled from both the sides. So, we get this equation in terms of derivative of the function f. What is the f? f is this function $Asinh\alpha y + B \sin sin\ \beta y$.

Now, I have listed the derivative of f up to third order. So, there is no difficulty to apply these differential coefficients here to get the two equations after getting the two equations. We can write this equation in this form two homogeneous algebraic equation you are getting. But these are highly nonlinear equations because sinh $sinh\ \alpha b$ and sin $sin\ \beta b$ are all nonlinear terms.

Now, if I see the solution of these two equations how we can get? One possibility is that the two equations are satisfied by taking A = 0 and B = 0. But these two zero values of A and B will not serve our purpose of finding the buckling load because we are aiming to find the buckling load or critical stress in the plane. So, taking A = 0, B = 0 will not serve the purpose. So, therefore, other possibility is that the determinant of the equation must be zero.

So, what is the determinant of the equation? Determinant of the equation can be written with the column as this coefficient of A first row first column; this is the element coefficient of a first column second row, the element is this. When you go to this second column, the element is this, and this when you go to second column second row, the coefficient is this.

(Refer Slide Time: 42:45)

For non trivial solution of the

$$\begin{vmatrix} \left(\alpha^2 - \frac{\vartheta m^2\pi^2}{a^2}\right) sinh\alpha b & (\beta^2 + \frac{\vartheta m^2\pi^2}{a^2}) \sin\beta b \\ \alpha\left\{\alpha^2 - \frac{(2-\vartheta)m^2\pi^2}{a^2}\right\} cosh\alpha b & \beta\left\{\beta^2 + (2-\vartheta)\frac{m^2\pi^2}{a^2}\right\}\cos\beta b \end{vmatrix} = 0$$

Or,

$$\beta\left(\alpha^2 - \frac{\vartheta m^2\pi^2}{a^2}\right)\left\{\beta^2 + (2-\vartheta)\frac{m^2\pi^2}{a^2}\right\} tanh\alpha b = \alpha\left(\beta^2 + \frac{\vartheta m^2\pi^2}{a^2}\right)^2 \tan\beta b \qquad (7)$$

So, if I express the function in the determinant form, then we can write in this way. So, the determinants are of size 2×2 and the elements of the determinant can be found out from this equation. Now, after expanding the determinants when we expand the determinant, we will get this term. It is very easy to expand this because 2×2 determinant after expanding we can write this and after simplification we can write, $\beta\left(\alpha^2 - \frac{\vartheta m^2\pi^2}{a^2}\right)\left\{\beta^2 + (2-\vartheta)\frac{\vartheta m^2\pi^2}{a^2}\right\}$.

Then tanh $tanh\ \alpha b$ equal to $\alpha\left(\beta^2 + \frac{\vartheta m^2\pi^2}{a^2}\right)^2$ tan $tan\ \beta b$. You can see this equation is very complicated in nature, and these type of equation are the salient features of this type of problem buckling problem or the pre vibration problem in a continuous system. So, this equation is called transcendental equation or characteristic equation.

Because it may represent the combination of economical terms or polynomial terms or your hyperbolic function. So, solution of this equation is not straightforward one can obtain the solution one possibilities; they are by graphical means you can get the roots. Suppose, if you consider this, the left-hand side is a function f_1 and right-hand side as a function f_2. Now, if you plot along the x-axis, say α or β.

Because α and β both the terms contain the critical in-plane forces N_x N_y whatever maybe, in this case, it is N_x. So, uniaxial compression is there. So, α is containing N_x β is also containing N_x. So, if we find alpha and beta, we can find the N_x. So, if I plot along the x-axis, the alpha or beta or N_x and the along the y axis, the two function that is the left-hand side is one function f_1, and right-hand side is another function f_1.

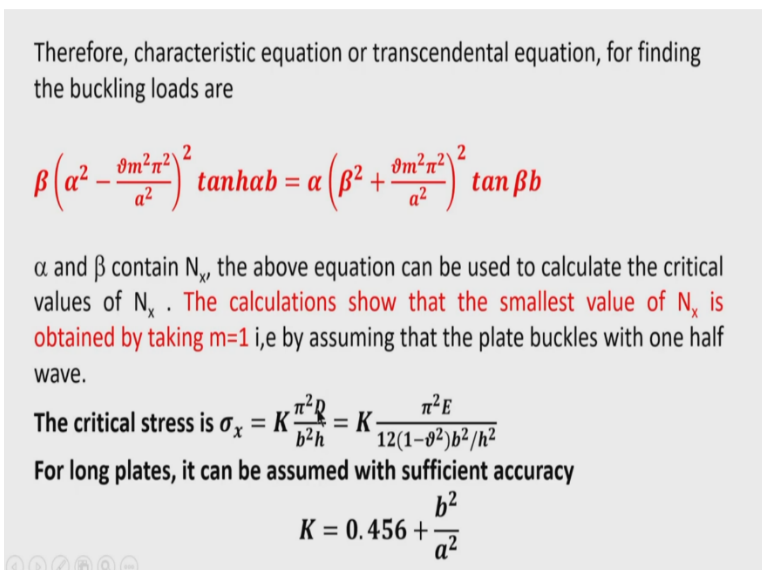
Then, when the two function or curves represented by two functions intersect, that is one root. Now, there will be multiple roots of such equation, and you will get multiple intersection point; of course, the graphical method will not yield you the accurate result. So, therefore, some numerical techniques have to be used. Various numerical techniques are available in the literature, or even nowadays, the matlab app contains so many sub routines that can be used to solve transcendental equation.

