Plates and Shells

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Module-05
Lecture-16
Finite Difference Method in Plate Bending

Hello everybody, again I welcome you in the MOOC massive open online course under NPTEL on plates and shell. And today I will deliver the lecture number 3 of the module 5. In module 5, I was discussing about the approximate method for the solution of plate problems. Actually, there are many boundary conditions or loading conditions for which the closed form solution are not readily available, and even if closed from solutions are attempted, it becomes very tedious.

So, therefore the people lose interest in solving the closed-form solution or finding the solution of the differential equation in exact manner. So, therefore there was a need for approximate method, and we have seen that based on the energy principle, we developed 2 very useful approximate method, which are known as Rayleigh-Ritz method and Gallerkin method.

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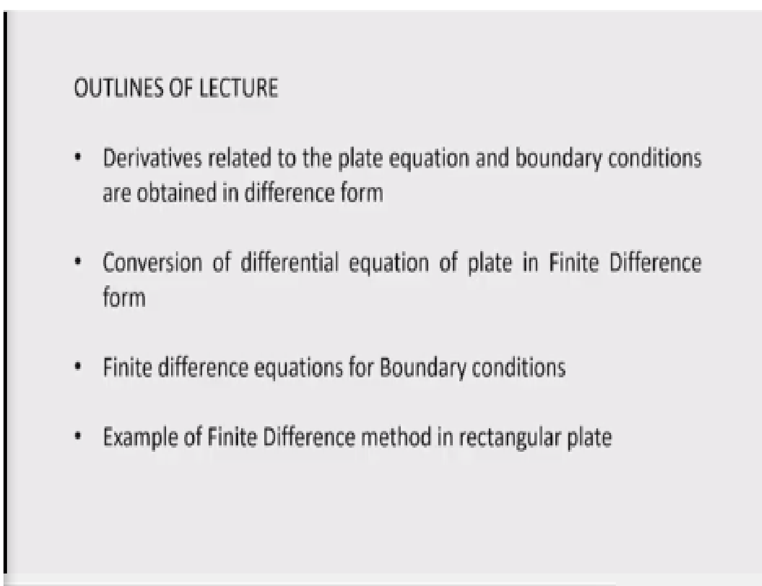

Today our discussion will be on the finite difference method in plate bending. Let us see what outlines of the lecture? First, I will find the derivatives related to the plate equation and boundary conditions in difference form. Then I will convert the differential equation of the plate in finite difference equations. Finite difference equations for boundary conditions also have to be developed. Then I will lastly illustrate an example of a rectangular plate whose deflections are found using the finite difference method.

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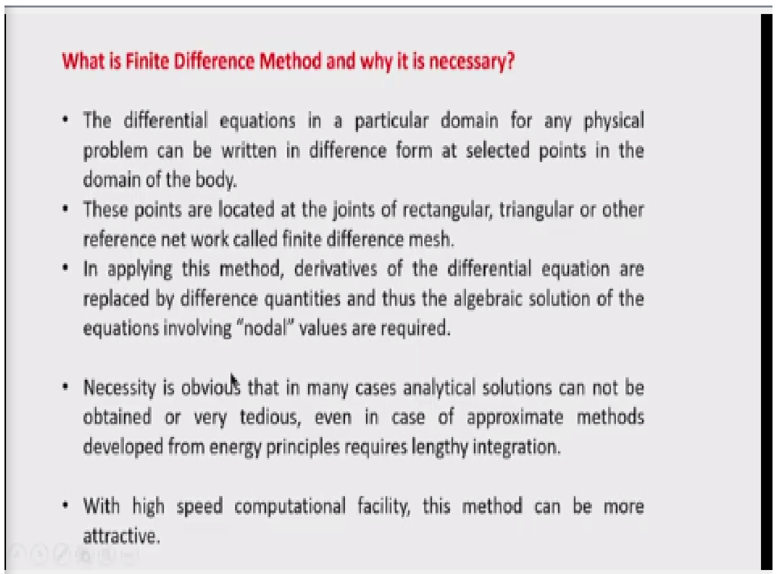

Now let us see what is the finite difference method and why it is necessary? You have seen the differential equations of equilibrium in case of plate is a 4th order partial differential equation. This differential equation may contain the constant coefficients, or it may contain variable coefficients if the thickness of the plate is not uniform. So, the complexity arises in case of such situation when we cannot find an analytical solution.

But in our earlier problems or whatever problems we discussed using the classical approach that is the Navier's method and Levy's method and also for circular plate, axisymmetrical formulation, where we assume that plate is of constant thickness. But sometimes, the plate may vary in thickness. For example, you have a cantilever slab which is of varying thickness, so in that case the modeling of plate with constant thickness will not be appropriate.

And the solution of such problem using the classical approach is difficult and is obtained in very few cases. So, there is a need for finding the alternative method, of course, the Rayleigh-Ritz and Gallerkin method are 2 alternative methods. But there also you have seen that lengthy integrations were required to find out or to arrive at the deflected surface. Hence, a numerical technique is necessary.

So, with the advance of computational facility in recent years, we can now use this type of method that finite difference method very easily to find the solution with reasonable accuracy. So, in this problem of plate the using the finite difference method, the differential equations are written in difference form at selected points in domain of the plate. So, what does it mean? Differential equation, for example, a simple example if I say that $dy/dx \ = q$, say any equation first-order equation.

But this is a differential equation I can solve it exactly, but if I want to express this in difference form, so I can write the dy as a difference of these y coordinate between 2 points 1 and say 2 and then divided by the distance between the points 1 and 2. So, it is a simple example that this simple differential equation can be converted into difference form that now it becomes a algebraic equation.

But the accuracy loses because the dx, that is, the distance that we are taking between 2 points if it is large, then the solution in this domain may not vary linearly. So, therefore the solution of differential equation by difference forms if the interval is very large and assuming the linear variation may not be appropriate. So, there is a source of error in case of difference form of equation, but this error can be controlled by suitable selection of these grid size or mesh size.

So, what is actually mesh? The whole domain is divided into rectangular or triangular or any other shape of this meshes small elements, and the intersection are joints of the rectangular triangular, or other reference network that we call is the finite difference mesh. So, these points are located at the joints of rectangular or other reference network, the grid may be a rectangular grid, or it may be triangular grid, or it may be any other shape of grid.

But the joints that we name that will be a nodes, so the differential equation has to be written in terms of the values available at the notes, so this is the principle. Now in applying this method, the derivatives of the differential equations are replaced by difference equations, and thus the differential equation is now converted to algebraic equations. So, solutions of algebraic equations are only necessary in that case.

So, the first task is to convert the differential equation into this difference form then we can apply these difference equations at the node covering the entire domain. So, we get a set of linear algebraic equations and then solving these we can get the nodal values. For example, in case of plate, say rectangular plate, say $a \times b$ if I divide these lengthwise direction say 100 division and width wise direction Eigen also divide 800 division or maybe other number also.

So, in that case the intersection of this the orthogonal lines will give you the nodes and at this node the value of the deflections are calculated from the solution of algebraic equation. How the algebraic equations are formed? The algebraic equations are formed after expressing the differential equation of the plate in difference form. So, the finite difference method is not only limited to this plate problem, it can be applied to any problem.

Of course, here we will consider the static analysis of the plate, but in case of dynamic analysis also the finite difference method can be conveniently used. Now necessity of the final difference method is obvious. Because suppose a plate which has 2 adjacent edges are fixed, and other 2 adjacent edges are free, so in that type of situation, we do not get any classical solution readily available.

So, therefore we adopt an approximate method, some approximate method that I told you these Rayleigh-Ritz or Gallerkin method can be applied. But again in that case choosing the shape function, shape function can be chosen like a beam Eigen function that I have discus, but that requires a very rigorous integration, and then we can get this result. So, naturally, the charm of solving the problem is lost by lengthy calculations or lengthy integrations.

So, therefore people try to add up some numerical techniques, and one of the popular numerical techniques is finite difference method. Other numerical techniques are also there; finite element method is also a numerical technique, it can be used in plate problem. But finite element method is not included in this syllabus. With high-speed computational facility, this method can be more attractive.

Because even say if I get 1000 simultaneous linear equations involving say nodal displacement and 1000 points in the grid points of the finite element meshes in a plate. So, there will be a very less time consumed because in computer, even in if you use the Matlab, the solution of the linear simultaneous equation is very easy. So, you can solve such type of problem with the high-speed computers available in nowadays.

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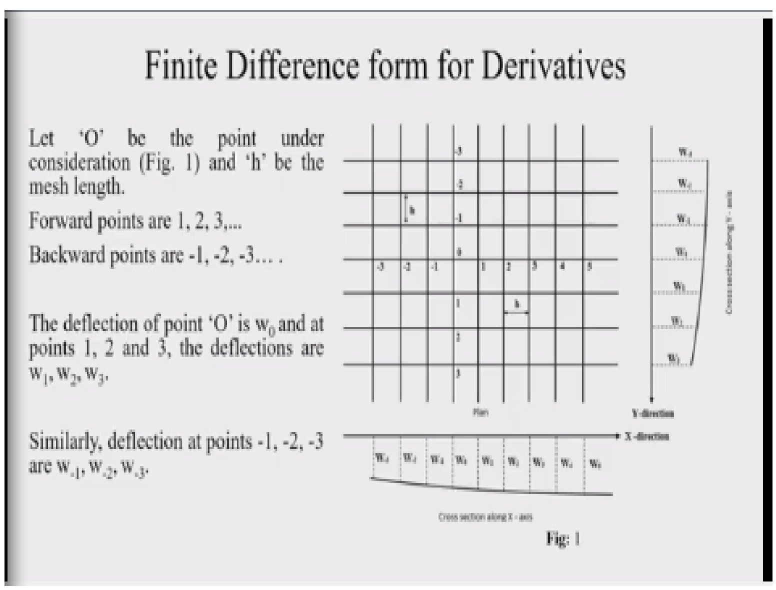

Now let us discuss the finite difference form of derivatives. The first and foremost thing in the finite difference method is to convert the derivatives involved in the differential equation of any physical problem to a difference form. Difference form is the difference between the nodal values and how it can be formed for 1st derivative, 2nd derivative, 3rd derivative because you require up to 4th derivative in case of plate equation.

