Plates and Shells
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Lecture - 17
Plate Subjected to Inplane Forces and Transverse Load

Hello everybody. Today, I am starting module 6. And in this module, we will study about the buckling of plates.

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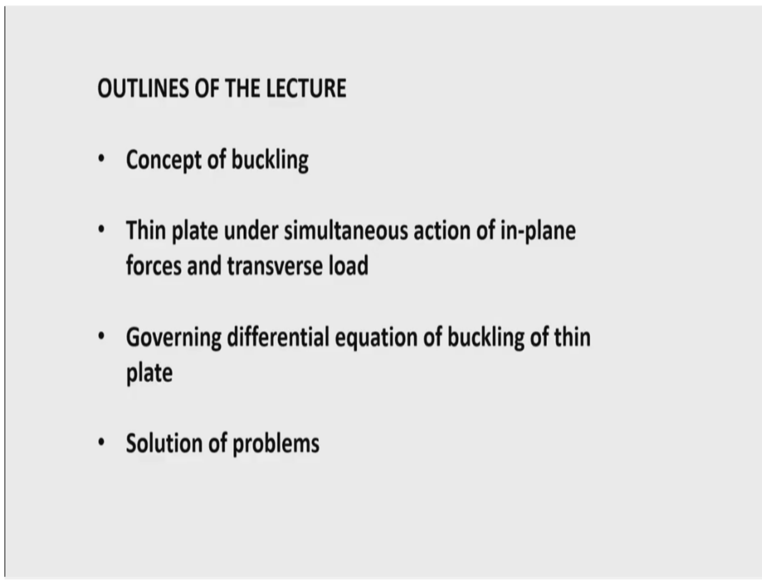

That is very important because many cases plates are compressed. And due to compression, the member is liable to be buckled. So, today my topic will be about the buckling of the plate. And before that, we must note that the thin plate equations that we have so far used and accustomed to; we neglected the in plane forces. But when the buckling occurs there must be a force in the axial directions that means in-plane forces must be present.

So, here first the emphasis will be given to derive the equations of equilibrium of the plate, including the in-plane forces and then we will go for buckling. So, in this lecture we will discuss the concept of buckling. Then we will derive an equilibrium equation of the thin plate under the simultaneous action of in-plane forces and transverse load. The in-plane forces in the derivation I will take as a tensile force which is considered positive.

But in case of buckling the tensile force have to be reversed that is, it will be compression. So, just by reversing the sign, the algebraic sign of the inplane forces, we can get the equation of buckling of the thin plate. So, then solution of some problem I will discuss. The problem I will consider today is a simply supported plate that means plate, with all edges are simply supported and subjected to UDL, Uniformly Distributed Load.

That is the UDL and also the plate which is subjected to a concentrated load. So, two problems I will discuss in today's class.

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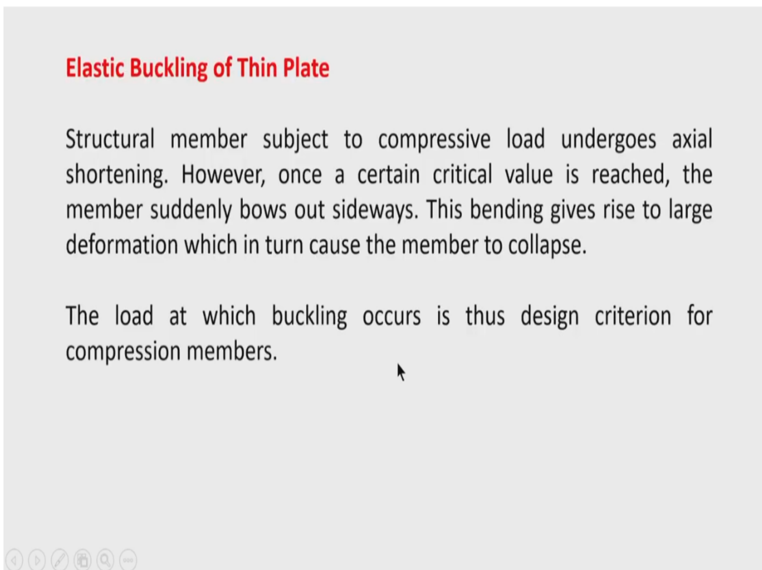

Now, let us discuss in brief what is buckling? Now, any structural member when subjected to compressive load is liable to be buckled when certain critical value is reached. So, when the critical value is reached and just it exits, the member suddenly bows out sideways. These bending give rise to large deflection, which in turn caused the member to collapse. So, buckling we can understand that is a condition when the instability arises.

So, that is a bifurcation problem. So, load at which buckling occurs is thus design criteria for many compression members.

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The following paragraph is quoted from a book "*Structure in Architecture*" authored by *Salvadori and Heller* (1963)

"A slender column shortens when compressed by a weight applied to its top, and in so doing lowers the weight's position. The tendency of all weights to lower their positions is a basic law of nature. It is another basic law of nature, that whenever, there is a choice between different paths, a physical phenomenon will follow the easiest path. Confronted with the choice of bending out or shortening, the column finds it's easier to shorten for relatively small loads and to bend out for relatively large loads. In other words, when the load reaches its buckling value the column finds it easier to lower the load by bending than by shortening."

The plate is also is not free from buckling when subject to in-plane forces

Now, the buckling phenomenon is understood by a paragraph that is written in a book by Salvadori and Heller, the name of the book is Structure in Architecture. So, he has explained very nicely what is buckling. So, buckling is seen in column as well as in the plate also. So, the paragraph from his book is quoted and I am reading the paragraph, then you will understand the inner meaning of these statements.

A slender column shortens when compressed by weight applied to its top and in so doing lowers the weight's position. The tendency of all weights to lower their position is a basic law of nature. It is another basic law of nature that whenever there is a choice between different paths, a physical phenomenon will follow the easiest path. Confronted with the choice of bending, bending out or shortening, the column finds it is easier to shorten for relatively small loads and to bend out for relatively large load.

In other words, when load reaches its buckling value, the column finds it easier to lower load by bending then by shortening. So, this is a nice quote from this book. That means, it is a basic law of nature, when there is a load put on the top of a column that, it resists the load by lowering its position that means it is compressed. But if you go on increasing the load at certain point, the member will no longer resist the load by just by compression.

So, it now requires the load to be resisted by slightly by bending in a slight manner. That is bending will generate the large deflection in that case and the member will choose instead of the shortening the bending is easiest for the member. So, therefore, the buckling phenomenon is understood as the transition from straight configuration to the bent configuration, this transition point we have to determine and this is nothing but the critical load.

So, plate is also not exception. So, there are also the chances of buckling in plate when subjected to in-plane forces.

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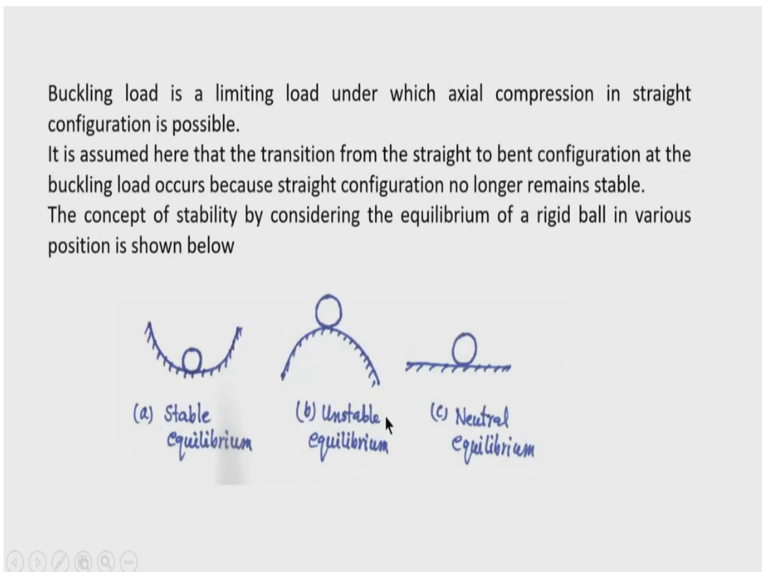

Now, there are certain difference between the plate buckling and column buckling. However, let us examine the three equilibrium positions that are important to study the buckling phenomenon. Buckling load is the limiting load under which the compression in state configuration is possible. So, it is a limiting load up to which the members remain straight, and just in case of load above that or beyond that will lead to an unstable configuration.

So, the column will now then bend and these bending will produce the large deflection. So, it is assumed that transition from the straight to bent configuration at the buckling load occurs because the state configuration no longer remains stable under the action of the load which exceeds the limiting load. So, the concept of stability can be explained by considering the equilibrium of a rigid ball in various positions.