One method that is very popular and it is found in Numerical Recipes book, Numerical Recipes, is a book on computer programming. It is written in C++ language, or it is written in FORTRAN also. So, in this book, you will get there are two sub routines one is jet break, and another is RTB's. Jet break is a subroutine which bisects or which a bracket the routes that means bracketing of route is first carried out that means, in which range the route will lie.

Say first route may lie in the range, say a_1 to a_2. So, this is one bracket, a_1 to a_2. Then second root may lie in another brackets b_1 and b_2. So, this is another bracket, but bracket is only identified; it is not the accurate value of the root because bracket is only the route may lie in this region. Then is sub routine call RTBs Root Bisection to be used to accurately find out the route in the bracket. So, in this way, the number of roots of the transcendental equation can be found out.

And the from which the buckling load can be calculated. Buckling load is N_x, and it is embedded inside the parameter alpha or beta. So, this is the way how we can get the transcendental equation and solution of transcendental equation gives you these roots of the equation gives the buckling loads, and there are infinite numbers of buckling loads. But for design purposes, we will take the safe load that is the lowest load which is known as critical load.

(Refer Slide Time: 48:15)



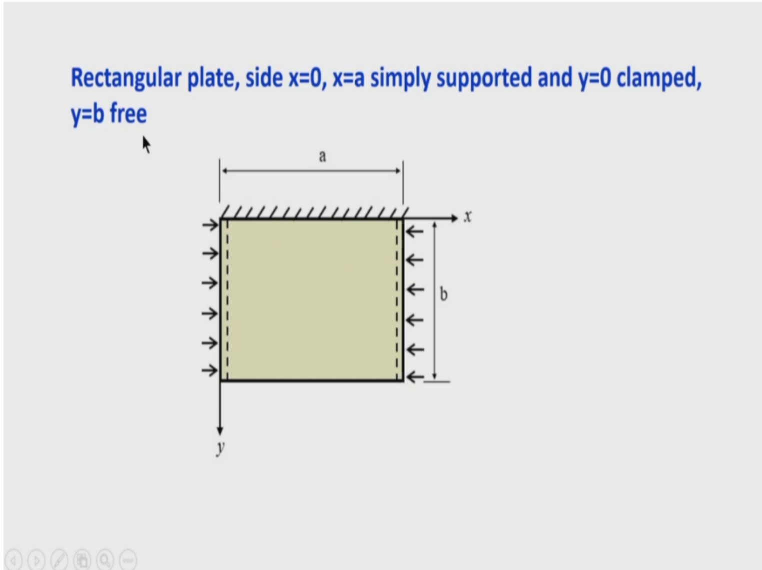
So, characteristic equation is this, and calculation shows that the smallest value of N_x is obtained by taking m = 1. Now, where is N_x? In this equation, N_x is found in α and β. So, if you find α and β, you can find N_x by assuming that played buckles with one half-wave that is m =1 that is only one-half wave in the x-direction. Then the critical load can be written, or critical stress can be written as $\sigma_x = K\frac{\pi^2 D}{b^2 h}$.

And substituting $D = \frac{Eh^3}{12(1-\nu^2)}$, we can express the critical stress as $\sigma_x = K\frac{\pi^2 E}{12(1-\vartheta^2)b^2/h^2}$. Now, you can note here that critical stays sigma x is inversely proportional to the square of the ratio B to h. That is, the critical stress depends on indirectly that is inversely proportional to the b/h ratio square. So, if I plot a graph, say between, say, critical stress and critical stresses along the y axis and b/h ratio is along the x-axis.

Then we will get the design curve that is decreasing with the increasing value of b/h. So, that type of graph that is the variation of the buckling stress with the slenderness ratio. This is analogous that we find in the design of column. In case of plate, the slenderness ratio is not directly involved instead of length, we are now dependent on the width of the plate B, and B square by a square is a factor which must be taken for different plates to calculate the critical value of sigma x.

Now, for very long plates with sufficient accuracy, it can be assumed that critical stress factor is $0.456 + \frac{b^2}{a^2}$.