So, first, let us discuss the finite difference form for the derivatives. Here we take a rectangular grids, and we divide the plate into with the grid, and we have selected the equal mesh size in both direction x-direction and y-direction. But it is not mandatory one can choose unequal mesh size also. Now here, you can see the deflected surface if you see along the x-axis or a section along the x-direction, you will see this is the deflected surface.

Similarly, if you see the deflected surface along the y-direction, you will see this is the deflected surface. Now at the nodal point suppose this is the deflected surface drawn along this line, where the nodes are numbered 0, 1, 2, 3, 4, 5 in the positive direction of x-axis and -1, -2, -3, etcetera in the negative direction of the x-axis. Similarly, if I take the downward direction as the positive direction of the y-axis, then notes are numbered is 1, 2, 3, etcetera along the positive direction of the y-axis.

And in the negative direction of the y-axis is 0, -1, -2, -3, our positive direction of y-axis is downward. So, therefore the negative numbering is given in the upward direction. So, you can see the deflection along this line at different nodes are say here is node is w_0, then w_1, w_2, w_3, w_4, w_5. In the negative direction, you will get w_{-1}, w_{-2}, w_{-3} . So, this -1, -2 subscript that are used here only indicates the nodes along the negative direction of the x-axis, similar thing is existing along the y-axis.

So, you are getting the deflected surface along the y axis, but you can see that node 0 is common to x and y-axis, so you are getting here w_0. Now here also in number say 1, 2, 3 along the y axis and here -1, -2, -3 along the negative direction of the y-axis. So, you are getting a deflected shape. So, our aim is to evaluate the deflection at the different grid points because the exact expressions of the deflected surface say $w(x, y)$ as a mathematical function is not known for certain cases of boundary conditions and loading.

So, therefore we need to discretize the domain into several grid points, and then we evaluate this displacement at the grid points using this numerical technique which is finite difference method. And then, from the displacement values, we can find the other stress resultant like, say, bending moment in x-direction, bending moment in y-direction, twisting moment, then shear force, edge shear, etcetera.

Of course, when we consider the finite difference form of the derivatives, we have to also form the boundary condition because boundary conditions also involve the derivatives of the function. For example, this bending moment, bending moment is related to 2nd derivative; similarly, shear force is related to 3rd derivative. So, we require the derivative also to be converted in the finite difference form.

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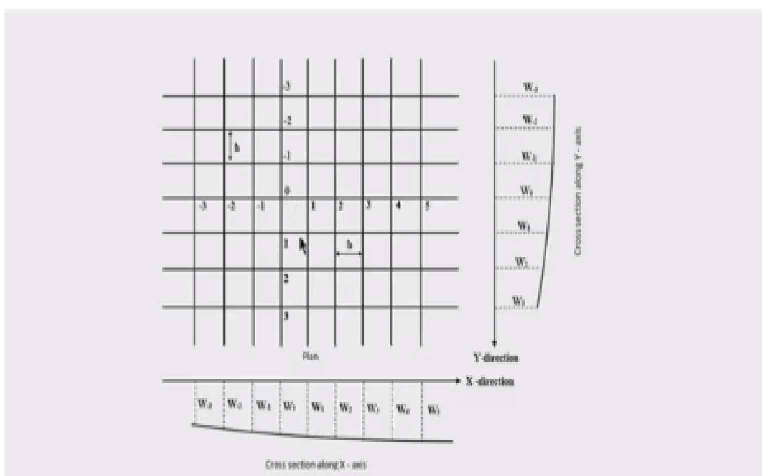

Division of plate with gridlines of equal size in x and y direction

So, this is the finite difference grid point that I discussed, and plate is divided. Here, of course, for illustration purpose I have taken the equal mesh size; that is, the size of the mesh is h in x-direction as well as it is h in the y-direction.

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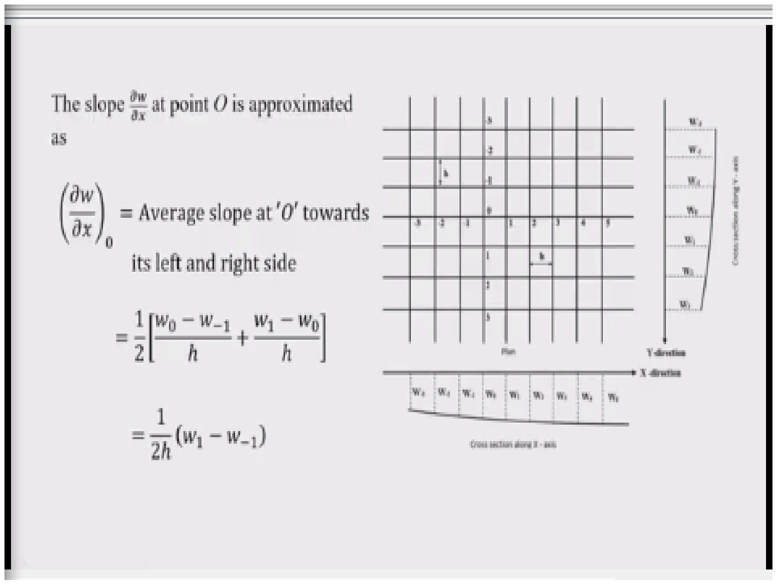

Now let us find first the slope, slope is nothing but the first derivative of the function. So, now at node O, or that is the 0 node here, we want to find the slope. So, this is found say in the forward direction we have say the quantities are 1, 2, 3 etcetera from 0 and in the negative direction the quantities are - w_{-1}, w_{-2}, w_{-3}. So, we have taken the dw/dx at node O is nothing but the average slope at O towards says left and right side.

Now let us calculate what is the slope if we consider the displacement value on the right-hand side? And what is the slope if we consider the displacement value on the left-hand side? So, this is the displacement value on the right-hand side that is w_1 and w_0. So, difference of w_1 and w_0 divided by the distance will give the slope of the line from this line that you are seeing 0 to 1. Similarly, on the left-hand side, that is the backward difference; this is the forward difference, this is the backward difference.

So, in the backward side that is on the negative side, the slope will be say w_{-1} that is $w_0 - w_{-1}$, so this is divided by h. So, these 2 quantities are added and divided by 2 to take the average slope. So, after doing this algebra that is summing these 2 slopes and divided by 2, we get ultimately the slope at O in the nodes. If the deflection at the nodal point 1 and -1 is known, then it can be calculated as $(w_1 - w_{-1})/2h$. So, this is the finite difference form of this slope, first derivative.

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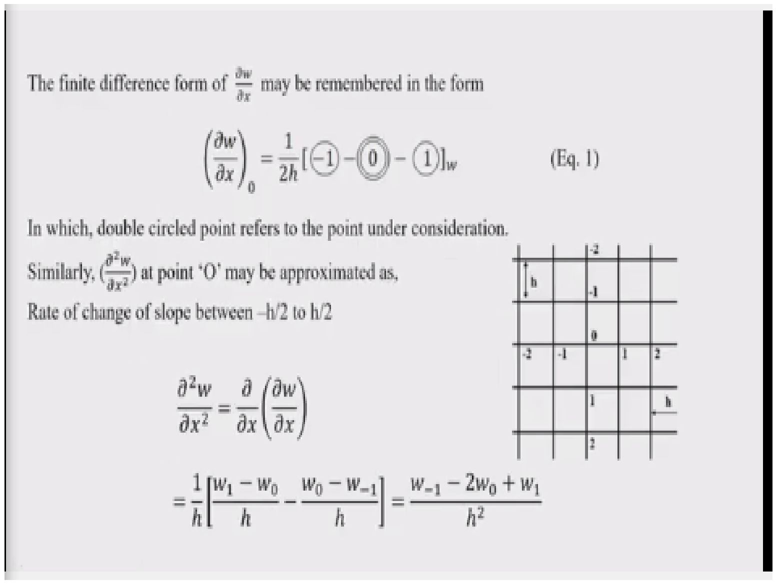

Now let us discuss this second derivative because before discussing the second derivative, we write the finite difference form of each of the derivatives in this form. Because this will be easy to remember, you see the double circle here represent the node, so at the node, coefficient is 0. So, here you can see in the earlier case, the final result is here, the $\left(\frac{dw}{dx}\right)_{at\ 0} = \frac{1}{2h}(w_1 - w_{-1})$. So, there is no value of the w at 0 appearing in this equation.

So, therefore if I write with 3 quantities that means coefficient of $w_0 = 0$; that is the centre point. So, I can remember that this point is the node, so I will place this stencil; this is called stencil. If I place this at O, then I will get the right-hand side deflection value has to be multiplied by 1 and left-hand side deflection value has to be multiplied by -1. So, these are summed up and divided by 2h; then I will got this slope.

So, double circle refers to the point under consideration, that is we call it node point. Similarly, the second derivative that is the curvature at point O, can be calculated because I have used the word approximated because finite difference equations are approximate equation, so these values are approximate. So, similarly, second derivatives are calculated, second derivative is nothing but the rate of change of slope between centre of this site say $O - 1$ the centre and other centre is considered O to 1.

So, between these 2 points, the curvature is calculated. So, curvature is calculated say $\frac{\partial^2 w}{\partial x^2}$ is nothing but the rate of change of slope. So, slope is this and rate of change that is $\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)$. Now here we know that $dx = \frac{1}{h}$, so directly we substituted $\frac{1}{h}$ but change of slope $\frac{dw}{dx}$ is the rate of change of displacement that is the slope, slope between the 2 points $-\frac{h}{2}$ to between $-\frac{h}{2}$ to $+\frac{h}{2}$.

So, we get this $\frac{1}{2h}$, so here instead of this $2h$ because $-\frac{h}{2}$ to $+\frac{h}{2}$ distance is h. So, we have kept here h, and the value is $w_1 - w_0$ and minus because change we are evaluating the change, so $\frac{1}{h}(w_0 - w_{-1})$. So, after simplification, we get $\frac{1}{h}(w_{-1} - 2w_0 + w_1)$. So, this equation has some implications that means it is the second derivative and this equation when you use you have to place the stencil, stencil I will show in the next slide.

That is the stencil for this slope, but for derivative also stencil can be constructed. So, in that case if I place the node here this stencil here centering at O, the coefficient here is -2. And then other 2 values here will be 1 and 1, so -2 is only the coefficient of central point that is the node, and towards right and towards left, the coefficient is 1.