Consider a hemispherical ball where a rigid ball is rested at and the equilibrium position you can see the equilibrium position is here. Now, if you slightly disturb the ball, then it will oscillate and then come to rest again at the equilibrium position. Consider this position here the hemispherical ball is shown in such a way that convex side is upward. So, if the ball is at the equilibrium position and slight disturbance of its position lead to the unstability.

So, this configuration is unstable equilibrium position then there is another position in neutral equilibrium position, say consider a plane surface, where the ball is placed and if it is displaced to a certain position, it will remain at the new position. So, this is what is neutral equilibrium position. Now, the buckling is the change of configuration that is from stable equilibrium position to the unstable equilibrium position, but in between a neutral equilibrium position has to be there.

So, stable equilibrium to neutral and then unstable. So, in between stable and unstable a neutral equilibrium is there and we have to find this neutral equilibrium position that gives you the critical load of the compression member. And compression member is your maybe your plate or maybe your this column also. Because you can see in steel structures whether it is a column suppose a column may consist of a box section where the sides of the boxes are plates. So, in that case the plates will buckle according to the manner in which plate buckles takes place. So, suppose for example, the plate girder which has the flange and vertical web. So, in that case the vertical web will buckle like a compression member. So, any steel structures if you see these are composed of plates. So, therefore, the plate buckling is important for the design of steel structures.

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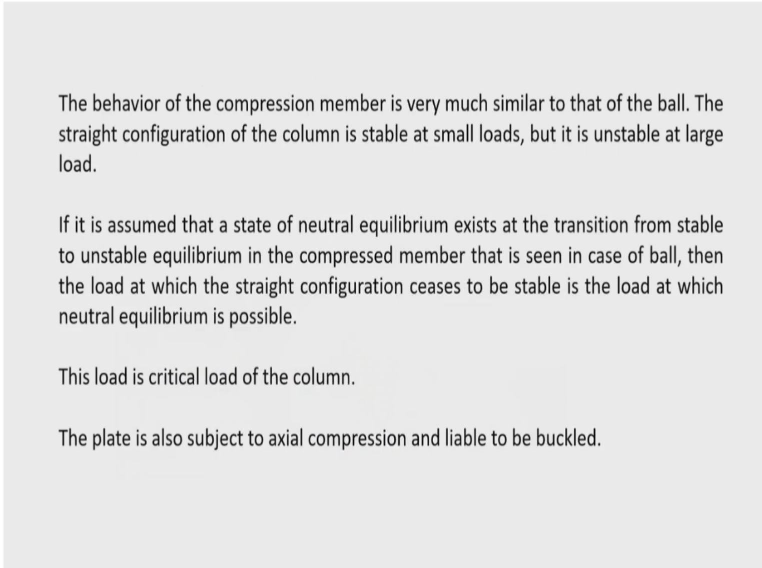

Now, we shall try to find out the buckling load of the plate. But before that I want to find out the differential equations of equilibrium considering the inplane forces. Because buckling is only possible when there is an application of axial forces and these axial forces must be compressive. So, we shall today derive a general expression for the differential equation of plate considering the in-plane forces in addition to the transverse load.

After that, I will show some applications and then buckling can be considered or analysed with the help of this equation, that we will derive today just by changing this sign of the inplane load. So, I am taking the lateral load to be zero, because buckling is always a homogeneous boundary value problem. So, there will be no external load in case of buckling only the axial forces that is applied in the buckling analysis and we have to find the value of this axial force as the critical value.

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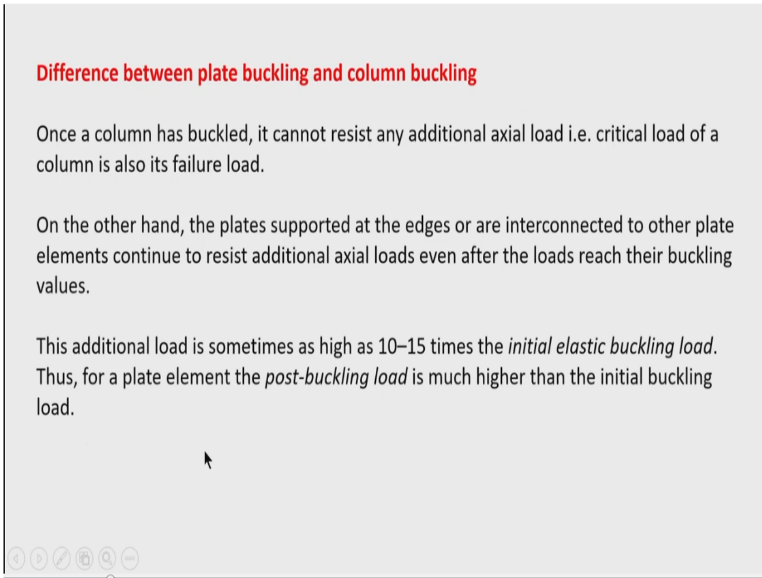

Now, let us see the difference between plate buckling and column buckling. Now, once column has buckled, there is some difference that can be noted very easily or it can be observed after detailed analysis. So, some of the statements that I have written here are from general observations of the buckling equations of the plate and column and also there is a note from the research paper. So, research papers, this observation that is the strength of the plate in the post buckling cases that I have brought here just to illustrate.

This post buckling behaviour is not included in this syllabus. So, once a column has buckled, it cannot resist additional load. So, that means, this axial load at which the column buckles can be taken as a critical load of the column or failure load, but this is not the case in case of plate. So, plates supported at the edges or are interconnected to other plate continue to resist additional axial load even after the load reach their buckling values.

So, this phenomenon is observed in case of plate. For example, a plate has dimensions which are important in both directions x and y direction. So, when the buckling takes place in case of line elements a column that we have studied, we have seen that the bending that bent shape that is formed is in the form of a single halfway. That gives the lowest value of the buckling load that is the critical load.

So, number of half waves can increase when the buckling load also increases, but for the design purpose we take only the lowest load. But in case of plate, it is seen that the buckling load is determined not only by considering the same half waves in both the directions. So, in that case in this case of plate buckling, the number of half waves that are necessary for buckling to take place varies in both direction.

So, it may happen that for support some support conditions the number of half waves in x direction is m, but number of half waves in y direction may not be m. So, this may be different. So, that we have to find out by detailed analysis. The additional load that plate resist even after the buckling takes place, may be 10 to 15 times higher than the initial elastic buckling load. So, for a plate element post buckling load is much higher than the initial buckling load.

Now, the concept of plate buckling the critical stress that we find for the plate has been adopted in the design code. I am mentioning the Indian Standard Code of steel design steel structure that is IS 800 where the critical states of the plate that we will find by solving the differential equation has been adopted in the code for checking the buckling resistance of the plate specially for the plate girder web.

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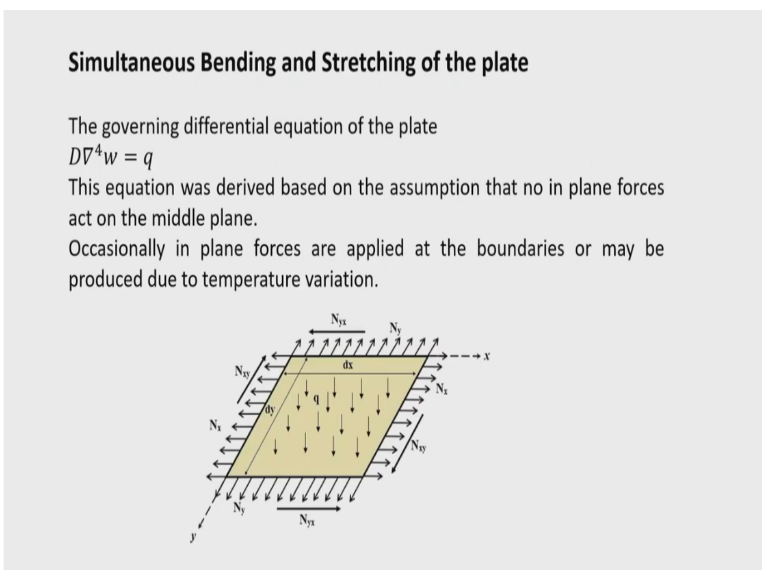

Now, let us discuss the differential equation. That is the most important requirement for solving the buckling problem in our case, because the solution of buckling or bending or even vibration, whatever you consider can be solved either in a closed form deriving the closed form expression or can be solved by taking approximate method. But, first let us learn the closed form solution which will establish the benchmark results.

That in some cases we can verify. Even the complicated cases when we model the plate within discrete method like finite difference or finite element. Then to check the model results of the discrete methods, we can verify the results of this or we can compare the results of the closed form expression. So, that is why closed form solutions are necessary in case of mechanics problem. So, governing differential equation of the plate ignoring the inplane forces can be written say $D\nabla^4 w = q$.