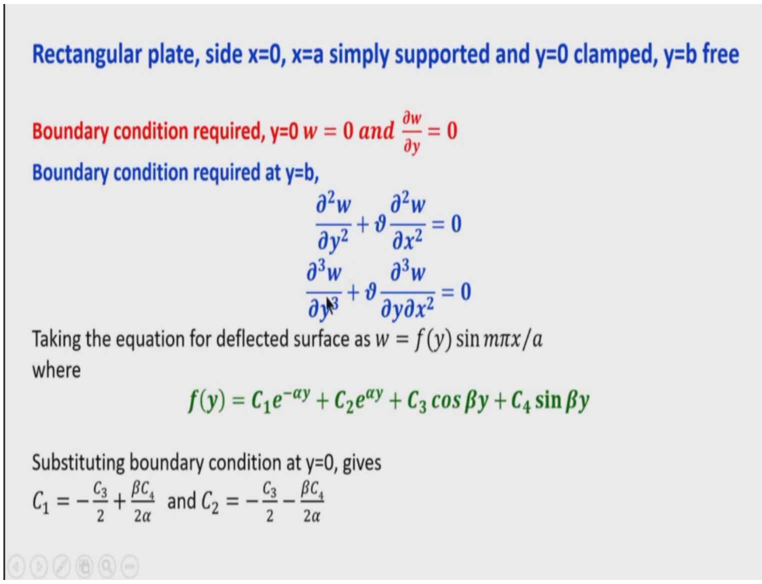
(Refer Slide Time: 50:53)



Now, let us see another condition when rectangular plate side x = 0 and x = a are simply supported. These two edges are simply supported fine, and y = 0 is claimed, and y = b is free. So, in this condition, let us proceed how to obtain the buckling load. Again, I am telling that buckling load is not so straightforward here, in case of these Levy's condition. As we have found in case of Navier's condition, buckling load can be found with ease.

But here, the buckling load is calculated only by solution of the transcendental equation, which contains this hyperbolic function, trigonometrical function, etcetera.

(Refer Slide Time: 51:47)

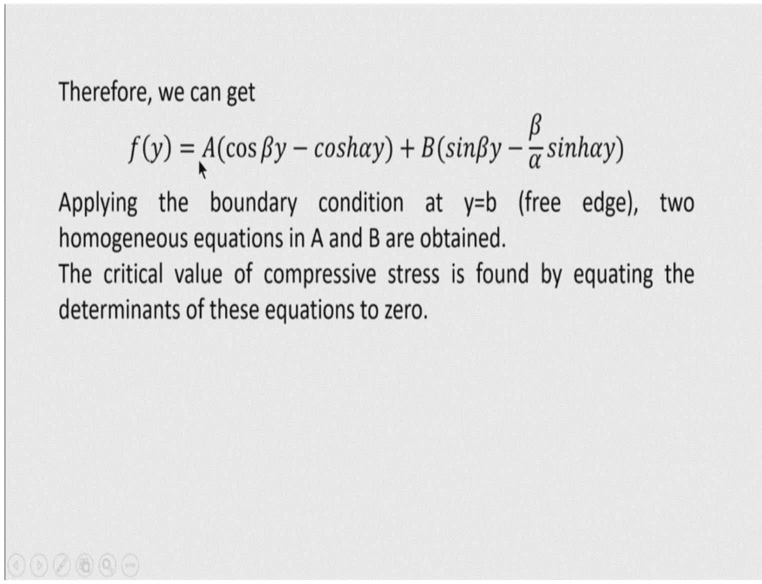


Now, substituting these boundary condition at x = 0 of x = a simply supported and y = 0, y = 0 clamped and y = b it is free edge. These two-boundary condition have to be applied at y = 0 clamped and w = 0 and slope $\frac{\partial w}{\partial y} = 0$. At the other end, y = b, the free edge the boundary condition should be $\frac{\partial^2 w}{\partial y^2} + \vartheta \frac{\partial^2 w}{\partial x^2} = 0$ because this is the bending moment along y-direction.

So, curvature along the y-direction is fast and then the curvature along the x-direction and is contribution is taken after multiplication with Poisson ratio. Then edge shear along the y-direction is zero so, $\frac{\partial^3 w}{\partial y^3} + \vartheta \frac{\partial^3 w}{\partial y \partial x^2} = 0$. So, blue colour equation gives the condition for the free edge y = b, and the red colour equations give the condition at the clamped end.

Now, our solution was this $C_1 e^{-\alpha y} + C_2 e^{\alpha y} + C_3 \cos cos\, \beta y + C_4 \sin sin\, \beta y$. These two terms can be combined in terms of sin hyperbolic and cos hyperbolic. Now, using utilizing the boundary condition at y = 0, that is the clamped end for zero condition, zero deflection condition and zero slope condition, we can get the relation between C_1, C_3 and C_4 as well as C_2, C_3, C_4. So, these two equations are obtained after utilizing the boundary condition at y = 0.

(Refer Slide Time: 53:57)



Then we get the equation in simplified form because this constant C_1 are expressed in terms of two constant. Similarly, constant C_2 are expressed in terms of other two constant. So, therefore, utilizing these two equations, we can express the solution in this form $A(\cos cos\ \beta y\ -\ cosh\alpha y)\ +\ B(sin\beta y\ -\ \frac{\beta}{y}\ sinh\alpha y)$. These two constants are only because of this condition that we obtained earlier.

