Plates and Shells

Prof. Sudip Talukdar
Department of Civil Engineering
Indian Institute of Technology-Guwahati

Module-05
Lecture-15

Applications of Rayleigh-Ritz and Gallerkin's Method

Hello everybody, welcome to the massive open online course MOOC and today I will start the lecture 2 of module 5. In the last class I have introduced the approximate method for the solution of plate problem. There we have seen that among various approximate methods the major principle or the major rule that is followed is the variational principle or you can call it the principle of least work. That based on that 2 formulations were derived, that is first I have discussed a problem in a beam to apply the variational methods.

And then we discuss the problem of a plate. Today, let us discuss further applications of the Rayleigh-Ritz method in case of plate. First I will bring to you a problem of rectangular plate clamped along all edges which were not solved in earlier class using the exact method. Then we will discuss a problem of circular plate and then we will go for another variational method which is also approximate method known as Gallerkin method.

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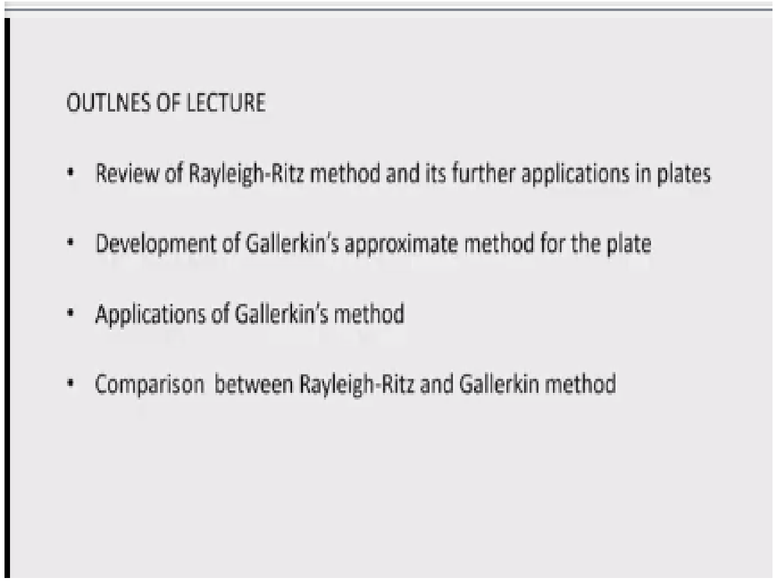

So, outlines of the today's lecture will be review of the Rayleigh-Ritz method and it is further applications in plates, development of Gallerkin's approximate method for the plate, application of Gallerkin method, then comparison between Rayleigh-Ritz and Gallerkin method.

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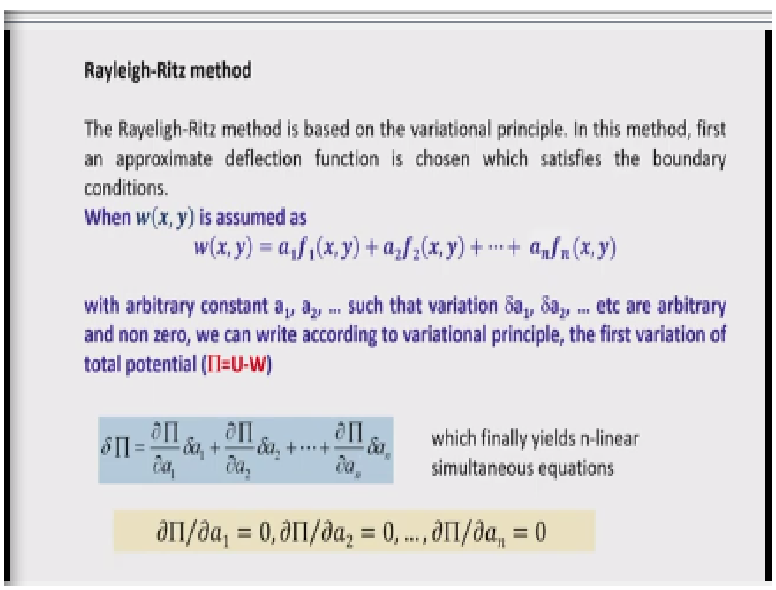