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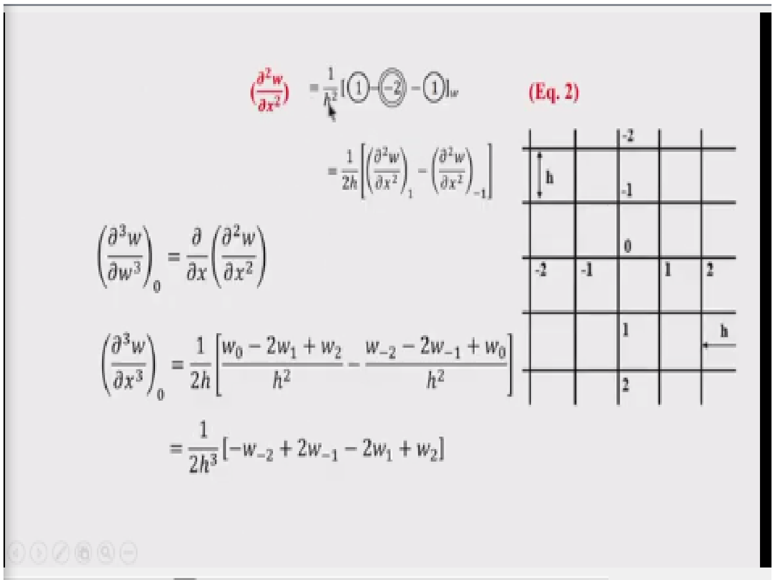

So, in that fashion we can now express this. Now you can see this is the stencil for curvature, so centre point is coefficient of deflection, or this at the node concern node that is the node 0 is -2. And towards right it is 1, and towards left it is -1, at towards left also +1, so these are the coefficients. So, that indicates that the curvature quantity that we have taken is between the change we have taken 1 to and -1 and divided by 2h.

So, this h actually the distance between 2 points that we have considered is from $-\frac{h}{2}\ to\ \frac{h}{2}$. So, generally, h comes, so therefore ultimately, the equation becomes in terms of h. Let us now calculate the 3rd derivative. In the 3rd derivative, once this is known, so we can make use of this equation, and then we take again the rate of change. So, $\frac{\partial^2 w}{\partial x^2}$ now can be written because at the 2 points if you consider covering say 0, 1 and 2 in the forward direction.

If I write this quantity in the forward direction and other quantity in the backward direction. So, in that case, if we write this second derivative, we can write $w_0 - \ w_1 + w_2$ and here we can write $w_{-1} + w_0 + \ w_{-2}$, so all this quantity will be divided by h^2. So, after simplification, you get the difference form of the 3rd derivative as $\frac{1}{2h^3}\left[-\ w_{-2} + 2w_{-1} - 2w_1 + w_2\right]$.

Now here you can see in the 3rd derivative equation, this quantity w_0 get cancelled. So, there is no w_0 appearing here, that means coefficient of w_0 will be 0

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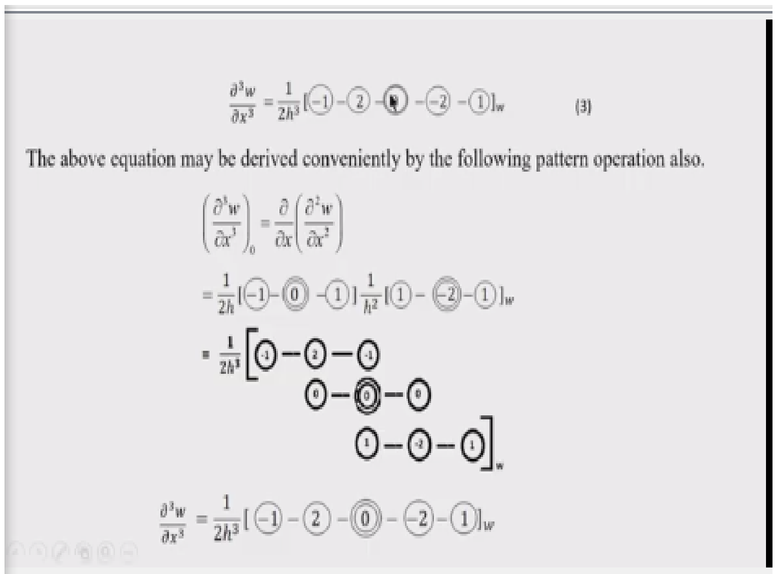

So, if I write in the stencil form, you see this is the node we are considering the node at O, so this is the coefficient at the node, which is 0 towards the right side the coefficient is -2, -2. And then on the further side, it is +1, so here it is $+ 1w_2$. So, similarly, it is also given towards the left-hand side it is +2, and it is -2. Now one thing you can note here it is very interesting to note that the odd derivatives that are antisymmetric.

If I consider this is the line of symmetry, so on. The positive side, you are getting -2, 1 coefficient. On the negative side, you are getting 2 – 1, so this is completely antisymmetric, so our calculations are correct. The above equation may be derived conveniently by following pattern operation also. So, what is pattern operation? That means we know $\frac{\partial^2 w}{\partial x^2}$, and also we know this operator because we have calculated the slope.

So, we write this operator $\frac{1}{2h}$, -1, 0, +1. This is the operator that I have written. Then second is this $\frac{\partial^2 w}{\partial x^2}$, so this operator is this $\frac{1}{h^2}$1, - 2, 1. So, pattern operation means here you just expand it. So, I multiplied it say this with -1, so -1, -2 into -1 +2, then -1 into 1 is -1, then I am multiplying this all the terms by the second term 0. So, that is 0, 0, 0, and it is starting from the second column.

Then this 3rd term here I am multiplying it 1, then -2, then 1, then if you add all this column-wise then we will get here $2h^3$ is the factor bracket this is -1, this is 2, and this is 0 +1 - 1 will be 0 then again it is -2 and 1. So, this is same as this, and you can see here that antisymmetric pattern is observed, antisymmetric pattern is obvious here, and it is true because it is a 3rd derivative odd number of derivatives, so antisymmetry should be reflected in the expression.

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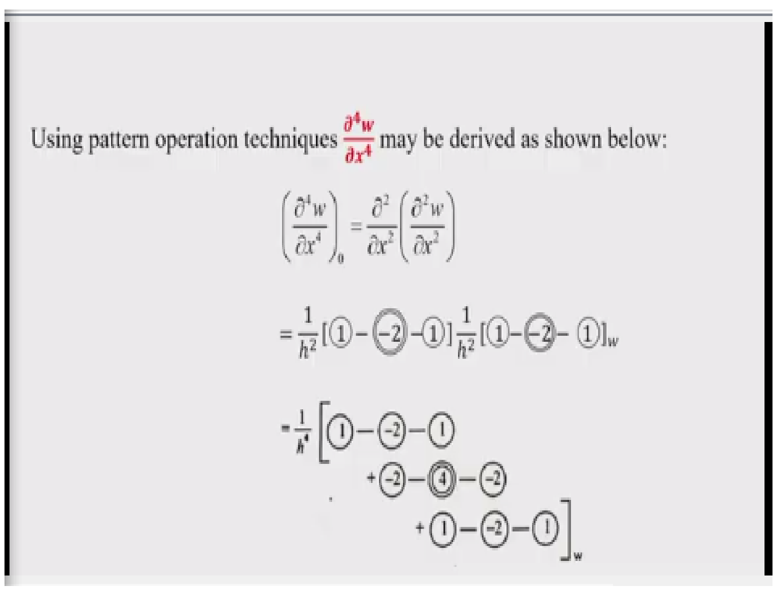

Next, we consider the last derivative that is because our plate equation is of 4th order, so we must go up to the 4th order derivative. So, if I calculate or if I want to express the 4th order derivative in finite difference form, I can write that $\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2}\right)$. So, again I use the pattern operation, so pattern operation is used here, and then this is actually this operator for curvature I have written here.

Then, here again, this is the curvature between the points that I have written here. The w is associated, but here it is only operator $\frac{\partial^2 w}{\partial x^2}$, so you can distinguish. So, first thing is the operator that I have written $\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2}\right)$ in which the subscript w is associated. Now, again you do the same operation that I have done in the earlier cases multiply each term, so 1 then -2, then 1, then -2, +4, -2, then 1, -2, 1.

So, 4th derivative is now obtained, so 4th derivative will be $\frac{1}{h^4}$ bracket 1 if you add this 0, then add it 6 and again add -4 and again add 1. So, this quantity will be $\frac{1}{h^4}$ **(**1, -2 – 2) addition, not subtraction. So, addition -2 and -2 -4 and 1 + 4 + 1 that is 6, then this term is -2 -2 is -4 and lastly is 1. Now here you can see that because it is even derivative, so the symmetry is obeyed, symmetry pattern is obeyed, so how it can be obeyed? Say here the central value is 6, 1 + 4 + 2, 6 on the left-hand side you are getting -4 and 1, right-hand side also you are getting -4 and 1.

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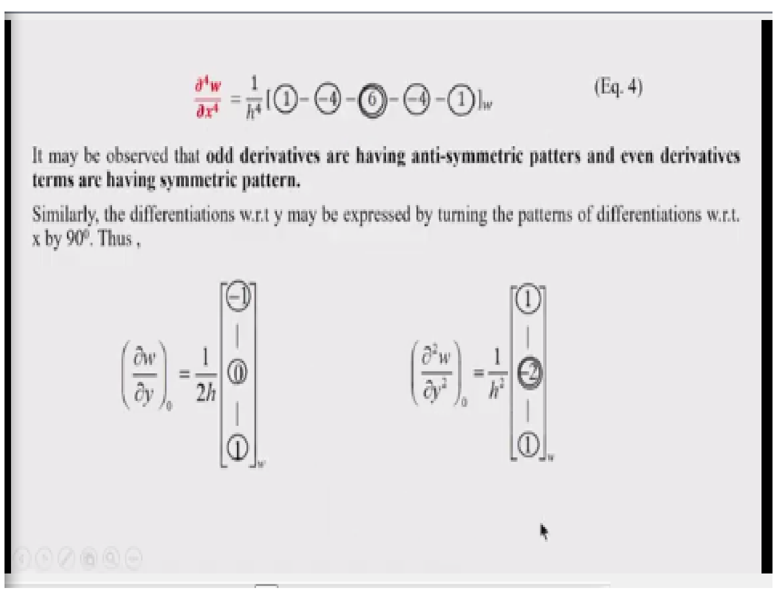

So, in this stencil form, I can write it 6, -4, 1, -4, 1, so it is completely symmetric alright. So, we have obtained the finite difference form of the derivatives with respect to x. So, if I want to write or express the derivative with respect to y similar, this procedure can be adopted without any difficulty. And here, in case of writing the derivative in stencil form, we just rotate it 90 degree. So, we get the difference form of the derivative in the stencil pattern with respect to y when we differentiate it, 1st derivative, 2nd derivative, 3rd derivative and up to 4th derivative; we just obtained the previous results by rotating it 90 degree.