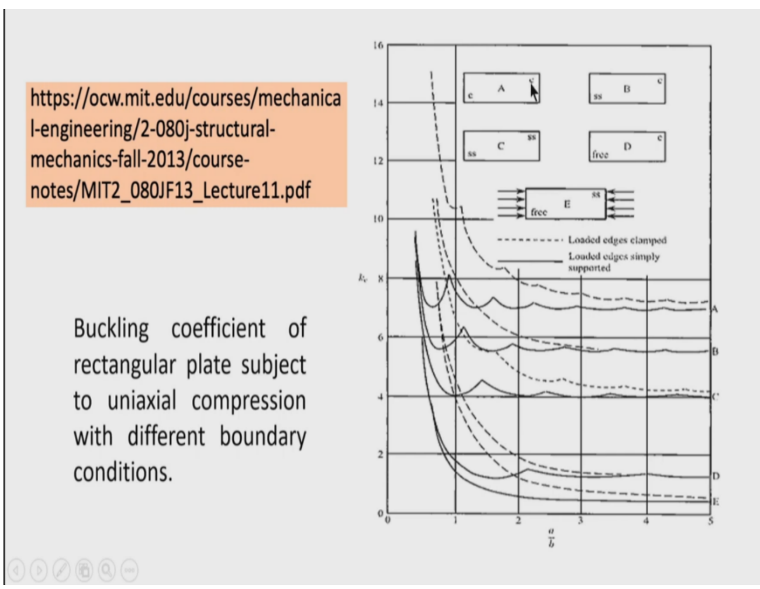
And this condition is related to the clamped end condition, that is, y = 0, your deflection w is 0 and your slope $\frac{\partial w}{\partial y} = 0$. So, this is the condition, then equation is obtained after imposing the boundary condition at y = 0 that is the clamped condition, but still, we are not completed the problem is not completed. So, the problem here again to find the constants A and B. And how can we find the constants A and B?

This is only possible if we apply at the two-boundary condition at the other edges. Other edge that is the opposite edge y = b. Now, two constants are there, A and B. So, we need to apply two boundary conditions there. So, two boundary condition at the free edge are this condition of zero bending moment and condition of zero edge shear. These two conditions are applied at the edge y=b, and we can get a transcendental equation that can be solved.

So, critical value of compressive strength is found by equating the determinants of this one function is that y = 0 you are getting the deflection is 0 slope is 0. And y = b one condition is bending moment equal to 0. So, in the bending moment, you will get the second derivative of f and also with f. So, this expression is composed of earlier as I have shown this expression is composed of your; this $\frac{\partial^2 f}{\partial y^2} - \nu \frac{m^2 \pi^2}{a^2} f$.

So, substituting the derivative of this f properly at y = b, we can get the one equation from that. And another equation is by substituting the zero edge shear condition at y = b other equation is obtained. So, when two equations are obtained, then we equate the determinants of these two equations to zero, and after expanding the determinant as earlier, we have seen. We get a transcendental equation that have to be solved by numerical techniques.

(Refer Slide Time: 57:26)



Now, to help the theory in design problem. We now take the help of a chart available in reference. So, buckling coefficient of rectangular plates subject to uniaxial compression with different boundary conditions are given in this chart. So, you can see here these five plates are there and these are marked as A, B, C, D and E., And you can see suppose this A plate and, in the corner, upper corner it is written as C.

In the lower corner, it is C that is it is clamped both A and C represents the clamped end and S S the present; this simply support end and free is free end. Now again, there are two lines one is dashed line that is the loaded edge is clamped. This is the loaded edge of the plate, and if these edges clamped then the card is represented by dashed line. And if the loaded edge is simply supported, then it is represented by solid line.

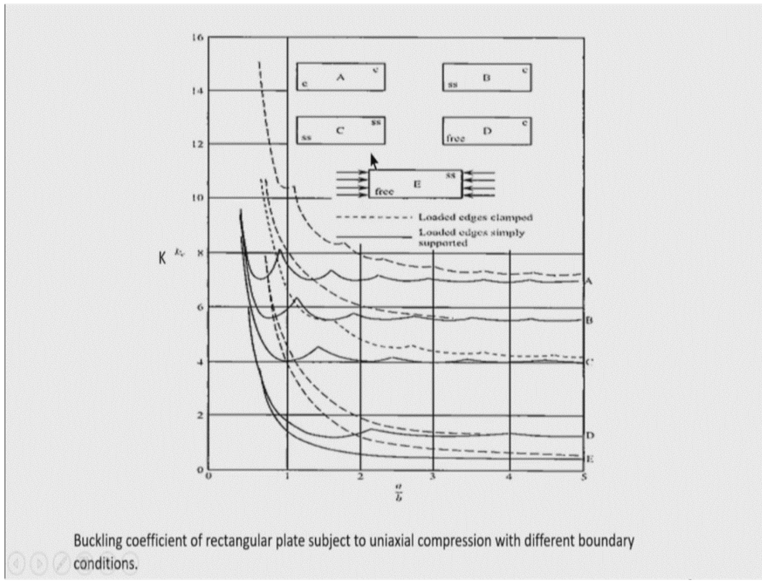
Now, let us see the curves. Earlier in the first class of buckling, we obtain the buckling coefficient for plates which simply supported condition along the four edges. And we have seen that this condition is possible this condition you can see the conditions C, the plate C two edges are simply supported, and these two edges may be claimed clamped or free, but C curves we have to consider. So, you can see the curve C here two curve, or they are in each category.