Now, in the last class I discussed Rayleigh-Ritz method. Rayleigh-Ritz method was derived based on the variational principle. That we have seen that first variation of the total potential is 0 if the system is in stable equilibrium. That means, if we find the strain energy and the work done by the external load, then we can form an expression for the total potential. So, one of the total potential is found then taking the first variation, we can write an equation expression in this form $\delta \prod = \frac{\partial \Pi}{\partial a_1}\delta a_1 + \frac{\partial \Pi}{\partial a_2}\delta a_2 + \cdots$ will be there and the nth term is $\frac{\partial \Pi}{\partial a_n}\delta a_n$

Now here you can see the δa_1, δa_2 etc., these are actually arbitrary variation of the coefficient that are used in finding or in assuming the deflection function preliminary. That means, when you use the Rayleigh-Ritz method or when you go for evaluating the strain energy expression then you must know the deflection function. Deflection function, before starting the problem or in many cases is not exactly known.

So, therefore we assume functions which satisfy the boundary conditions. So, boundary condition actually there are geometrical boundary condition, that means condition on slope and deflection and other conditions are on bending moment and shear force. So, if both the condition that is geometrical and force boundary conditions are satisfied then you will get the exact solution. But it is not necessary that the deflection function that we find out from Rayleigh-Ritz method should satisfy the differential equation.

It satisfies the boundary condition exactly you will get the result which is acceptable in practical applications. So, here you can see that deflection function is assumed in the form of a series. That means, I can take f_1 x_1 that is a function of x and y and also f_2 is a function of x and y. So, these functions are in the form of series, that means one can take a polynomial expression or one can take a trigonometric expression, in both cases the formulation can be done.

Now question arises, if the boundary conditions are not satisfied, then what will be the accuracy? Accuracy in deflection in most of the cases is obtained within reasonable limit, if the geometrical boundary conditions are satisfied. But if the force boundary conditions are satisfied, you will get the accurate results in deflection and in it is derivative that means, slope, bending moment and shear force that is the second derivative as well as third derivative.

Now, let us see for any arbitrary variation of the total potential, the equations are written $\frac{\partial \Pi}{\partial a_1}\delta a_1 + \frac{\partial \Pi}{\partial a_2}\delta a_2$ there are other terms and finally $\frac{\partial \Pi}{\partial a_n}\delta a_n$. Now, since δa_1, δa_2 up to δa_n are nonzero arbitrary constants, so that means individual coefficient that is $\frac{\partial \Pi}{\partial a_1}$ = 0, then $\frac{\partial \Pi}{\partial a_2}$ = 0, like that $\frac{\partial \Pi}{\partial a_n}$ = 0.

So, here you can see we are getting n number of simultaneous equation, but I mentioned it as a linear equation, how it becomes a linear? You can see that energy expression that is used for finding the total potential contains the deflection and square of the deflection, actually energy is a positive quantity it contains the square of the deflection. So, when a square term that is a_1 square a_2 square term or the product of a_1 a_2, all this term will be mixed up in the total potential.

Then when you differentiate a square term you will ultimately arrive at the linear term. So, therefore, you will get a n number of linear equations with this operation $\frac{\partial \Pi}{\partial a_1}$ = $\frac{\partial \Pi}{\partial a_2}$ = 0. Now, if you want to increase the accuracy of the solution, you can increase the number of terms in the series otherwise you can truncate it up to a limited number of terms.

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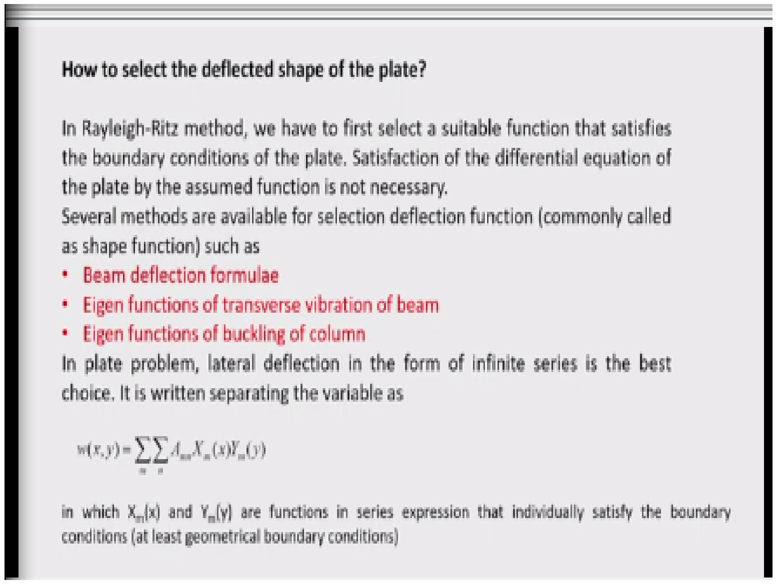