D is the flexural rigidity of the plate which is given by $Eh^3/12(1-v^2)$, where E is the Young's modulus of elasticity, h is the thickness of the plate and v is the Poisson ratio. So, D is actually depends on the material properties and the plate thickness. Boundary condition is not influencing the value of D. ∇^4 is the operator, which consists of three terms $\frac{\partial^4}{\partial x^4} + \frac{2\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$.

So, this operator when we operate with w, you will get the full differential equation of the plate and q is the distributed transverse load on the plate that is shown here. Now, in this figure you are seeing that the plate is subjected to in-plane forces. So, what are the in-plane forces? One is the force N_x that is the axial force along the x direction that is acting on a side which is parallel to y axis, that is N_x the axial force and we take here the N_x is uniformly varying or uniform along this direction.

So, at $x = 0$ and $x = a$ edge, the inplane forces are denoted by N_x and $y = 0$ and $y = b$ edge, the in-plane forces are denoted by N_y. And this is also constant that means, uniform along the edges. So, here along the x axis the length of the plate is a, along the y axis the length of the plate is b, transverse force is q and we have shown here an element of the plate not the full plate, so, dx and dy is the element of the plate.

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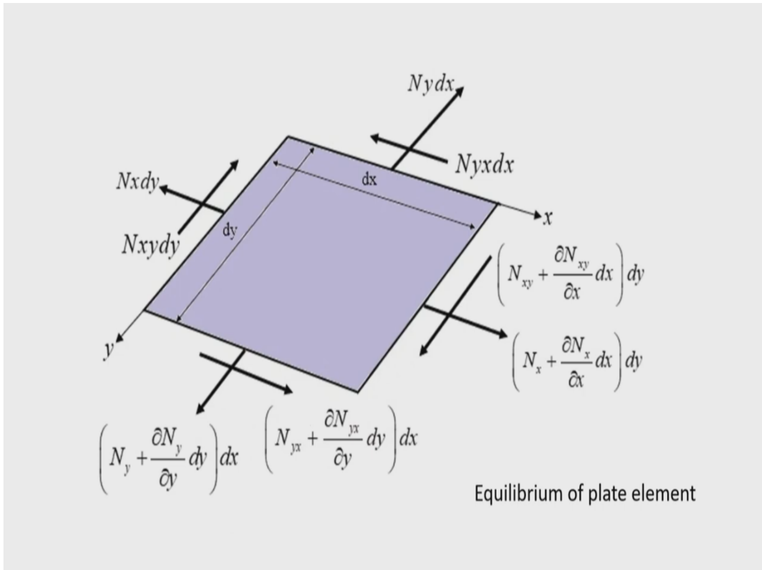

But, when you consider the equilibrium of the forces that is acting on the plate specially, I am showing here in the free body diagram, the in-plane forces acting on the element dx and dy, what you got it is the inplane forces on this face this $N_x \times dy$, dy is the length of this face or edge. So, total forces $N_x \times dy$. On the opposite face, this force N_x plus the increment. Though increment will be $\frac{\partial N_x}{\partial x} dx$ and the length of the side is also dy. So, it is multiplied by dy. So, in the other edges, you can see $N_y \times dx$ is the force, total force acting along the edge $y = 0$. Similarly, on the edge $y = b$ you are seeing the inplane force that is $\left(N_y + \frac{\partial N_y}{\partial y} dy\right) dx$. This is the in-plane force acting along the $y=b$. Now there are in-plane shear forces, that gives rise to compressive force in along the one diagonal and the other diagonal there is a tensile force.

So, you can see here if you take the resultant of these two inplane shear forces that is acting along the edges tangential to the edges. Then you will get the resultant is the tensile force along the diagonal. Similarly, in this case here you will get the resulting force is tensile. So, along one diagonal there is tensile force and other diagonal there will be compressive force. So, the plate is under the action of inplane direct forces and inplane shear forces.

So, inplane forces are actually N_x, N_y and N_{xy} and by virtue of the complementary nature of the shear force, $N_{xy} = N_{yx}$, so, that will take in in case of derivation. Now, here in this figure I have shown only the in-plane forces no other forces have shown because I want to find out the interrelation between this N_x and N_{xy}. So, if I consider the equilibrium of the forces along the x direction, I can establish an equation of equilibrium.

Then, if I write the equation of equilibrium that means, resolving the forces along the y direction, then also I will get other equations of equilibrium.

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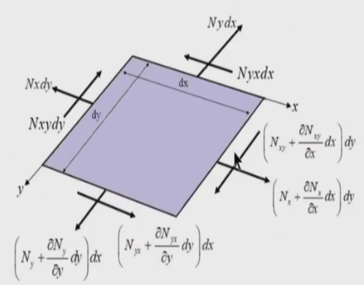
Equilibrium of the plate element under the action of in plane forces



Let us consider the the equilibrium of the plate element dxdy as shown in Figure. Resolving the forces in X direction

$$\left(N_x + \frac{\partial N_x}{\partial x} dx\right) dy - N_x dy + \left(N_{yx} + \frac{\partial N_{yx}}{\partial y} dy\right) dx - N_{yx} dx = 0$$

So, by doing so, here say along the x direction. What are the forces? One forces is this along the positive direction of the x axis this force is $\left(N_x + \frac{\partial N_x}{\partial x} dx\right) dy$. So, that force is written here. Then minus the acting in the opposite direction so, $-N_x dy$. So, these forces written here. Then plus other force let us see in the x direction. Positive reaction is this force N_{xy} or $\left(N_{yx}+\frac{\partial N_{yx}}{\partial y}dy\right)dx$.

On the opposite direction and $N_{yx}dx$ equals this is the force. So, total force I have summed up and equated to 0. So, now, after cancelling some terms that we will get say $N_x dy$. This is $N_x dy$ from here it is positive and here you are getting $-N_x dy$ is negative quantity. So, this term will get cancelled. Similarly, other terms you will get an $N_{yx}dx$, and here you will get $-N_{yx}dx$. So, this term will also get cancelled.

So, then after dividing both sides by the area of the elements that is *dx dy* that is not to be treated as 0.

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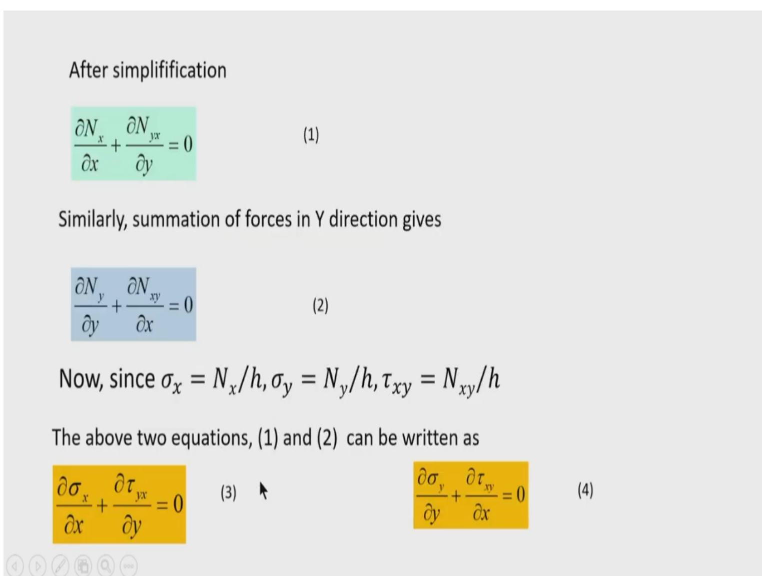

So, we ultimately get an equation of equilibrium involving the membrane forces that is $\frac{\partial N_x}{\partial x}+\frac{\partial N_{yx}}{\partial y}=0$. Similarly, summation of forces in y direction gives $\frac{\partial N_y}{\partial y}+\frac{\partial N_{xy}}{\partial x}=0$. So, these two forces are the equations of equilibrium in the x and y direction and this is possible due to membrane action of the plates. So, plate will resist these lateral, the axial forces and this will establish the equilibrium of the forces.

Now, this form can be converted into this equilibrium equation of the two dimensional elasticity problem. That involving the σ_x, σ_y and τ_{xy}, that we are familiar this 3 stresses are important in case of plate. Now, if we take a unit width of the plate, and N_x is uniform then we can write $\sigma_x = \frac{N_x}{h}$. Similarly, we can write $\sigma_y = \frac{N_y}{h}$ and $\tau_{xy} = \frac{N_{xy}}{h}$.

So, substituting this $N_x = \sigma_x \times h$, $N_{yx} = \tau_{xy} \times h$, we convert these two equations say (1) and (2) in the form of say $\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}=0$ and then another equation is $\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}=0$. So, these two equations are equation of equilibrium in plane stress condition in absence of body forces. Generally, body forces in axial directions remain absent and body forces in the axial directions generally, it is produced due to thermal loading. So, here we neglect this and therefore, the equilibrium equations in x and y direction we have written and it is forming the conventional two dimensional equilibrium equations for plate. Now, we want to bring the stress function concept in this expression.