So, here you can see one line is dashed line, and other is solid line and solid line we are familiar earlier we have found that for simply supported plate along all edges the buckling coefficient for the plate for any integral value of a/b ratio was four. So, that is what you are seeing here. When a/b ratio is one, you are getting backlinks test coefficient is four. When a/b ratio is two, you are getting again buckling coefficient is four.

Again, when a by b ratio is three, buckling coefficient is four like that. When the plate is very long that is a/b ratio greater than four the buckling coefficient is approximately equal to four. Now, if I see other plates say, for example, if I consider a plate whose two opposite edges are simply supported that just we have completed, two opposite edges are simply supported, and other edge is simply supported other edges simply supported and free.

So, other edge is simply supported and free. So, this curve is given by E. So, we will now go to this curve E. This solid line represents the loaded deaths as a clamped edge. Whereas the solid line represents the loaded as simply supported, whereas the dashed line represents the loaded line to be clamped edge. So, two categories of lines are there. So, you should take it properly to calculate the buckling coefficient.

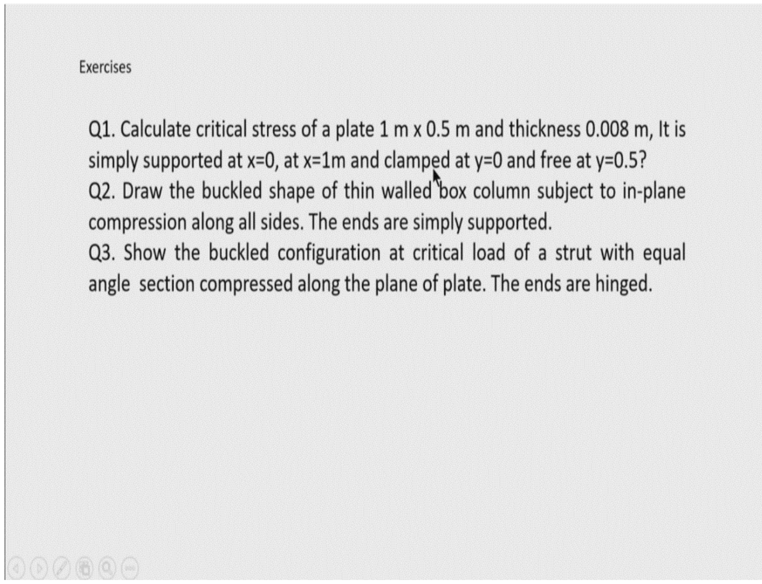
(Refer Slide Time: 01:01:17)



So, this is the graph again same graph, and you can see suppose a plate is there, where this the two opposite edges are simply supported and other two opposite edges one is clamped, and one is free. So, in that case, the condition is depicted by the plate D. So, in the D, we can find the curves related curves. So, curves are again of two types. One is dashed line dashed curves. That is, in this curve, the plate loaded edge is clamped.

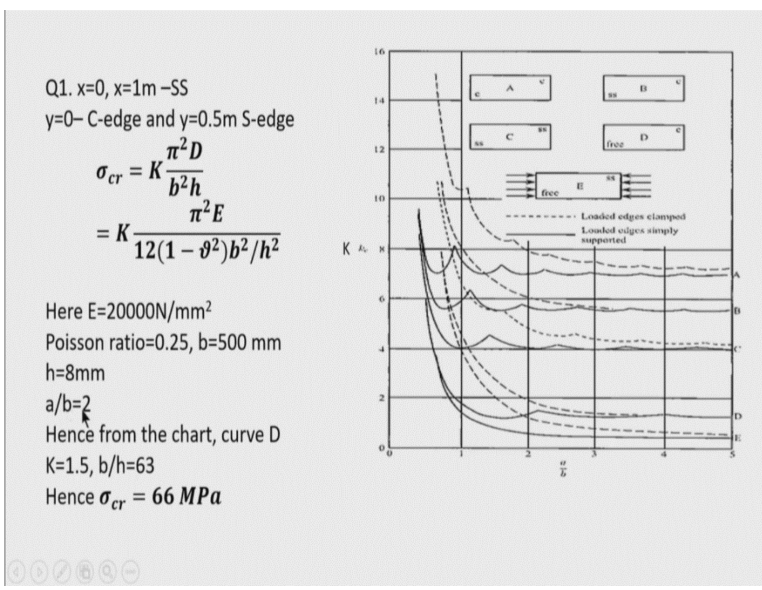
And solid line the plate in who is loaded edges are simply supported. So, loaded edges means to opposite edges are simply supported, and this condition is only for uniaxial compression in the plate.