So, first result that we got, say a slope, let us see slope we got this is the horizontal, horizontal line that means it indicates along the x-axis, so 1, 0, 1, so if I rotate this 90 degree I will get $\frac{\partial w}{\partial y}$. So, here this pattern is noted, $\frac{\partial w}{\partial y} = 0$ is the nodal value, and towards the negative side, it is -1, towards the positive side 1. Similarly, the second derivative, the value was $\frac{1}{h^2}$ and 1, -2, 1, so it is written after rotating it 90 degree.

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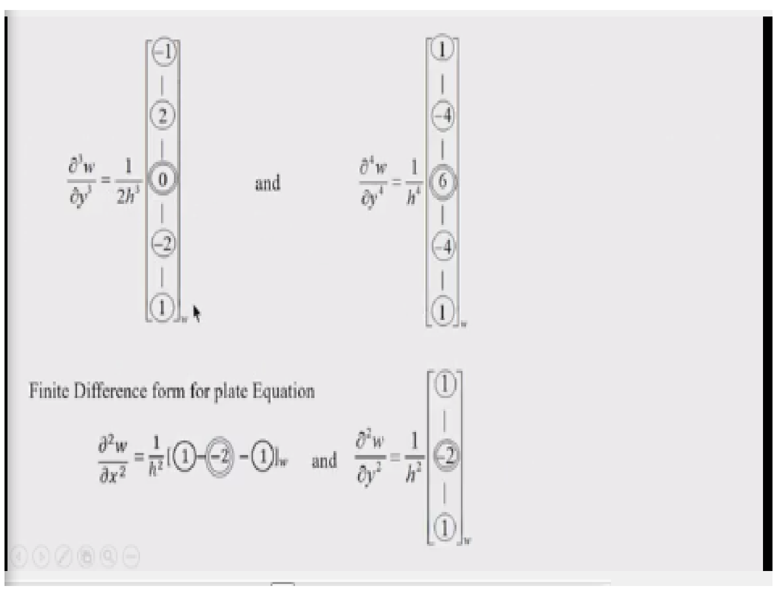

Then your 3rd derivative is written accordingly after rotating the stencil of the 3rd derivative, 3rd derivative stencil is this just to rotate this 90 degree, and we got this 3rd derivative difference form. Similarly, 4th derivative difference form is written $\frac{1}{h^4}$, and then the symmetric pattern is also observed along the y-direction. Now the finite-difference forms for the plate equation have to be obtained. So, we know different quantities that $\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial y^2}$, $etcetera.$

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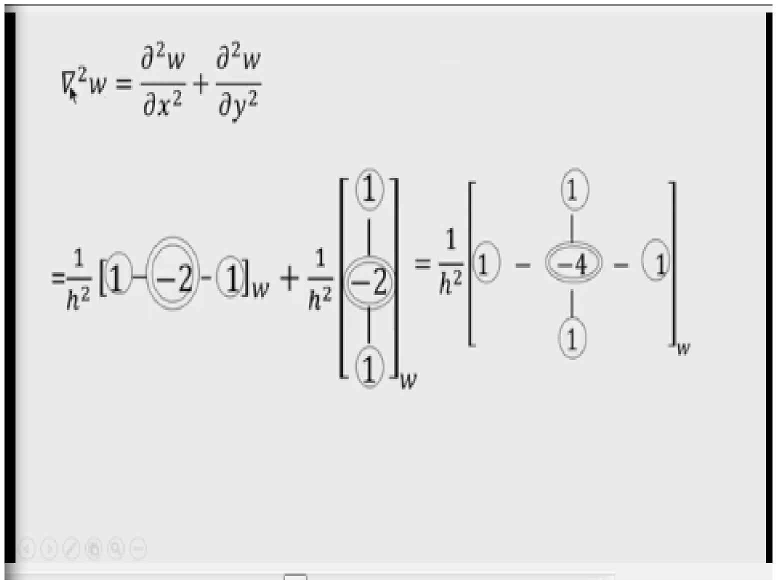

And then, if I express w with the Laplacian operator $\nabla^2 w$, I can write $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$. So, now writing the stencil form, this quantity means $\nabla^2 w$ I first written $\frac{\partial^2 w}{\partial x^2}$ I have written here + $\frac{\partial^2 w}{\partial y^2}$ I have written. Because this is the derivative with respect to y, and this is derivative with respect to x.

So, same thing is there, but 1 is rotated by 90 degrees upward. And here, you can see that the h is the grid size that is taken equal in x and y-direction. If the grid size is not taken equal in x and y direction, then suppose this is h_1, then it will be h_2. So, then you can express the equation in terms of h_1 by h_2 ratio, that is also possible; there is no difficulty if one take unequal rectangular meshes.

So, after adding, if these 2 are added very easily, you can see that $\nabla^2 w$ is nothing but $\frac{1}{h^2}$ 1, this term will be 1 and then -2 - 2 it is -4, then it is 1. Then in the vertical side, you are getting 1 and

1. Of course, this term will be 0; there will be no coefficient appearing at the corners. So, this is the $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$ equation in finite-difference from that is the Laplacian equation that very important equation.

Now here you can see if I remove w, then this term represents the operator ∇^2. Now it is possible for us to find out the finite difference form of the plate equation; how can we form? Because we know the Laplacian operator, so finite difference form of the plate equation is found writing this $\nabla^4 w = \nabla^2 \nabla^2 w$.

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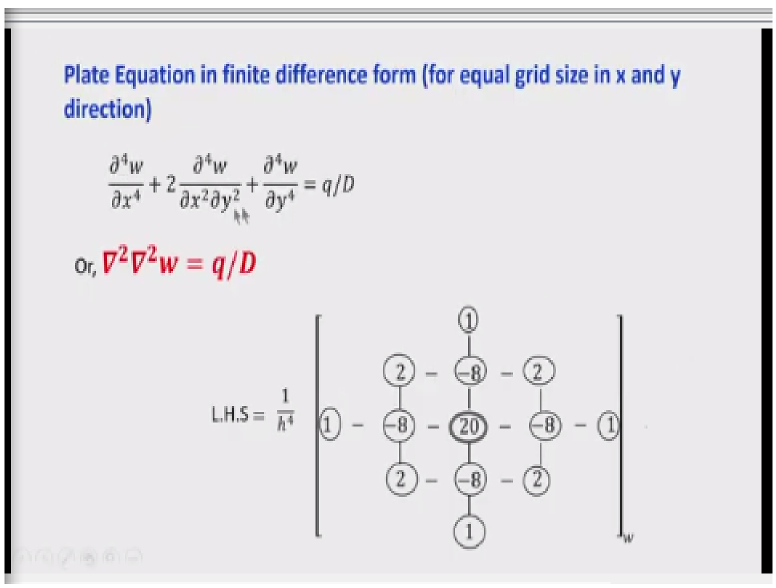

So, this operation can be written that is if expanded from the plate equation is like that $\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$, where q is the distributed load. Now D is the flexural rigidity of the plate; it has no connection with the grid size, it is a constant quantity depends on the material properties, and q is the distributed load.

Of course, when we convert the finite difference form in the distributed loading at each node, the load has to be expressed. So, left-hand side of this equation now becomes how it is found? It is found by this operational procedure that we have erupted earlier for this case say like that. So, here also, if we adopt this procedure, then you will find out the left-hand side of the plate equation that is $\nabla^4 w$; actually, $\nabla^4 w$ is nothing but $\frac{1}{h^4}$.

And in the central line along the horizontal direction, you are getting 5 terms. So, for example, here it is centering at node 0, so w_0, then w_1 and w_2, here w_{-1}, w_{-2}. In the y-direction, whatever value you take in the y-direction it can be represented. Now, these are the coefficients of the nodal displacement when you place the stencil at the nodes; the nodal displacement say at the node say centre node is say 0.

So, it will be expressed this quantity can be understood as $20\,w_0$; this quantity may be understood as $-\ 8w_1$; this quantity may be understood as 1 into w_2 like that any other quantities can also be interpreted.

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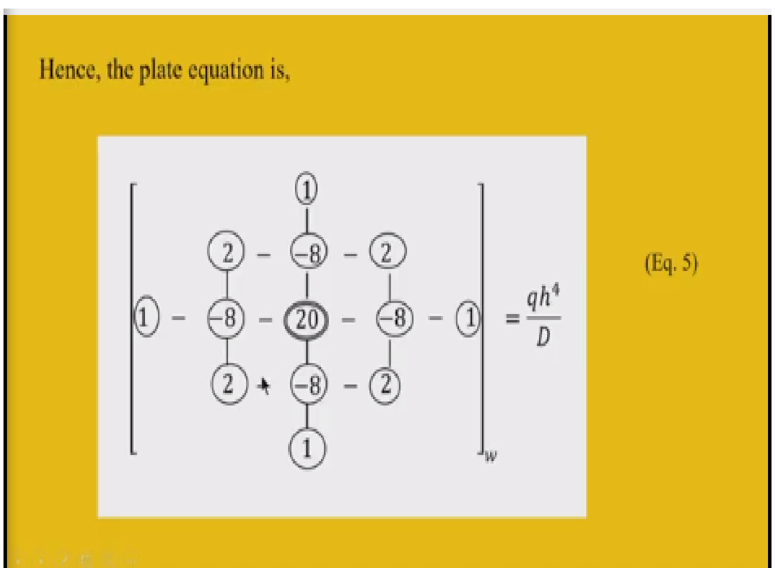

So, the plate equation in the stencil form, that is now our important tool to be applied in problem using the finite difference form. So, this stencil now you are seeing here the 20, -8, -8 and 1, 1. So, this when you place the stencil at a node, then it covers centrally these 5 points horizontally and vertically also 5 points then other 3 points. Then you can see this stencil is symmetric because the 4th order equation we are writing so it is a symmetric matrix.