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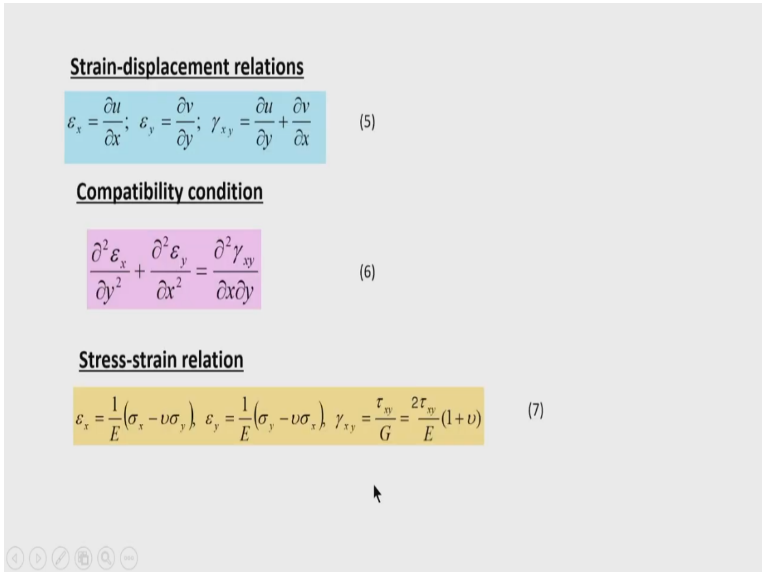

Now, let us discuss, let us introduce the stress strain relationship. Now, strain in the *x* direction is $\frac{\partial u}{\partial x}$, strain in the y direction is $\frac{\partial v}{\partial y}$ and shear strain is $\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$. Then compatibility condition for strain can be written as $\frac{\partial^2 \varepsilon_x}{\partial y^2}+\frac{\partial^2 \varepsilon_y}{\partial x^2}=\frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$. So, this is the compatibility condition in two dimensional elasticity problems. And stress strain relationship is also known to us. Stress strain relationship is $\varepsilon_x = \frac{1}{E}\left(\sigma_x - \upsilon\sigma_y\right)$ where υ is the Poisson ratio and then $\varepsilon_y = \frac{1}{E}\left(\sigma_y - \upsilon\sigma_x\right)$, υ is the Poisson ratio. Then γ_{xy} this is the shear strain is $\frac{\tau_{xy}}{G}$, *G* is the shear modulus of the elasticity.

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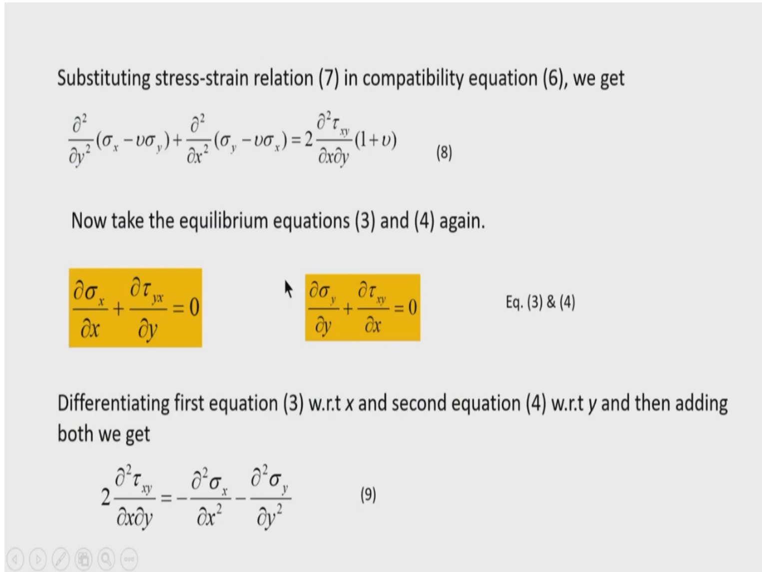

Now, substituting stress strain relationship in compatibility equation. So, compatibility equation is this equation. So, stress strain relationship we substitute here ε_x, ε_y and γ_{xy} we substitute here. Then we get this equation $\frac{\partial^2}{\partial y^2}(\sigma_x - \upsilon\sigma_y) + \frac{\partial^2}{\partial x^2}(\sigma_y - \upsilon\sigma_x) = 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y}(1+\upsilon)$. 1 + υ is coming here because of the expressing the shear modulus in terms of the Young's modulus of elasticity and Young's modulus of elasticity appears in both the sides of this equation 8. So, therefore, it gets cancelled. So, only we are living with this 1 + υ. Now, if I write again the equilibrium equation 3 and 4 that we have derived earlier. Here you can see this equilibrium equation in two dimensional elasticity.

That is in-plane stress problem is again written here $\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$. This is one equation, another equation is $\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0$. So, if I differentiate this first equation the first equation 3, this is the equation number 3. In second equation that is this equation. So, if I differentiate first equation with respect to *x* and second equation with respect to *y*, and then if I add both the quantities and separate these τ_{xy} terms in one side, then after adding when you differentiate this expression, that will be $\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \tau_{yx}}{\partial y \partial x} = 0$. If you differentiate this expression with respect to y, then you will get $\frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \tau_{yx}}{\partial x \partial y} = 0$.

So, then if I add this and keeping this $\partial^2 \tau_{yx}$ or $\partial^2 \tau_{xy}$ this is the same thing. So, $\frac{\partial^2 \tau_{yx}}{\partial x \partial y}$. So, one mixed derivative term will be coming. So, if I isolate this that is with after addition it will be $2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2}$. So, this is the relation that has been obtained. How it is obtained? This expression is differentiated with respect to x, this expression is differentiated with respect to y and then added. So, we ultimately arrive at this equation.

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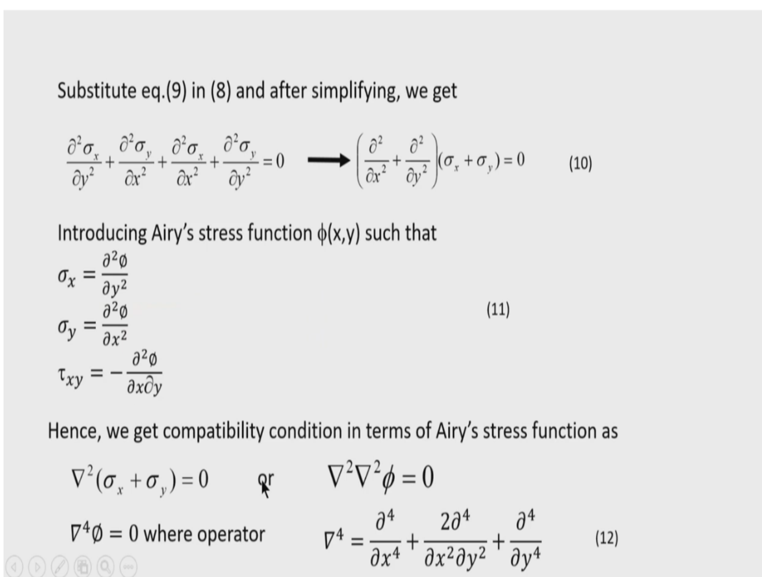

Now, this equation that is I numbered it 9. This equation is now substituted in equation 8. So, substitute equation 9 in equation 8. So, suppose the earlier slide say this equation 9 now, you substitute here directly you substitute here this equation. Then you will get equation like that $\frac{\partial^2\sigma_x}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial x^2}+\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}=0$.

You can note here this is the derivative of σ_x with respect to *y* and derivative of σ_x that is the second derivative with respect to *x*, both are appearing. Similarly, the derivative of σ_y with respect to *x* and second derivative of σ_y with respect to *y* appearing again. So, both are appearing derivative of this σ_x is here with respect to *y* as well as with respect to *x*. Similarly, derivative of σ_y is with respect to *x* as well as with respect to *y*.

So, these can be written in a convenient form say using this operator $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y)$, this can be written in this form. So, now we are familiar with this operator. This operator is nothing but your Laplacian operator that is ∇^2. So, we can now write this equilibrium equation compatibility equation as $\nabla^2(\sigma_x+\sigma_y)$, summation of $\sigma_x+\sigma_y$.