(Refer Slide Time: 01:02:20)



So, let us see some problems that you can now understand the application of this theory. Calculate critical stress of a plate 1 meter by 0.5 meter and thickness point 0.008 meter. It is simply supported at x = 0 and x = 1 meter along the longer direction and clamped at y = 0 and y = 0.5. This is the first question.

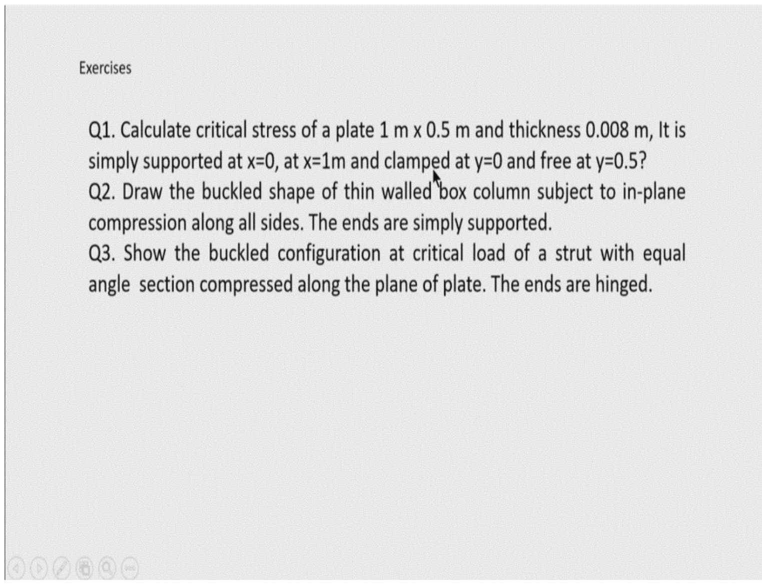
(Refer Slide Time: 01:02:48)



So, we shall utilize the curves now. Now since y = 0 is a clamp edge, y = 0 is a clamped edge and y = 0.5, this edge is free edge will equal 0.5 is free edge so, this edge is free. So, the curve is D. The curve is D, and from that, we can get this the buckling coefficient because your a/b = 2 here. So, ends from the chart corresponding to curve D, we can get K is equal to approximately 1.5 and therefore, b/h ratio it depends on this width of the plate and thickness it is 63.

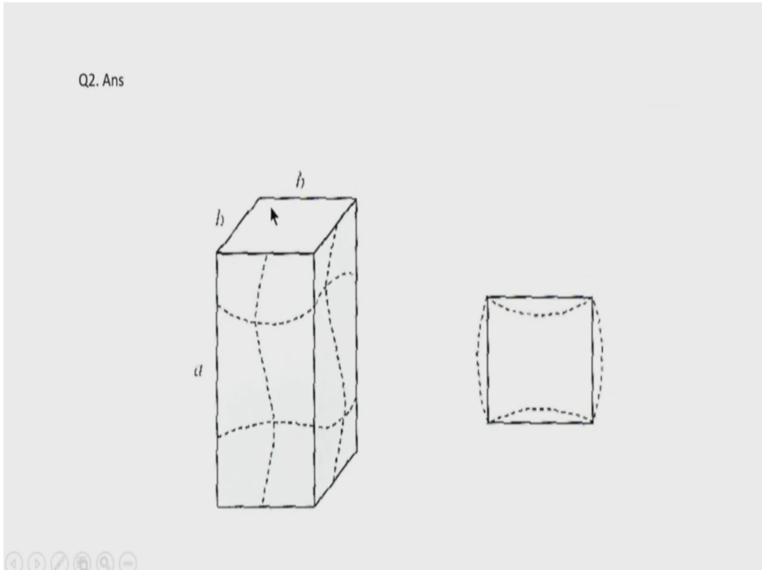
So, after substituting these below here, we can get the critical stress coefficient as a critical stress is 66 MPa.

(Refer Slide Time: 01:03:56)



Now, consider the second problem. Second question draw the buckled shape of a thin wall box columns subject to in plain compression along all sides the answer simply supported.