And $\frac{qh^4}{D}$, where q is the load at the nodal point, nodal point load has to be taken, h is the mesh size. If the mesh size is different, then, of course, this equation will be slightly modified in terms of, say ratio of mesh size in x to the y directions.

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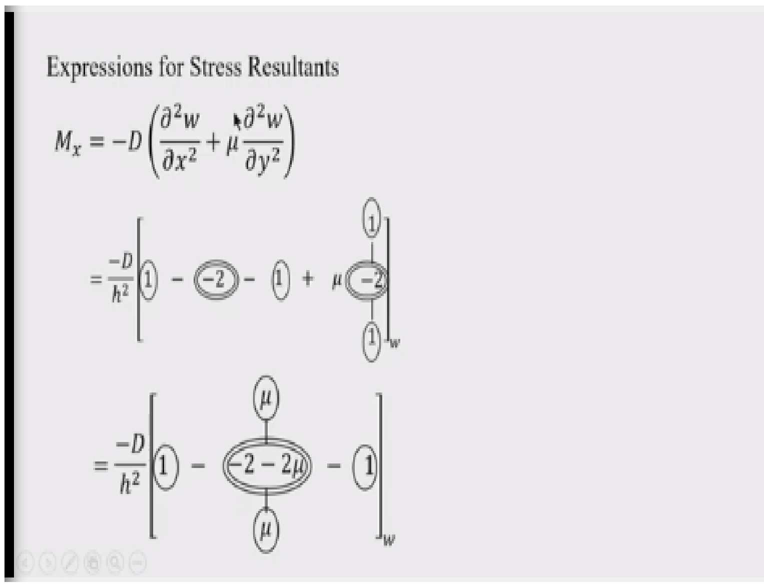

So, now expression for stress resultant let us see. The expression for stress resultant, first let us see what is M_x bending moment? Bending moment expression, you can see that it is 2nd derivative of w, $\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}$. Here you can see first what I have written? I have written this $\frac{\partial^2 w}{\partial x^2}$, so this quantity is taken from the earlier expression that I have derived.

$\mu\frac{\partial^2 w}{\partial y^2}$, $\frac{\partial^2 w}{\partial y^2}$ same quantity, but it is written in the vertical direction because the derivative is taken with respect to y, and the Poisson's ratio μ is associated, so that is must. Now, after adding because this is plus sign is there, so after adding, you are getting centrally this $-2-2\mu$. And then, this μ has to be multiplied with all the coefficients here with the difference form of $\frac{\partial^2 w}{\partial y^2}$.

So, therefore here vertically, this μ up and down μ +μ, + μ appears. Then on the other side that 1, 1 is there, so 1 and 1 is there because there is no coefficient or no other parameter involved with 1. So, this is the stencil form of the bending moment in x-direction.

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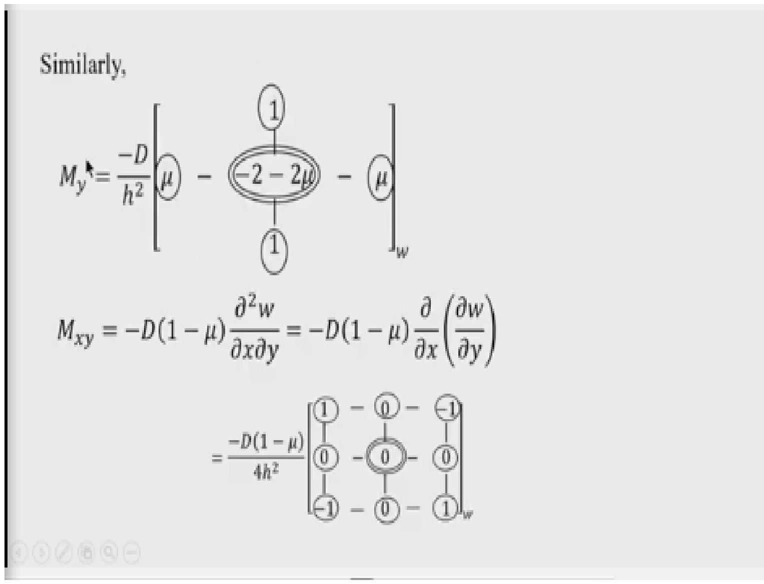

Bending moment in y-direction can be written just by rotating it vertically. So, here in that for derivative with respect to y, 2nd derivative this mu and mu comes here in the horizontal line. And this is del square w del x square it is coming in the vertical line. M_{xy} can also be written $- D(1 - \mu)\frac{\partial^2 w}{\partial x \partial y}$. Now here $\frac{\partial w}{\partial y}$ we expressed earlier in the difference form; now we are operating with $\frac{\partial}{\partial x}$.

So, it can be written in this form, and you are getting here the M_{xy} is very interesting expression $-\frac{D(1-\mu)}{4h^2}$and then 1, 0, -1, 0, 0, 0; the central value along row is 0 and along the column is also 0. Only you are getting the coefficient +1 or -1 at the corners, so 1, -1 and -1 and 1, so this is the value of M_{xy}.

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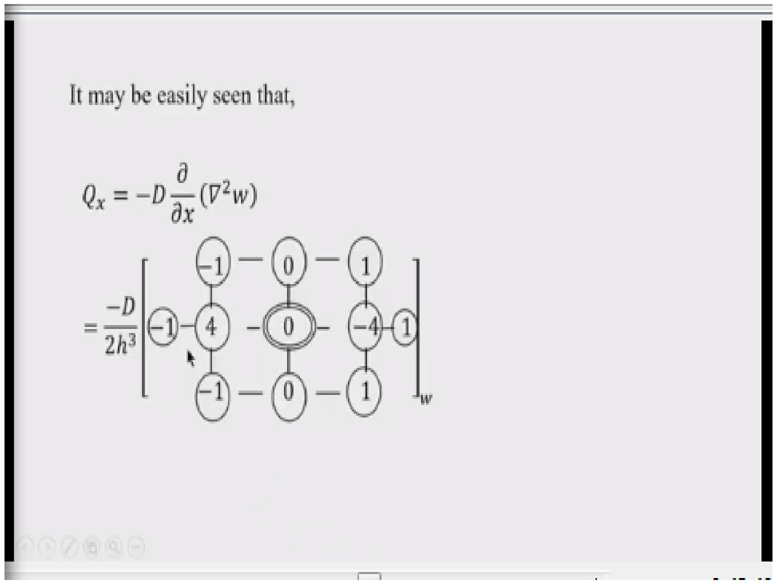

So, it may be easily seen that the shear force, shear force is nothing but $-\ D\frac{\partial}{\partial x}\left(\nabla^2 w\right)$; that is, if I expand it, it will be $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$. So, after operating with this because $\nabla^2 w$, we already found out, $\nabla^2 w$ is this quantity. So, once you find this $\nabla^2 w$, then after taking this rate of change of $\nabla^2 w$ will now get the shearing force Q_x as $-\ \frac{D}{2h^3}$ bracket -1, -1, 0, 1.

Then horizontal line you are getting -1, 4, 0, -4, 1, -1, 0, 1; one thing also you can note here that shearing force because it involves the odd derivatives. So, you are getting antisymmetric term, so here it is -4, here it is +4, here it is +1, here it is -1, so you are getting antisymmetric term.

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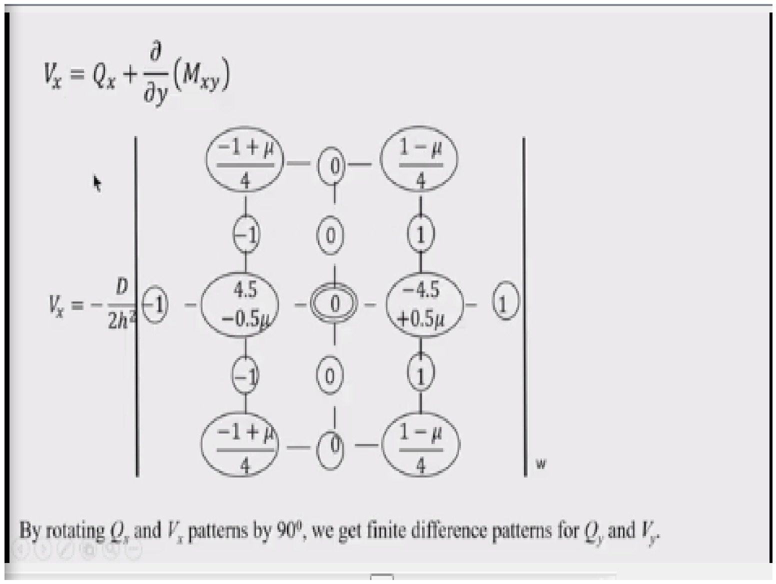

So, edge shear force, that is, we need to apply this conditions specially at the edges. For such edges where the boundary condition is that the 4th boundary condition has to be imposed. For example, free edge, free edge, we know that the shearing force is 0, then twisting moment is 0 as well as bending moment is 0. So, 3 conditions are necessary, and 3 conditions of free edges are given by Poisson that I told in my earlier lectures.

Now, the 3 conditions are combined by Kirchhoff mathematically to give 2 conditions. So, one condition is bending moment, and another condition is edge shear. So, in the edge shear expression, you will find the 2 quantities are involved that is Q_x and M_{xy}. So, twisting moment and shearing forces are combined to form the edge shear force expression. So, edge shear force expression is this, stencil is like that.

That is, in the horizontal line along this centre, you are getting centre value is 0 towards the right-hand side, you are getting $-4.5 + 0.5\mu$, μ is the Poisson's ratio, and then you are getting 1. Then, towards the left-hand side, you are getting 4.5, -4.5, then -1, here also you are getting this antisymmetric pattern; very clearly, it is observed. Then in the vertical side along these edges because it is the shear force along the edge where x is specified.

So, along this direction in the central line, you are getting 0 values, 0 coefficient. Then other columns are written as towards right it is $\frac{1-\mu}{4}$, 1, -4.5, $+\ 0.5$ μ, then, 1, $\frac{1-\mu}{4}$. In the left-hand column, you are getting $\frac{-1+\mu}{4}$, -1, then -1, $\frac{-1+\mu}{4}$, so anti-symmetric pattern is noted. Now by rotating Q_x and V_x pattern by 90 degree, now this thing we have obtained by pattern operation.