Now, this is a very popular equation in plane stress problem and then we want to express the stresses, the normal stresses and shear stresses in the form of Ariry's stress function. Because when we solve a equation of the plate simultaneously subjected to in plane forces and this transverse loading. Then, if we introduce this stress function then it will be convenient because, earlier we have got the equilibrium equation where it is $\frac{\partial N_x}{\partial x}+\frac{\partial N_{yx}}{\partial y}=0$ one equation. And another equation $\frac{\partial N_y}{\partial y}+\frac{\partial N_{xy}}{\partial x}=0$. So, these equations we obtained earlier and we have to find out simultaneously the inplane forces as well as the vertical deflection of the plate also. So, because of these bringing the Airy's stress function phi in the equation will be convenient. So, the Airy's stress function φ is introduced in such a way that second derivative of φ with respect to y will give you the σ_x. And second derivative of φ with respect to x will give you σ_y. You can note here the difference that when we required the σ_x we have to differentiate stress function φ with respect to y second derivative. Then when we require to find σ_y, we have to differentiate this Airy's stress function two times with respect to x. Then τ_{xy} will be given by this mixed derivative and with a minus sign.

Now, introducing this stress function in the compatibility equation the Airy's stress function now can be written as this is $\nabla^2(\sigma_x+\sigma_y)$. Now, introducing this if you add this $\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial x^2}$. Then again, we can write this is $\nabla^2\phi$. So, this equation can be written as $\nabla^2\nabla^2\phi=0$. So, this is again an equation of biharmonic nature homogeneous equation very popular equation in plane stress elasticity problem.

And where operator ∇^4 is expressed as $\frac{\partial^4}{\partial x^4}+\frac{2\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}$. So, this operator is similar to our operator that appears in the plate deflection equation in absence of this in plain force says that we derived earlier.

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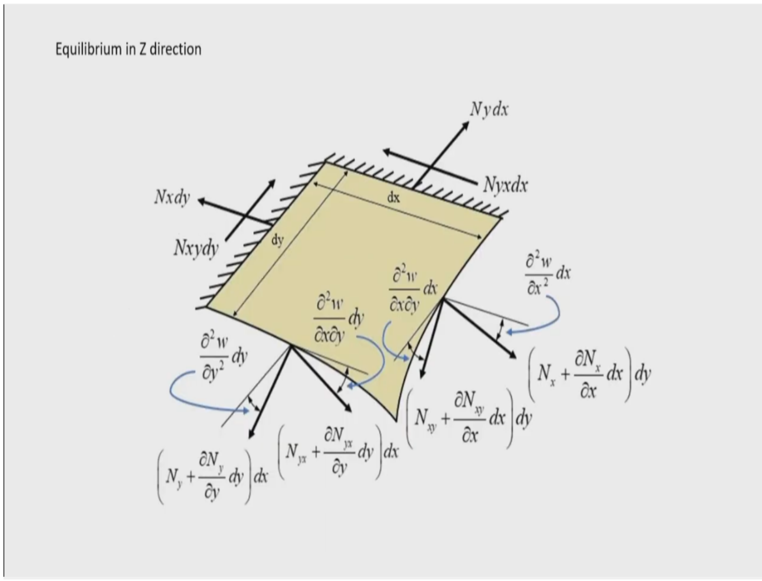

Now, the equation that we have obtained is considering the equilibrium of forces in the axial direction that is the x direction as well as y direction. Now, let us consider the equilibrium of forces in the vertical direction, vertical direction is the z direction. Again, we take an element of the plate of length dx along the x direction and length dy along the y direction. This direction is understood as the direction along the x axis. And this direction is understood as the direction along the y axis. So, here for simplicity we have taken these edges, two adjacent edges are fixed. It is not necessary for derivation purpose that one should take these edges are fixed. We are taking these edges fixed because we are taking the slope, the change of slope at opposite edges that is measured from a datum which has 0 slope. But it does not matter if the slope is other than 0 here also.

That means, if this is not fixed also then also the change of slope can be calculated at the opposite edges. But for simplicity of calculation, we have taken these two adjacent edges as the fixed. Now, let us see the forces that are acting along the edges. On this edge it is $N_x\, dy$ on the adjacent edges that is fixed that is $N_y\, dx$ membrane shear force is here $N_{yx}\, dx$. In this edge membrane shear force is $N_{xy}\, dy$.

Now, let us see what is the condition at the opposite edges? In the opposite edges if you see here first let us see along the x direction. You see due to deflection of the plate the slope will be changing and as well as curvature will be also generated. So, here if this is the change of forces that is the with the increment of N_x that I have written $\left(N_x + \frac{\partial N_x}{\partial x} dx\right) dy$. So, that is the incremented N_x in the opposite faces. And due to deflection of the plate, this force is now making an angle with the normal to the edge. Now, this angle is nothing but the change of slope. So, change of slope if I take the slope here is $\partial y/\partial x$ and the rate of change of slope will be derivative $\partial/\partial x(\partial w/\partial x)$ and this change occurs in length of dx. So, it is multiplied by dx. So, this is the angle. So, for small deflection this change of slope.

Because change of slope is net slope we are because here we have clamped this edge. So, therefore, we take this as this slope that is the angle that the normal to the deflected surface makes with the original x axis. That was before the deformation. So, therefore, if we take this as the angle then if I want to calculate the vertical component of the forces, then this force say this force is say P, P is here $\left(N_x + \frac{\partial N_x}{\partial x} dx\right) dy$ and this angle is say θ.

So, naturally we calculate the vertical component of these forces, because we are considering the equilibrium in the vertical direction that is z direction. So, vertical component of this force will be say $P \times \sin\theta$, P is this this term complete term and theta is this $\frac{\partial^2 w}{\partial x^2} dx$. Although I have not written here, but say for convenience, it takes this angle as θ this is understood as the angle and for small deflection $\sin\theta = \theta$.

So, generally you will get the vertical component of this force as $\left(N_x + \frac{\partial N_x}{\partial x} dx\right) dy \frac{\partial^2 w}{\partial x^2} dx$. Similarly, on the other edge because here the edges are clamped. So, there will be no slope here. So, naturally the vertical components of these two forces are 0. Similarly, vertical component of the shear force along these clamped edges will be 0. Now, consider the axial forces that is N_y on the other direction.

That is the along the edge opposite edge that we have considered first here $N_y\,dx$ along the edge dx. Now, here we are considering this incremented force $\left(N_y + \frac{\partial N_y}{\partial y}dy\right)dx$. So, in a similar fashion now, we can calculate the change of angle that is the angle here that is the slope because here we clamped it. So, $\frac{\partial^2 w}{\partial y^2}dy$. So, the intention of clamping the edges here only to get the change of slope as something minus 0, therefore, convenience of calculation no other purpose. So, $\frac{\partial^2 w}{\partial y^2}dy$ is the slope here is the angle. And vertical component of this equation incremented force $\left(N_y + \frac{\partial N_y}{\partial y}dy\right)dx$ will be this force multiplied by $\sin\theta$. θ is this angle so, this angle is again for small theta this will represent as $\sin\theta = \theta$. So, for membrane shear forces also we are getting the vertical component.

So, this is the rate of change of angle of twist, that you are getting here $\frac{\partial^2 w}{\partial x \partial y}dx$ and because these edges are fixed. So, we are getting this angle is the slope angle. So, the vertical component of N_{xy} is also found out by resolving the forces in the vertical direction and this vertical component of this shear will be $\left(N_{xy} + \frac{\partial N_{xy}}{\partial x}dx\right)dy\frac{\partial^2 w}{\partial x \partial y}dx$.

Similarly, here vertical component of the shear force will be $\left(N_{yx}+\frac{\partial N_{yx}}{\partial y}dy\right)dx$. So, these vertical components are known. Now we can sum up the forces in the vertical direction.

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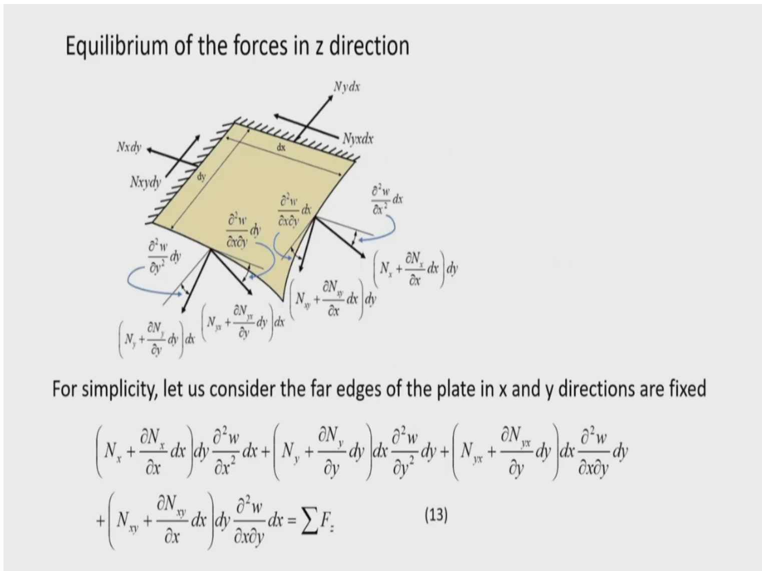

So, summing up the forces in the vertical direction we can write now, this is one vertical component of this that we have written here. Then vertical component of these forces we have written here then vertical component of membrane force we have written here. Vertical component of membrane shear force here we have written here and since these edges are fixed so there will be no vertical component. So, the summation is some force F_z.