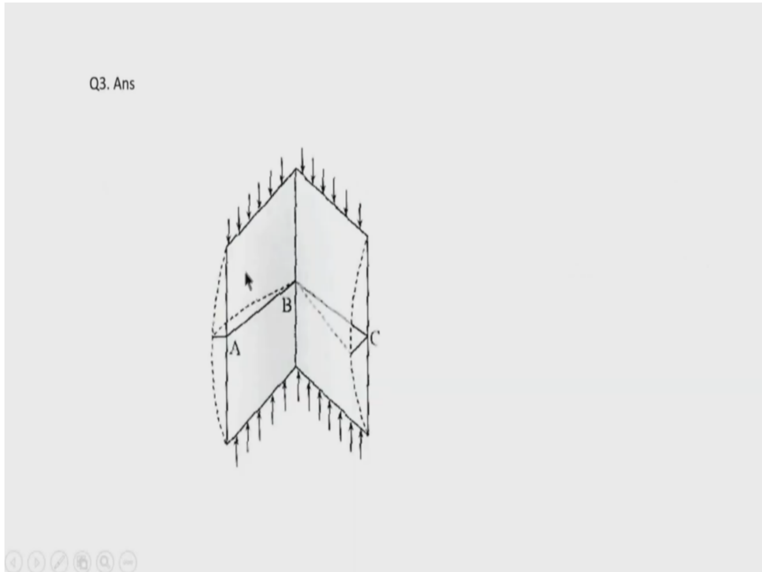
(Refer Slide Time: 01:04:09)



So, if I take a case of a box column that is form by four plates and it is compressed along one direction, and the ends of the columns are taken as fixed. Now, you can see if at the junction of these two plates, the condition of simply support exists, and this edge, the ends are simply supported a pin column. Then we have seen that the one half-wave will inform in the width direction. So, if this is a plate, for example, this is a plate whose length is L and some breadth, and some length is there.

So, one half with this form in the width direction and number of half-waves in the longitudinal direction depends on a/b ratio that is the aspect ratio of the plate. So, in this way, we can draw quality to believe the buckling shape. Now, if I consider this third problem, that is show the buckled configuration at critical load of a start with equal angles section compressed the along the plane of the plate.

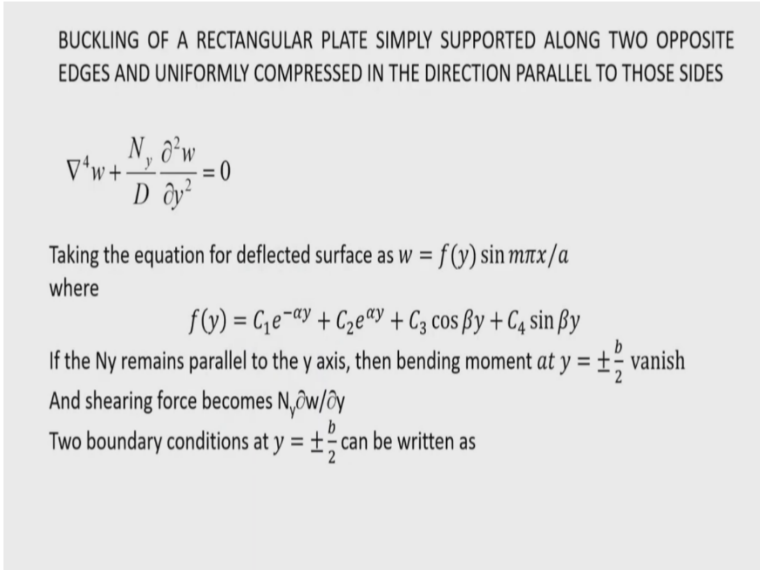
(Refer Slide Time: 01:05:33)



So, in that case, this is the angle section which has the free edge. So, the condition that we have now discussed that is a plate, and the column is pinned at the ends. So, that means, if this is your column so, answer simply supported or pin and then we are applying the compression in this direction, and you can see that the edges here is free. So, condition of the column a plate is that two opposite edges are simply supported and other and one edge is again simply supported.

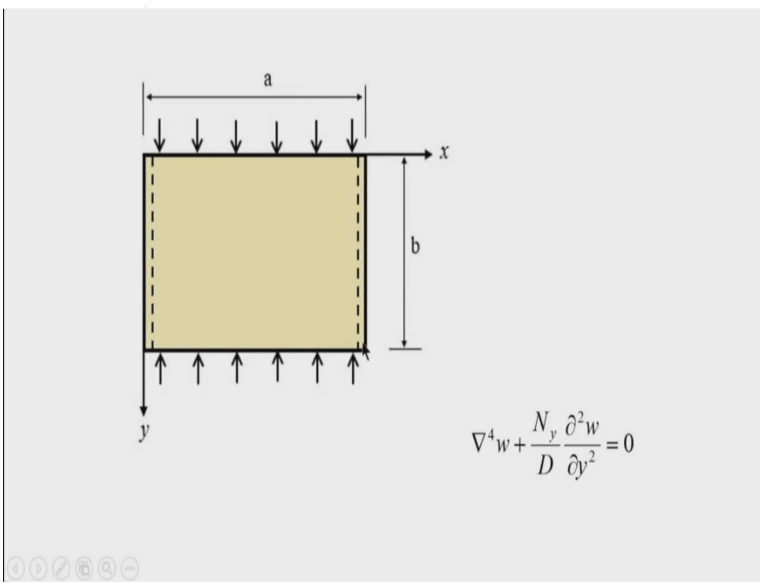
We are taking simply supported at the junction, and this edge is free. So, that we have obtained that the buckling critical states is obtained when one halfway is formed along the length direction. So, the buckle shape is drawn like that at the free edge.