But if you rotate it by 90 degree, you will get this Q_y and V_y, so it is easily obtain this rotating this stencil by 90 degree. Let us come to the boundary conditions. Now plate in general may have various common boundary conditions or very non-classical boundary condition what we call? Common boundary conditions or classical boundary condition that is either being that is simply supported or clamped or free, these are the classical boundary condition.

Non-classical boundary condition may be spring supported may be supported on the elastic beam, may be supported by rotational springs or in any other manner. So, that type of boundary conditions can also be expressed knowing the edge values. Now, here let us discuss the only classical boundary condition that is simply supported fixed and free edges.

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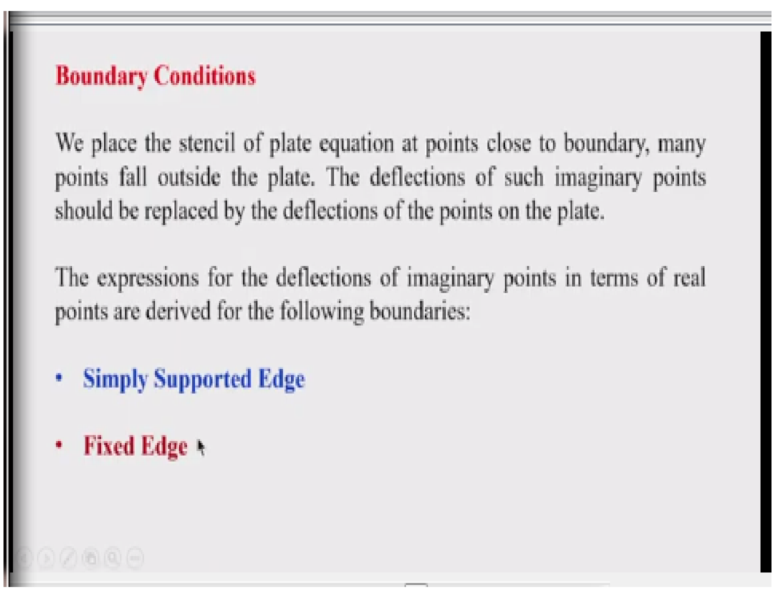

First we will give the expression for simply supported and fixed edges.

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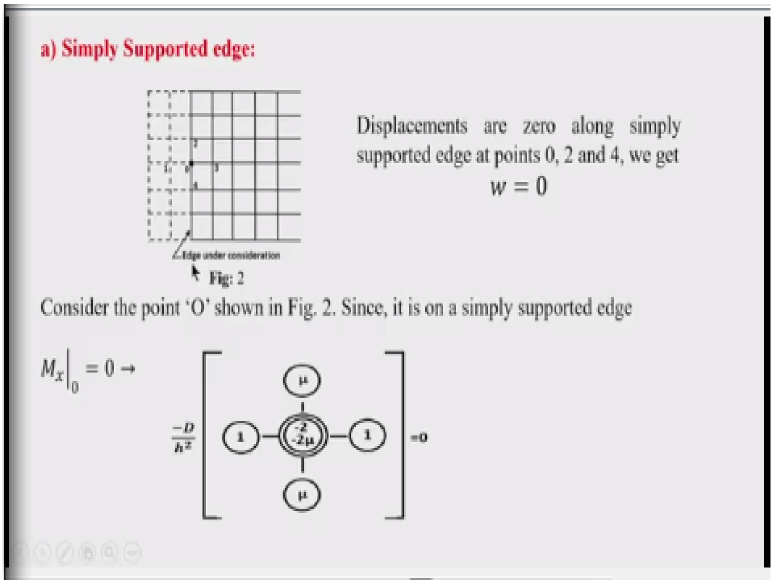

Then we will go for this free edge. Let us consider the simply supported edge. In the simply supported edge, this edge is simply supported, this vertical edge. So nodes are in the edge, the nodes falling are 0 to 4 are the nodes that are falling on this supported edge. So, naturally, the deflection values at 2, 0 and 4 are 0. Now we have to place the stencil for bending moment and the node 0 here at the edge.

So, if I place the stencil here, that means this centre value is now placed at 0 very carefully note it. So, you get one value outside the plate, so how to consider this? Now, this is given by taking the imaginary notes, towards the left or towards the right vertically or horizontally. And imaginary grids I have taken, whose size is same as the size. Now imposing this condition, if I place these grids, one point is on the grid point, and other is on the real grid point.

So, this point is called this is the image point and this is the imaginary point. Now if I express the deflection of a simply supported plate because this point deflection is meaningless because this is imaginary. So, we have to express the deflection of the point 1 that is on the imaginary grid to the deflection corresponding deflection of the image point, the image point is 0.3.

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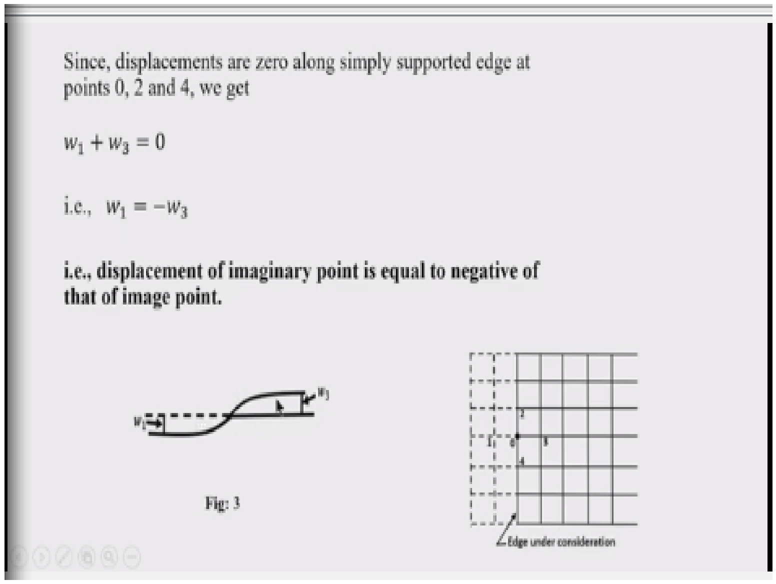

So, now, if I see the condition, here if you see the condition, simply supported case the deflection will be like that, because the rotation is permitted, so there you will get slope. Now seeing the grid size are equal, and from this condition, you can see if because it is falling on this 1, that is a imaginary grid, so $w_1 + w_3$. Because this is also filling on this real point, that is the image point, and coefficient is 1, so $w_1 + w_3$ is 0.

And these points are falling on the edges, so naturally this value will be 0, this deflection will be 0 here 2, 0 and 4. So, now we can express this boundary condition as $w_1 + w_3 = 0$, so $w_1 =- w_3$, that is the deflection of imaginary point is equal to the negative of the deflection of the image point, image point is 3 here.

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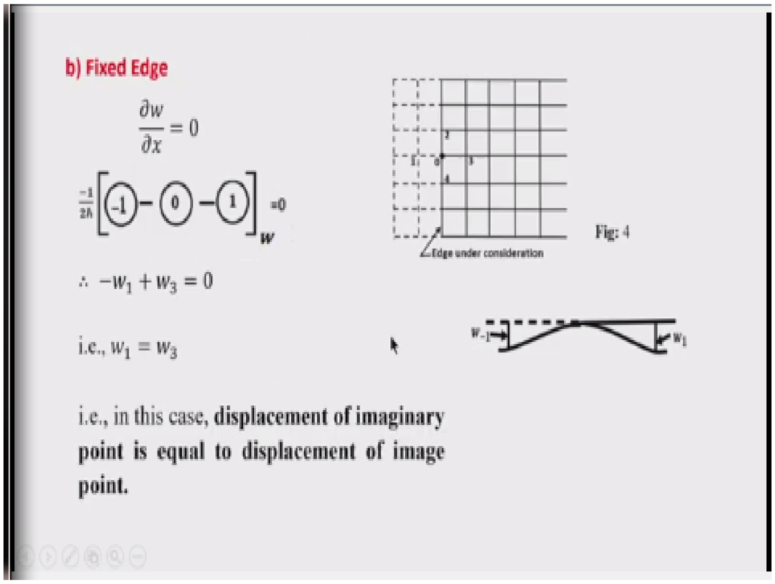

Now let us go to the fixed edge, fixed as condition is depicted by $\partial w/\partial x = 0$, and the deflection is also 0. Now when this stencil is placed here, then it is falling on the 0 is the centre, so other values are at the imaginary point 1 and 1 value at the image point. The value of the deflection at 2, 0 and 4 are 0 because this is falling on the fixed edge. So, now, if I express this, you can see here this coefficient is -1, so $-w_1 + w_3 = 0$, so, therefore, we are getting w 1 = w 3. So in this case, the displacement of imaginary point is equal to the displacement of image point.

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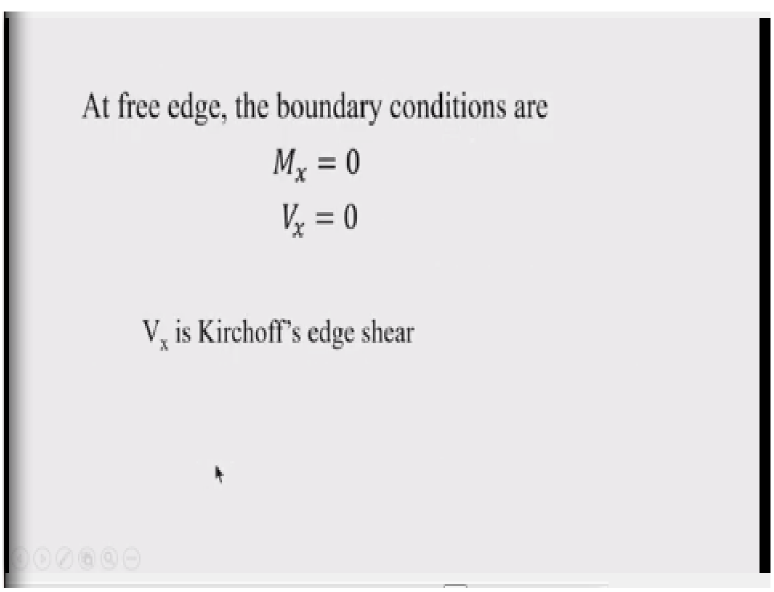

So, at the fear edge imposing this condition $M_x = 0$, $V_x = 0$, where V_x is the Kirchhoff edge shear, one can write the boundary condition.