So, question arises how to select the deflected shape of the plate? So, the most important thing in Rayleigh-Ritz method or that is based on the variational principle is to select a deflected shape which may be called as a shape function. So, to select a shape function, there are different guidelines available. So, one is that beam deflection formula can be used, that means suppose a plate, we are now focusing on the plate problem.

So plate, if you look at the plate deflected surface, you will get a function in x as well as in function in y. So, in beam deflection formula, suppose if you consider the deflection in the beam in one direction, you can take this formula satisfying the boundary conditions of the beam in another direction, orthogonal direction. If the boundary condition of this strip of the beam is different, then you can take another function in terms of y. So, the separation of variable is possible.

So, what is seen here that deflection function w (x,y) can be written as by separating the variable that is X_m is a function of x, Y_n is a function of y. So, and then A_{mn} is the constant associated with product of these 2 functions. So, this is one method that this can guide the analyst to choose the shape function or the deflected surface of the plate to be applied in Rayleigh-Ritz method.

Then another possible method is the Eigen function of the transverse vibration of beam. So, that means transverse vibration of beam, generally it is represented by a 4^{th} order partial differential equation. Now, when you want to find the Eigen functions of the beam vibrations, then we will again use the separation of variable technique and we arrived at the ordinary differential equation of 4^{th} order in terms of one variable.

So, this differential equation when it is solved and the constants of integration are found applying the boundary conditions then we get the Eigen functions. But Eigen functions whatever you get has no absolute magnitude. So, it is expressed in relative terms, but this will take care by the coefficient A_{mn} whatever is there in the assumed deflection function. So, this is one possible method to use the Eigen functions in Rayleigh-Ritz method as a assumed deflected shape. Then Eigen functions of buckling of column that can also be used.

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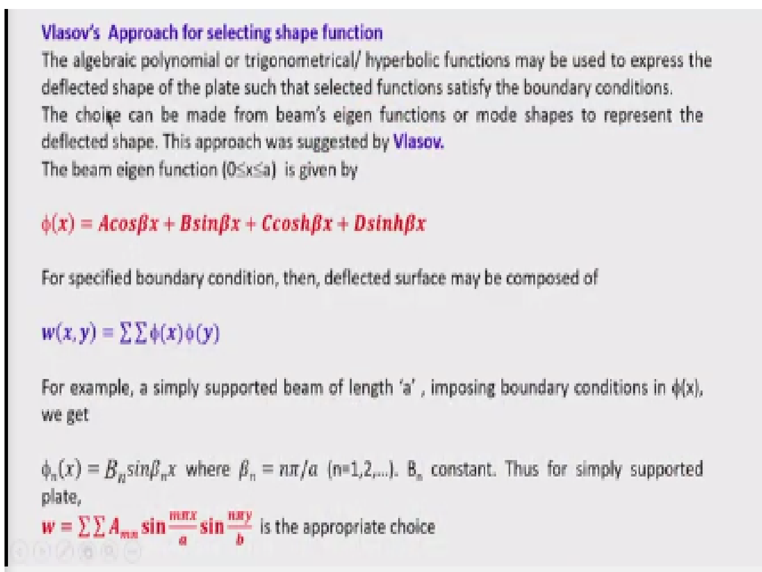

Now, let us see the Eigen function method that are used in choosing the deflected shape of the plate in formulating the problem using Rayleigh-Ritz method was first proposed by Vlasov. Therefore, this approach is sometimes known as Vlasov. Now it is argued that the Eigen functions are the function whose satisfy exactly the boundary conditions of the beam. Then if it is taken properly, then it can be present the exact solution of the plate problem.