So, after cancelling some common term, you will get some common term and neglecting the square of this small term because here you will get suppose if you multiply term by term then in the first expression you will get dx^2 term. Here $\frac{\partial N_x}{\partial x}dx^2dy\frac{\partial^2 w}{\partial x^2}$. So, when the square of the small quantity comes then when should neglect because we are considering this small deformation theory.

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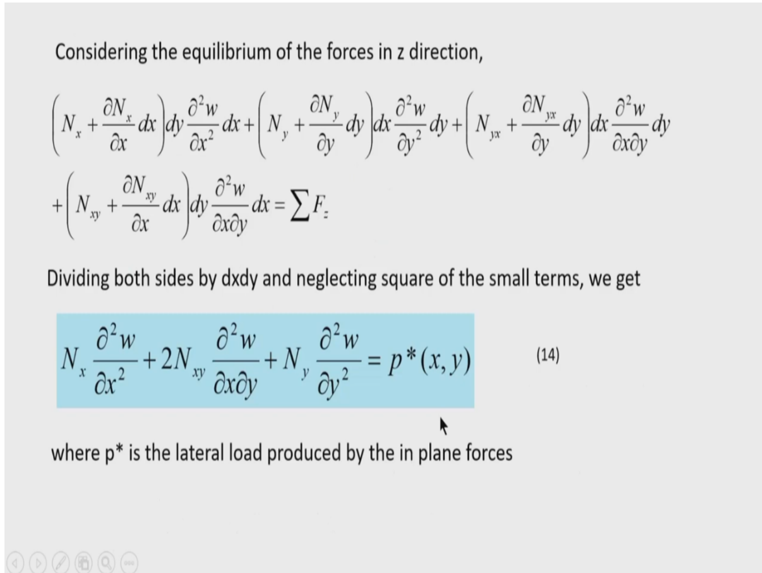

So, considering the equilibrium in the z direction and dividing both sides by *dx dy* and neglecting the square of the small terms. We get this equation in this form $N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} = p^*(x,y)$. p^* is a lateral load produced by the in-plane forces. So, that force we have taken as a fictitious load on the plate.

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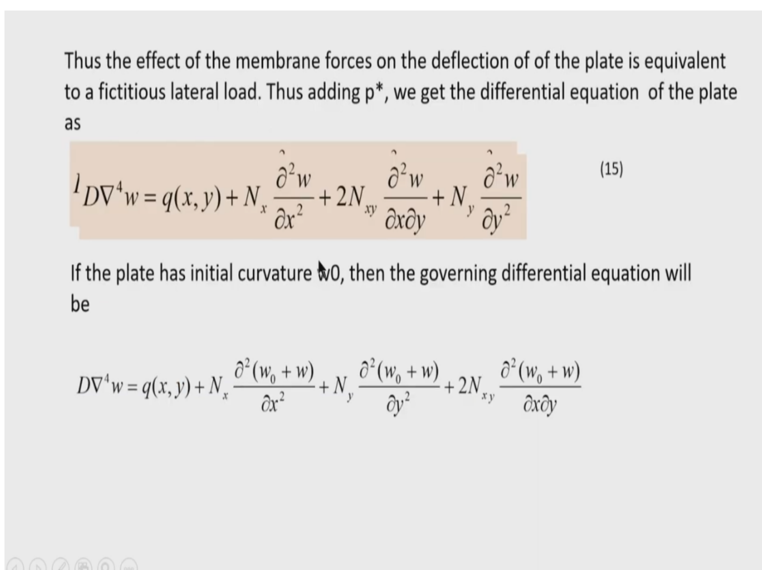

Thus, the effect of membrane forces on the deflection of the plate is equivalent to a fictitious lateral load. Thus, adding p^*, p^* is a fictitious adding load which is nothing but

$N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} = p^*(x, y)$. So, adding this p^* to the plate equation now, we get the plate equation in this form $D\nabla^4 w = q(x, y) + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}$.

If the plate has initial curvature sometimes plate may have even like column the initial imperfection is also common in many cases. So, initial curvature is there say in the form of w_0 it may be a continuous function or it may be some localized imperfection also. Then we can add this to this w and we get this equation say $D\nabla^4 w = q(x, y) + N_x \frac{\partial^2 (w_0 + w)}{\partial x^2} + N_y \frac{\partial^2 (w_0 + w)}{\partial y^2} + 2N_{xy} \frac{\partial^2 (w_0 + w)}{\partial x \partial y}$. w_0 is the initial imperfection or initial curvature or initial deflection of the plate.

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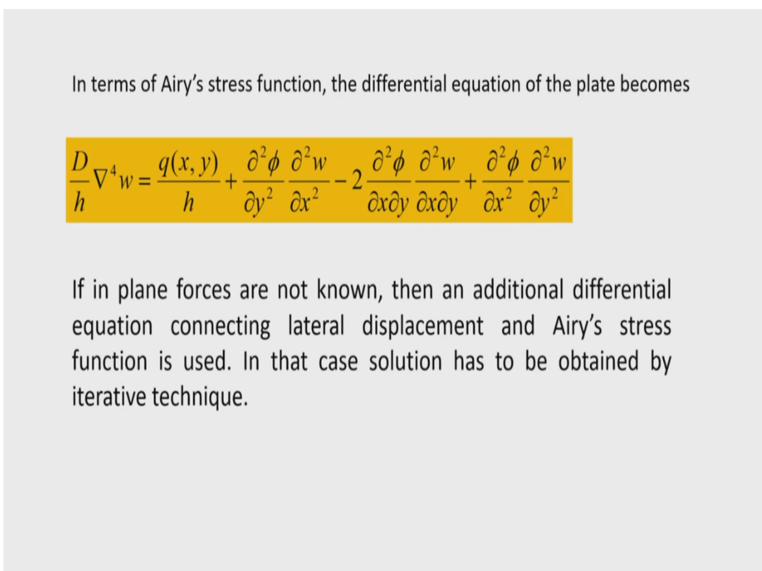

Now, introducing the Airy's stress function that we have earlier obtained. What was the Airy's stress function? ϕ is the Airy's stress function and ϕ is such that $\frac{\partial^2 \phi}{\partial y^2}$ is your σ_x. And similarly, $\frac{\partial^2 \phi}{\partial x^2}$ is σ_y and $-\frac{\partial^2 \phi}{\partial xy}$ is ${}_{xy}$. So, introducing this quantity now, we can write this equation.

The plate equation in presence of the transverse load, as well as the in-plane forces like that D/h, h is the thickness of the plate, ∇^4, ∇^4 is the operator biharmonic operator that consists of three terms. The terms are $\frac{\partial^4}{\partial x^4}+\frac{2\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}$ = q/h, q is the distributed load $+\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$.

This is coming due to conversion of N_x in terms of this stress function. Because N_x is related to σ_x and σ_x x is related by the second derivative of the Airy's stress function. So, if in-plane forces are not known sometimes in-plain forces are not known, then additional differential equation connecting lateral displacement in Airy's stress function is used. So, that is the additional deflection that we have used.

So, in that case, solution has to be obtained by iterative technique, no close form solution may not be possible in that case.

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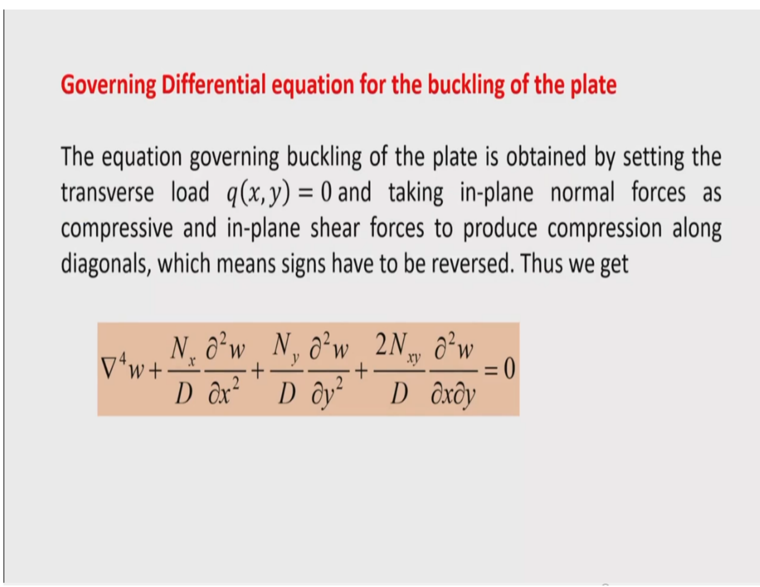

If we want to study the buckling phenomenon of the plate, that is our intention. So, first we have derived the differential equation of the plate subjected to in-plane forces. So, this is the differential equation of the plate subjected to the in-plane forces. Now, when we study the buckling, buckling is a homogeneous problem. So, where q should be absent. So, we shall put q = 0, and then the sign of N_x, N_{xy} and N_y will be reversed to produce the effect of compression on the plane.