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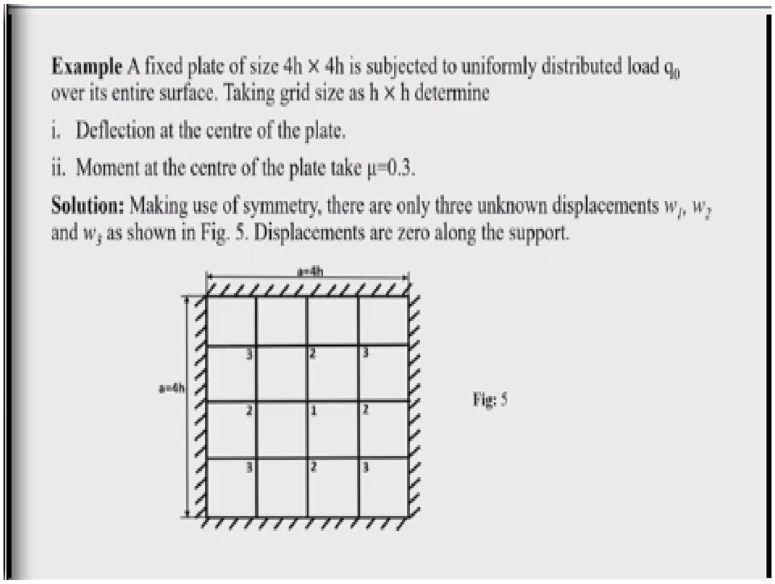

Now let us illustrate a problem of rectangular plate that has to be solved by finite difference method and fixed along the boundaries. So, for simplicity, I have taken the equal grid size and I have taken a square plate for illustration purpose, but it is not mandatory; you can take any unequal grid size also. So, we have to find the deflection at the centre of the plate and moment at the centre because moment at the centre will be equal in both direction because of symmetry, and the plate is subjected to UDL. So, at each node, the value is say q.

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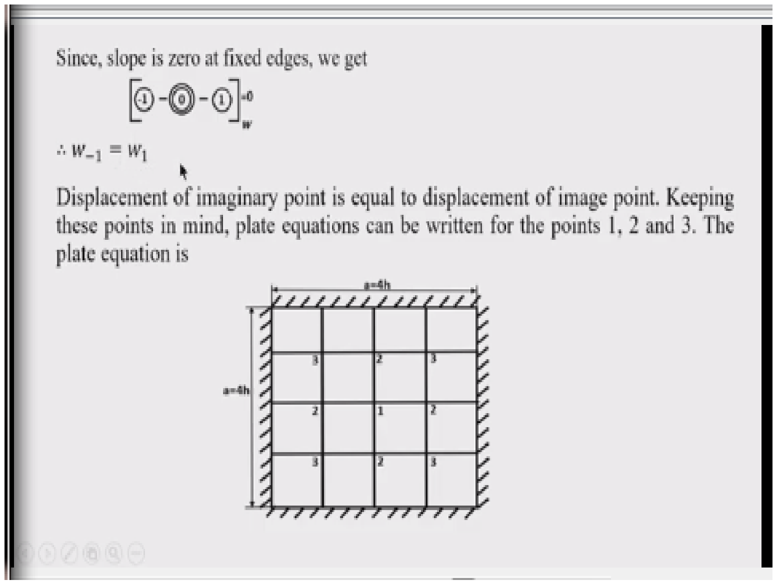

Now since the slope is 0, we get $w_{-1} = w_1$, so that relation has to be valid from viewpoint of boundary condition.

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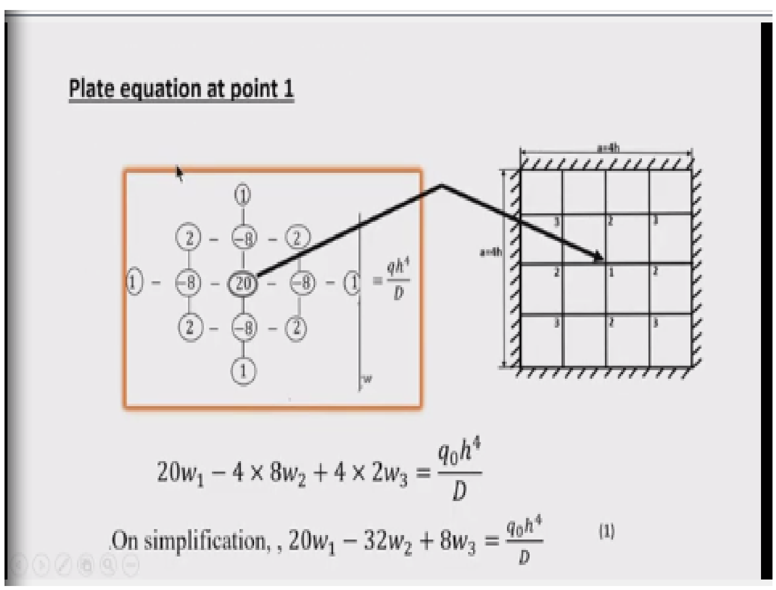

Now first, we plate number the nodes. Now because of symmetry, we can see here if this is the centre point that is numbered as 1. You can see these are the points of symmetry 2, 2, 2 and 2, so here the w values are equal. So, these are also symmetry point 3, 3 and here 3 and 3, other nodal points are falling on the edges in which deflections are 0, so I have not numbered; naturally, it will be substituted at 0 value.

Now let us place the stencil for the plate equation, this is the stencil for the plate equation that I have shown earlier, and this is placed here at the node 1. Now you can see after placing the stencil at the node 1 the 20 is here and -8, 1, this 1 is falling on the edges, so naturally, there will be 0 value. Then again, this is - 8 and 1, this is falling on the edges, this is also falling on the edges.

And these are the points here coefficient 2, so naturally $20w_1$ minus because of symmetry now these values are equal, deflections are equal. So, this we can write this 4 into $2w_3$, these are the point 3. Then this coefficient is -8, so 1 - 8, 1 - 8, 1 - 8, 1 - 8 where the deflections are equal, so we write -4 into 8w 2. So, this is the equation, and right-hand side is $\frac{q_0 h^4}{D}$. So, after simplification, we get this.

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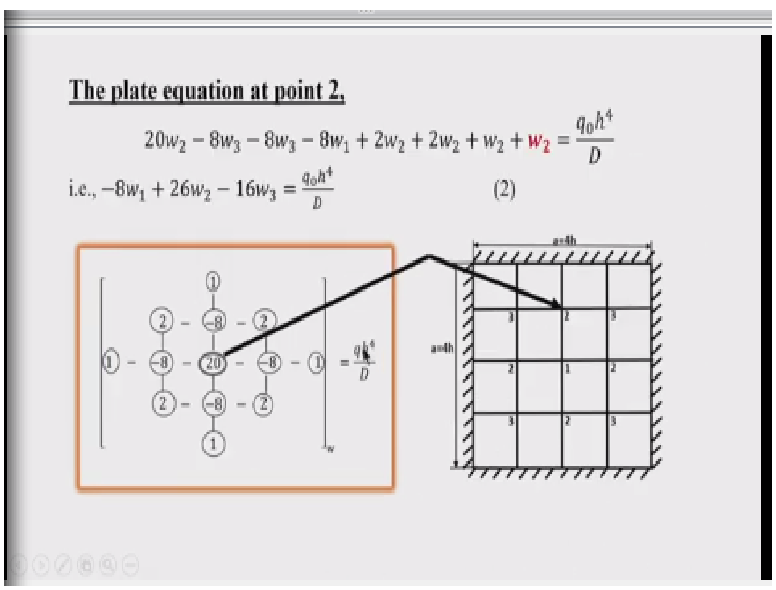

Now next place the stencil at the node 2. After placing the stencil at node 2, you can see that this -8 here and 1 is falling on the edge. So, naturally the deflection this will go to 0 because there will be no deflection at the edges similarly here. And here you are saying -8 and here 1. But now you can see -8 is falling here, but another point is falling on the imaginary point. So, imaginary point if you see at a same distance here.

So, imaginary point deflection will be equal to deflection of point node 2 because of the condition that we have derived. So, therefore one imaginary point deflection I have written with a red color that you can understand it and other values are easily written these $20w_2$, then $20w_2 - 8w_3 - 8w_3 - 8w_1 + 2w_2 + 2w_2$, so another w 2 will come here and then imaginary point w 2. So, this is q 0 h to the power 4 divided by 2 because this load is uniformally distributed, so after simplification, we get this value.

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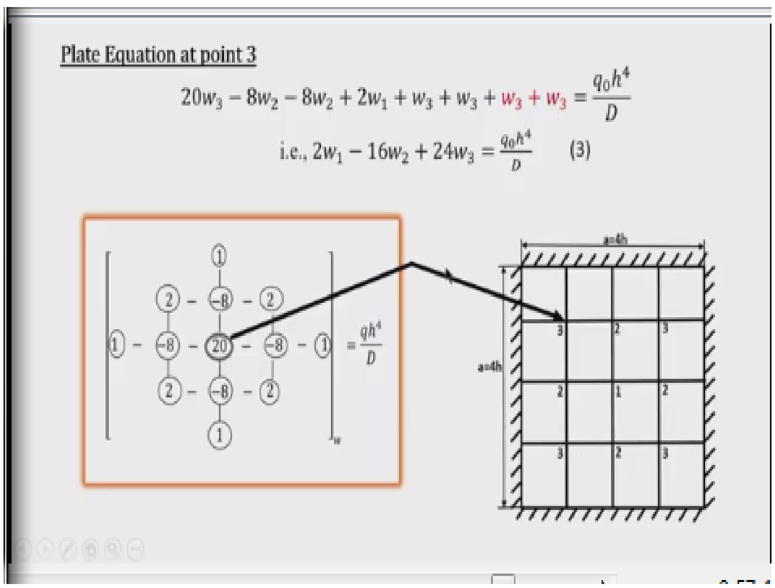

Next, be played the equation is to be written at nodal point 3, any of the nodal points you can choose because these are symmetrical points. So, after placing the stencil at 3, we can now write the nodal equation. So, nodal equation if you write, you can see if I placed the 20 here, one node imaginary node you are getting in the vertical direction and one imaginary node you are getting in the horizontal direction.