(Refer Slide Time: 01:06:31)



Now, let us see a buckling of rectangular plates simply supported along two opposite edges and uniformly compressed in the direction parallel to those sites.

(Refer Slide Time: 01:06:42)



So, consider the plate here x-direction. This is the x-axis, and this is the y-axis. So, length of the plate is a and width of the plate is b. So, two opposite edges at x = 0 and x = a are simply supported, and other two edges may have other boundary conditions. That is now the plate is obeying the Levy's condition. And the plate is compressed in the direction that direction is taken as uniformly compressed in the direction parallel to those sides.

That means the sides which are simply supported parallel to this sides the compression is applied. In earlier cases, compression is applied along this side. But here, the direction of compression is reversed. So now this buckling equation is this instead of N_x, we will write $\frac{N_y}{D}$ and $\frac{\partial^2 w}{\partial y^2}$ this term is coming. And there is no other in-plane forces. So, this represents the buckling equation of the plate of uniform thickness.

(Refer Slide Time: 01:08:00)

Two boundary conditions at $y = \pm\frac{b}{2}$ can be written as

$$\frac{\partial^2 w}{\partial y^2} + \vartheta\frac{\partial^2 w}{\partial x^2} = 0$$

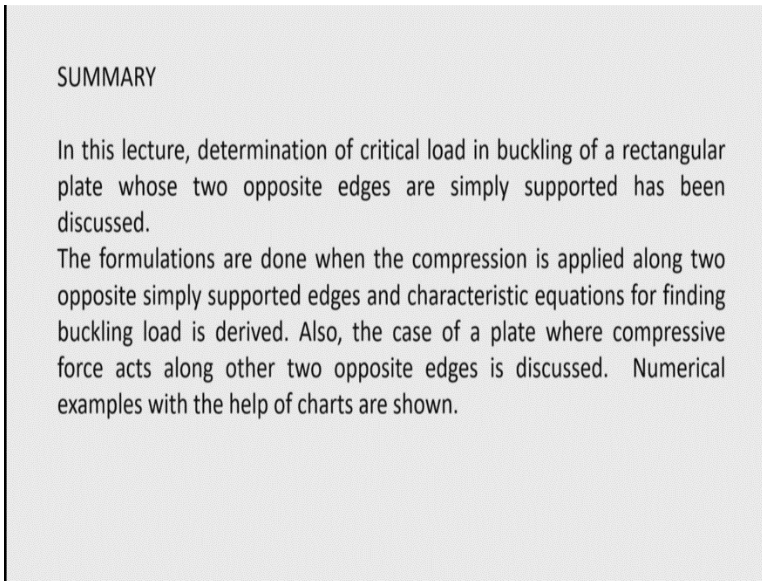
$$\frac{\partial^3 w}{\partial y^3} + (2-\vartheta)\frac{\partial^3 w}{\partial x \partial y^2} = -N_y\frac{\partial w}{\partial y}$$

So, this can be solved only the boundary condition that you have to take. The series the same w is equal to $f(y)$ sin $sin\,\frac{m\pi x}{a}$ that will be the series. That have to be substituted in this equation to get this $f(y)$ and $f(y)$ is again same. But thing is that boundary conditions they do apply at the edges at y = 0 and y = b will be now like that. Suppose two boundary conditions are there; one is the suppose this edge when the compression is applied along the edge parallel to x-axis.

Then the zero bending moment condition along the y-direction is written, and edge shear per unit length has to be banished by the component of the axial force. So, there are two conditions are utilized to find out the transcendental equation in such a cases. So, this case may arise when the compression is applied parallel to these simply-supported edges. Solution of this $f(y)$ is this, but boundary condition that you are apt to apply this y = 0, y = b.

For example, if you take a symmetrical condition of the plate, then you can take the x-axis here. And boundary condition can be applied at $y = +\frac{b}{2}\ or -\frac{b}{2}$.

(Refer Slide Time: 01:09:38)



So, let us see what we have learned today. So, in this lecture, determination of critical load in buckling of a rectangular plate whose two opposite edges are simply supported has been discussed. The formulations are done when the compression is applied along to opposite simply supported edges, and characteristic equation for finding buckling load is derived. And we have shown how to solve the characteristic equation by mean some numerical techniques that I named some techniques that you can use it.

Also, the case of a plate where compressive force acts along other two opposite edges is discussed. That means compression is applied parallel to these simply-supported edges. Numerical examples are solved with the help of charts that I have given you the chart from the reference that can be used to solve the numerical problems in designing plate in which the buckling load becomes critical. Thank you.