So, if it is reversed the sign of this N_x, N_{xy} and N_y are reversed and q is taken to be 0. Then the equation for buckling of the plate is given by this $\nabla^4 w + \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2} + \frac{N_y}{D}\frac{\partial^2 w}{\partial y^2} + \frac{2N_{xy}}{D}\frac{\partial^2 w}{\partial x \partial y} = 0$.

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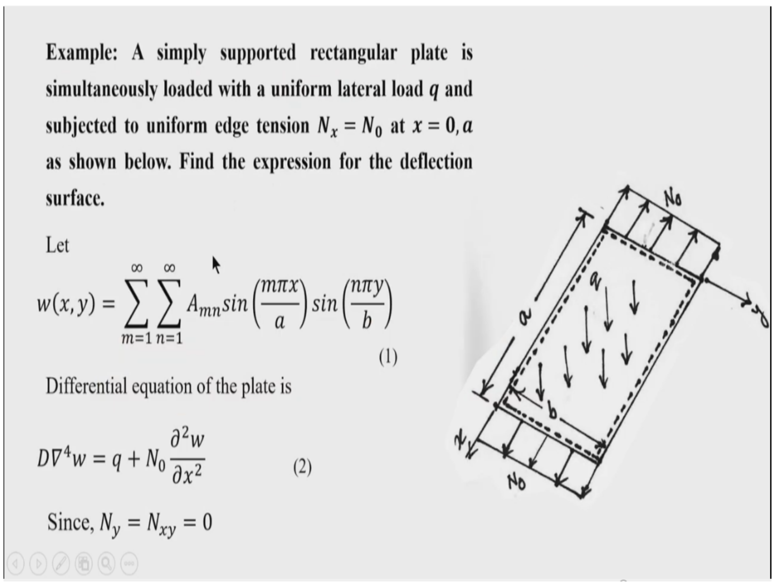

Now, let us consider a problem, illustrate a problem. We have taken a simply supported plate all edges are simply supported as seen in this figure. And it is subjected to transverse load q as well as this axial compression here N_0. So, N_0 is the compressive force, but for the time being we will take it as a tensile force because we have now just derive the equation of equilibrium with the tensile.

When we consider the buckling of the plate, then we will consider this N_0 as compression. But here we consider the edge force as the tensile force. So, if it is a tensile force, then the differential equation is this $D\nabla^4 w = q + N_0 \frac{\partial^2 w}{\partial x^2}$. Now, you can take here a Navier series because the plate is simply supported along all edges. So, a double trigonometrical series will satisfy the edge condition.

That is the deflection and bending moment at edges $x = 0$ and $x = a$ and $y = 0$ and $y = b$ will be satisfied by taking the double trigonometrical series $sin\left(\frac{m\pi x}{a}\right)\, sin\left(\frac{n\pi y}{b}\right)$ with coefficient A_{mn}. So, this double sum can be expanded and the series can be computed for different number of terms. Now, here we are taking this membrane shear force and the other in-plane forces in the y direction to be 0. And this force in the x direction is taken as constant which is N_0. So, differential equation now becomes this.

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Assume, Navier's series and substitute in the differential equation,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)^2+\frac{N_0 m^2\pi^2}{Da^2}\right\} sin\left(\frac{m\pi x}{a}\right) sin\left(\frac{n\pi y}{b}\right)(A_{mn}) = \frac{q(x,y)}{D} \qquad (3)$$

Multiply both sides by $sin\left(\frac{m'\pi x}{a}\right) sin\left(\frac{n'\pi y}{b}\right)$ and then integrate in the domain of the plate,

$$A_{mn}\left[\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)^2+\frac{N_0 m^2\pi^2}{Da^2}\right]\frac{ab}{4} = \frac{1}{D}\int_0^a\int_0^b q(x,y) sin\left(\frac{m\pi x}{a}\right) sin\left(\frac{n\pi y}{b}\right) dxdy \qquad (4)$$

So, assume in a Navier solution method, we substitute the series in the differential equation and then we get this. If I substitute the series in this differential equation, then in the left-hand side we will get this term and in the right-hand side, we will get this term. Now, multiply both sides by $sin\left(\frac{m'\pi x}{a}\right)$ and $sin\left(\frac{n'\pi y}{b}\right)$ and then integrate in the domain of the plate. So, integration is done in the domain of the plate and due to orthogonal condition, the $sin^2\left(\frac{m\pi x}{a}\right) dx$ integration will be $a/2$ and another integration $\int_0^b sin^2\left(\frac{n\pi y}{b}\right)$ will be $b/2$. So, this term is this.

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Hence,

$$A_{mn} = \frac{4q_{mn}}{D\left[\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)^2 + \frac{N_0 m^2\pi^2}{Da^2}\right]ab} \tag{5}$$

where

$$q_{mn} = \int_0^a \int_0^b q(x,y) sin\left(\frac{m\pi x}{a}\right) sin\left(\frac{n\pi y}{b}\right) dxdy \tag{6}$$

For uniform loading q, we get

$$q_{mn} = q\frac{4ab}{mn\pi^2} \quad \text{for m,n=1,3,5,...} \tag{7}$$

So, therefore, we arrive at the value of A_{mn} is this and hence q_{mn} can be written as this expression as $q(x,y)sin\left(\frac{m\pi x}{a}\right)sin\left(\frac{n\pi y}{b}\right)dxdy$. For a uniform loading q has been obtained in earlier cases also. So, we get q_{mn} as $q\frac{4ab}{mn\pi^2}$, where m and n are odd integers. So, if I take q as the uniform load, then q can be taken outside the integral, sin and these two terms can be integrated.

And after a substitution of limit, lower limit and upper limit you will find that the value of integral exists only for odd integers that is all value of m and n 1, 3, 5 up to ∞.

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$$A_{mn} = \frac{16q}{D\pi^6 mn\left\{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 + \frac{N_0 m^2}{Da^2\pi^2}\right\}} \tag{8}$$

Hence, deflection surface becomes,

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{16q sin\left(\frac{m\pi x}{a}\right) sin\left(\frac{n\pi y}{b}\right)}{D\pi^6 mn\left\{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 + \frac{N_0 m^2}{Da^2\pi^2}\right\}} \tag{9}$$

At the centre, $x = \frac{a}{2}$ and $y = \frac{b}{2}$

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{16q}{D\pi^6 mn\left\{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 + \frac{N_0 m^2}{Da^2\pi^2}\right\}} \tag{10}$$

So, this A_{mn} is found and after finding the A_{mn} we can now write down the deflected shape. So, deflected shape we have taken this and A_{mn} is known after finding this series solution and then the complete deflection surface is written like that. So, maximum deflection occurs at the centre and therefore, by substituting $x = a/2$ and $y = b/2$, and for odd value of integers we get this maximum deflection as this value.

You can see in presence of axial tension; the maximum deflection reduces. So, the maximum bending moment will also get reduced.

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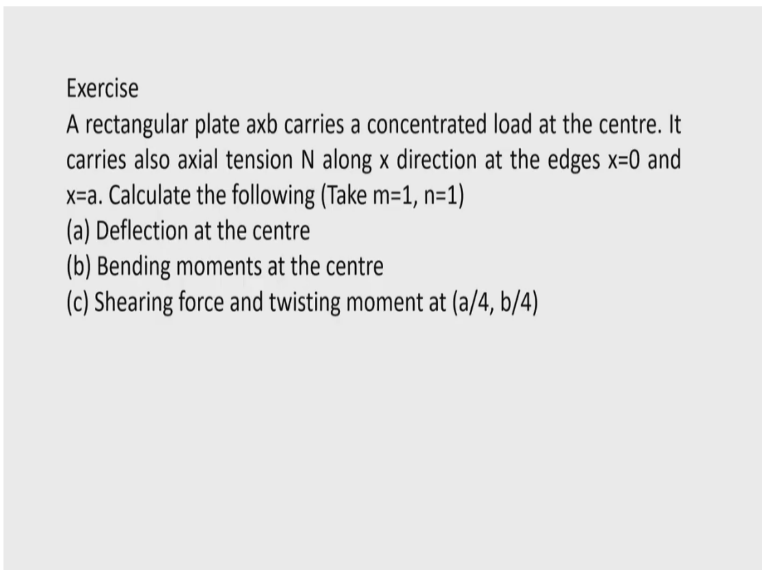

So, an exercise can be solved in a rectangular plate $a \times b$ carries a concentrated load at the centre. It carries also axial tension N along x direction at the edges x =0 and $x = a$. Calculate the following we take $m = 1$, $n = 1$, deflection at the centre, then bending moment at the centre and shearing force and twisting moment at $a/4 \times b/4$. So, let me give some steps of this exercise and you can complete this and get the results.