So, therefore 2 imaginary nodes because the image point will be 3; one imaginary node goes here whose image is 3, another imaginary node goes here whose image is also 3. So w_3, w_3 are written which are red colour, to understand that these are the image points and corresponding imaginary point deflections are related to the image point deflection. Image points are node 3 and other values that can be easily written $20w_3$, $20w_3 - 8w_2 - 8w_2 + 2w_1 + w_3 + w_3$.

Only these reflect the image point corresponding to imaginary point in the vertical as well as horizontal direction. So, we get 3 equations and our unknowns are 3 because of symmetry we get this advantage.

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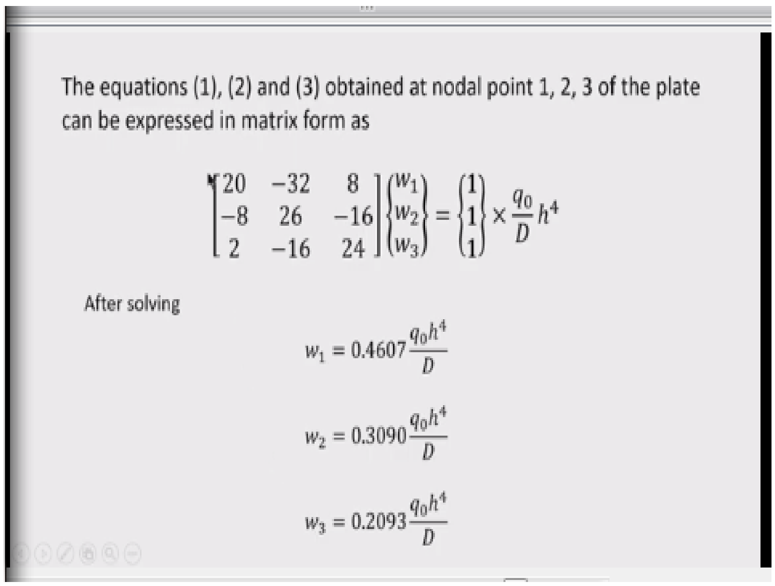

And now solving these equations by any method this can be expressed in matrix form like that. The $[20\ \ -32\ 8\ \ -8\ 26\ \ -16\ 2\ \ -16\ 24]$ this is a 3×3 matrix, coefficient matrix and the unknown vectors are w_1, w_2, w_3, unknown variables. And this is the right-hand side is written after solving we get w_1, w_2, w_3 as this.

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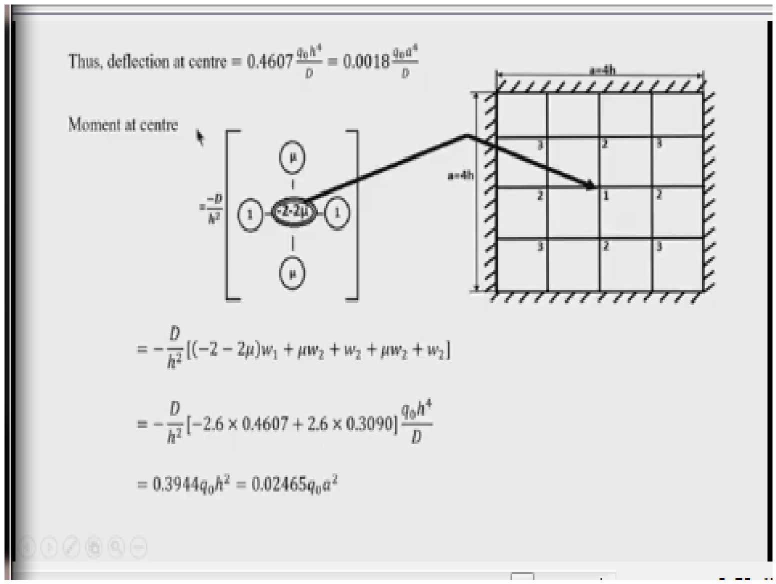

Now let us calculate the moment because another question was that 2nd part was to compute the moment at this centre. Because of symmetry of the support and symmetry of the loading, we found that the $M_x = M_y$ at the centre. And M_x into M_y at the centre can be calculated; this is the stencil for M_x or M_y whatever you call. Now, this is to be placed at the node 1, 1 is the central node.

So, if I place node 1, stencil here 1 at 1, then what we get? We get this say $-\frac{D}{h}$, this is ok, then we get this central node $-2-2\mu$ this is the coefficient into w_1. Then on the right-hand side, we are getting this w_2 is there; on the left-hand side, again w_2 is there. And on the vertical direction, we are getting μ into w_2 and $\mu \cdot w_2$. So, since w_1 and w_2 are known. So, substituting this and simplifying, we get the expression for bending moment like that at the centre it is equal to $0.02465\ q_0a^2$, h is the grid size.

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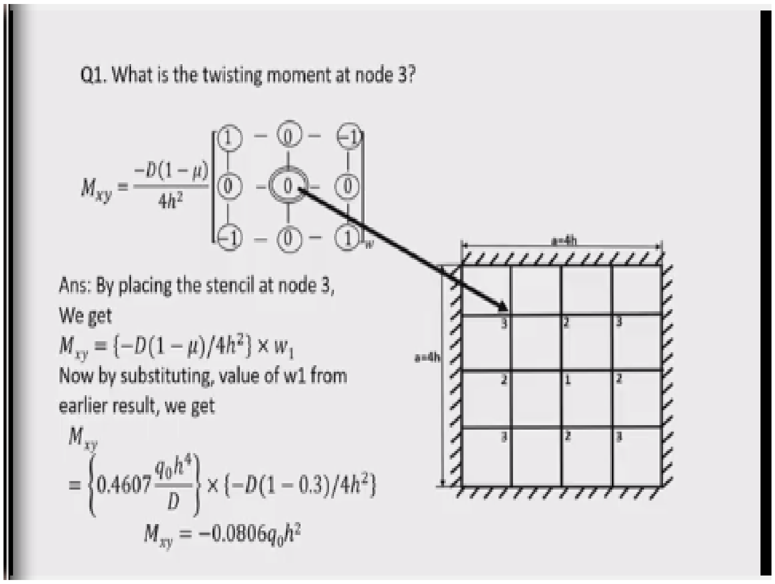

Now, if you see that of course, f this is the written in terms of h, it can be converted into original size of the plate, because h we have taken as $\frac{a}{4}$, so it is written like that. And now let us give a question to you, if you understand this you will be able to answer. What will be the twisting moment at the node 2? Because now we found the deflection value at the nodal points, we can now calculate any stress resultant.

So, let us proceed to calculate the twisting moment at 3, twisting moment at 3 is calculated, this is the stencil for twisting moment, it is placed here at 3 and node 3. And you can see it covers this point right-hand side one point, left-hand side one point horizontally. And also, in the vertical side that is one point this is another point this and here again another point, this is other point, and these are other points.

Now you can see because the nodes here if I place the stencil, these are the edge nodes, so naturally, no deflection is there. And again, these are the earth nodes; these, these and these edge nodes, so there will be no deflection. Even if there is a coefficient here 1 - 1, there will be no deflection. Central node has no geo coefficient, so no deflection, vertically if I see along the central line, so there is also 0 coefficients, so no deflection. Only you are getting here the meaningful coefficient that is 1, so this deflection is w 1. So, therefore the M_{xy} is written, this coefficient into w_1.

Now substituting the value of w_1 that we have obtained earlier, we now write this is the value of w_1. And now this is $-D(1-0.3)$ 0.3 is the Poisson's ratio 4h by $4h^2$. So, it is kept in terms of grid size, but however, if you want to express this in terms of plate dimension, then h can be replaced by a by 4. So, this is the application of finite difference method in case of plate problem. We have taken an example of fixed plate, but other boundary condition can also be taken.

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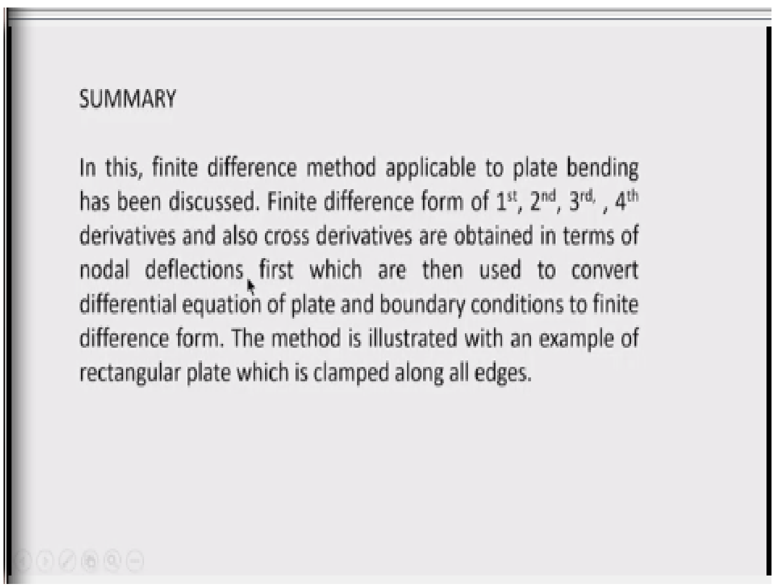

So, let us summarize the lecture what I have delivered today. In this lecture, finite difference method applicable to plate bending has been discussed. Because finite difference is a very vast topic, it can be applied to beam bending, buckling of column, vibration problem but I have taken only the plate problem. So, discussion was carried out related to only plate bending. So, finite difference forms of 1st, 2nd, 3rd and 4th derivatives and also cross derivatives are obtained in terms of nodal deflections.

First which are then used to convert differential equation of plate and boundary conditions to finite-difference form? The method is illustrated with an example of rectangular plate which is clamped along all edges. Now you have noted that because of difference form. There was no need of solving any differential equation. Ultimately differential equation is converted in a algebraic equation, and this is very easily solved.

Because the conversion that took place say; linear simultaneous differential equation in variables are nodal deflections. So, depending on the number of node points you can take, that number of equations will increase. But nowadays, with the help of this computational facility, there will be no difficulty to obtain to solve n number of simultaneous linear equations. Thank you very much.