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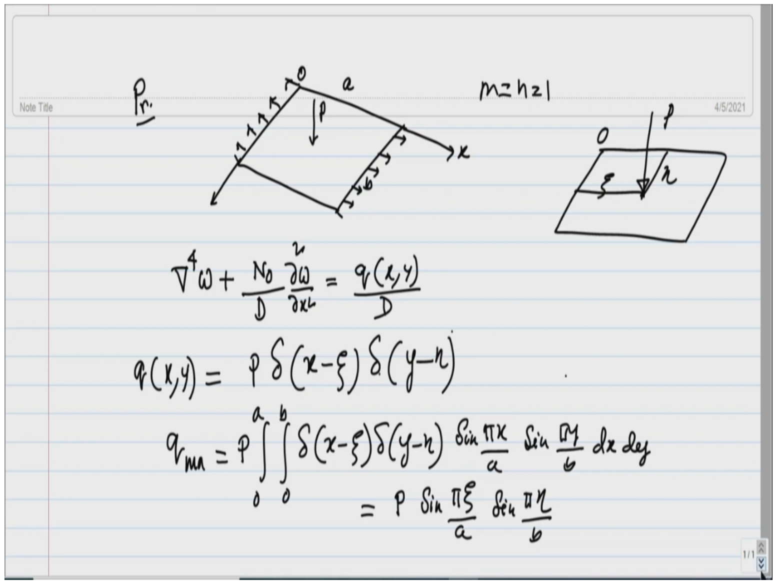

So, I am discussing now a problem where I have taken a rectangular plate side a and b, a is along the x axis and b is along y axis. It carries a concentrated load at the centre say centre coordinate is $a/2$ and $b/2$, I have taken the origin at o and also it is subjected to in-plane forces uniformly distributed along the edges. So, now the differential equation of the plate can be written as plus $\frac{N_0}{D}\frac{\partial^2 w}{\partial x^2}$.

Other terms will be absent because there is no other in plane forces = $q(x,y)/D$. Now, you can see here the load is this concentrated load not the distributed load, but we know that concentrated load can be expressed in the equivalent distributed load by making use of Dirac delta function. So, here I can write $q(x,y)$ because it is suppose the concentrated load is acting at any point along any plate whose coordinate is say ξ and η measured with respect to origin.

Then we can write, so, $q(x,y)$ we can express like that. So, therefore, the expression q_{mn} that is required to find out is nothing but the double integral 0 to a 0 to b and instead of $q(x,y)$ we now write this, P I can take outside and $\delta(x-\xi)\delta(y-\eta)$ and the function sin because we are taking first term, so, let $m = n = 1$, so, $sin\left(\frac{\pi x}{a}\right)sin\left(\frac{\pi y}{b}\right)dxdy$. So, this integral can be easily evaluated and it forms that $Psin\left(\frac{\pi\xi}{a}\right)sin\left(\frac{\pi\eta}{b}\right)$.

I have taken a general coordinate ξ and η but in the final result you can put $\xi = a/2$ and $\eta = b/2$. Now, when q_{mn} is found then we can find the A_{mn}. Now here of course, the A_{mn} will be your; this first term consisting of only the first term of the series. So, m=1 n = 1.

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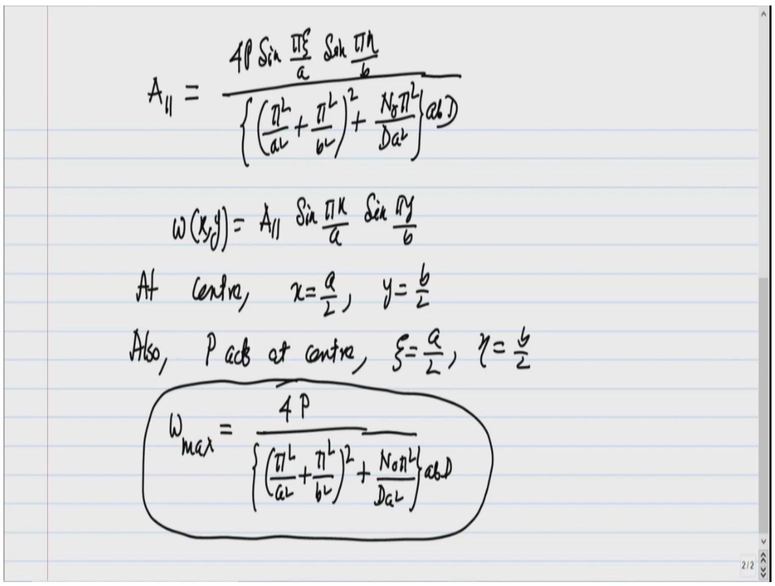

So, now if I write the expression of this A_{11}, A_{11} will be because m =1 and n = 1, A_{11} will be $4Psin\left(\frac{\pi\xi}{a}\right)sin\left(\frac{\pi\eta}{b}\right)$ divided by the terms that contain the plate parameters say $\left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}\right)^2$. Then it will contain the effect of axial forces $\frac{N_0\pi^2}{Da^2}abD$. So, this is your A_{11}. So, once you find the coefficient of deflection that only one term is needed, so, there is no difficulty to find the centre deflection. Now centre deflection term is written here $w(x,y)$ is a function of this $A_{11}\, sin\left(\frac{\pi x}{a}\right)sin\left(\frac{\pi y}{b}\right)$. So, once you get this then at centre $x = a/2$, $y = b/2$ and also P acts at centre. So, $\xi = a/2$ and $\eta = b/2$. So, maximum deflection that you get is at the centre and its value is $\frac{4P}{\left\{\left(\frac{\pi^2}{a^2}+\frac{\pi^2}{b^2}\right)^2+\frac{N_0\pi^2}{Da^2}\right\}abD}$.

So, this is the value of the maximum deflection that you require to find. Then you can see in presence of axial tension the maximum deflection decreases. Now, once you get this maximum deflection then the deflection expression then you can proceed to find the bending moment and shear force also. Now, let us find what will be the bending moment and shear force at some at any point x, y.

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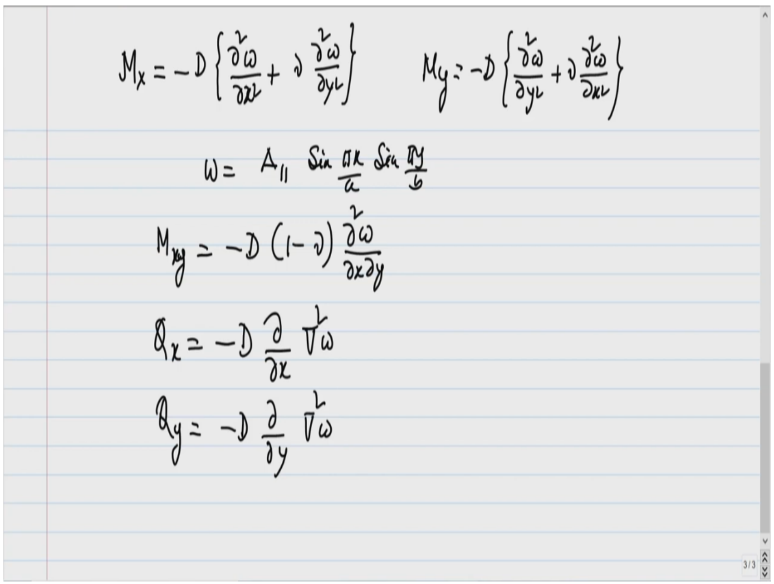

If you see the expression for bending moment the bending moment in x direction is nothing but $-D\left\{\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right\}$. And bending moment in y direction will be $-D\left\{\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right\}$. Now, since the expression for w is known, w is A_{11}, that we have found and $sin\left(\frac{\pi x}{a}\right)sin\left(\frac{\pi y}{b}\right)$. We can calculate this quantity without any difficulty.

Because only in that case you have to substitute the second derivative of this expression. So, now the twisting moment M_{xy} is calculated as $-D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$ and shear force Q_x at any point can be calculated $-D\frac{\partial}{\partial x}\nabla^2 w$ and Q_y is calculated $D\frac{\partial}{\partial y}\nabla^2 w$. So, once the w is found then other quantity can be easily calculated. So, as an exercise you can complete this.

So, this type of problem you may get in the assignment so, their I will get the complete solution. So, what we learnt today? Today we have learned the differential equations of bending of plate subjected to in plane forces which we neglect earlier. Now, here we consider the in plane forces and obtain the differential equations of the plate. And then we discuss the concept of buckling, because this equation will be used in case of buckling also, reversing the sign of these in-plane forces. Then we solved some problems to illustrate the use of this equation. Thank you very much.