But difficulty with the Eigen functions are that in some Eigen functions of the beam especially for boundary conditions which are not found frequently. For example, Eigen functions which is like, that it is a fixed at one end and free at other end, that is a cantilever type of Eigen functions containing the hyperbolic and trigonometrical terms. So, in that case the integration becomes difficult however this approach can be adopted using the integration of the strain energy expression, so that the accuracy of the results can be increased.

So, now generally the Eigen functions of the beam is expressed as combination of trigonometrical cos and sine function as well as cos hyperbolic and sine hyperbolic function.

$$\phi(\boldsymbol{x})=\boldsymbol{A}\cos\boldsymbol{\beta x}+\boldsymbol{B}\sin\boldsymbol{\beta x}+\boldsymbol{C}\cosh\boldsymbol{\beta x}+\boldsymbol{D}\sinh\boldsymbol{\beta x}$$

From above equation we can see that there are 4 terms, one with cosine and another with sine and another with cos hyperbolic and another with sine hyperbolic. And 4 terms contain 4 independent constants of integration; these constants of integration can be found applying the boundary conditions of the beam.

Now for specified boundary conditions, the deflected surface can now be composed of that $\phi(x)\phi(y)$

i.e $$w(x,y)=\sum\sum\phi(x)\phi(y)$$

So, here you can see that if I use the Eigen function for the simply supported beam, the boundary condition of the simply supported beam can lead to a Eigen function in this form

$$n(x) = Bn\sin\beta nx$$

where $\beta n = n\pi/a$ is a non-dimensional parameter. So, here n = 1, 2 etc. and then it can go up to infinity and B_n is a constant.

That is for simply supported plate we can write, that simply supported plate means simply supported along all edges, we can write a deflected shape as like

$$w = \sum\sum Amn \sin sin \frac{m\pi x}{a} \sin sin \frac{n\pi y}{b}$$

These functions or this form of deflection is familiar to you because here now covered the analysis of plates simply supported along all edges by Navier's method. And this double trigonometrical series is used in case of Navier's method for the solution of plate problem. So, the Vlasov approach for selecting the deflected function or deflected shape with the beam Eigen function is also a very popular choice among the analyst.

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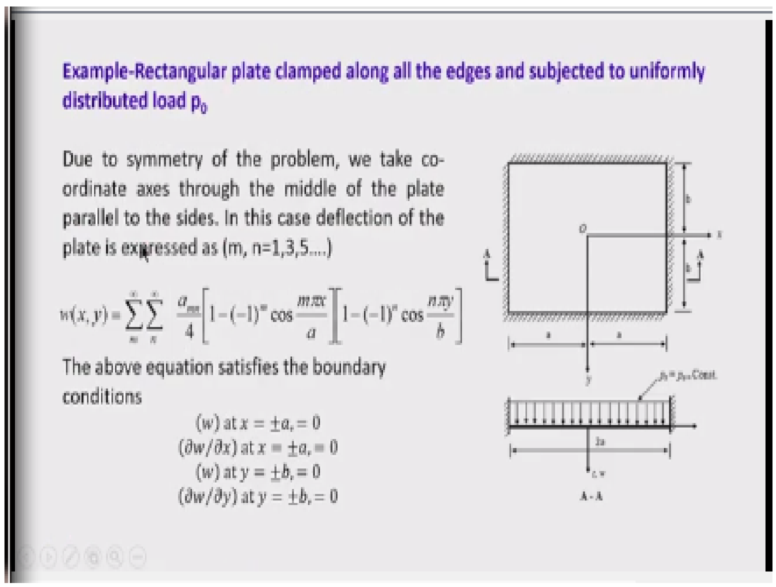

Now, let us give an example of rectangular plate clamped along all the edges and subjected to uniformly distributed load P_0. Now, here you can see the plate here is clamped along all edges, that is along all edges there cannot be any deflection and there cannot be any slope in any direction, any 2 orthogonal directions. Now, this problem actually was not solved analytically and the analytical solution was not found readily for that kind of problem.

However, this problem can be solved by the method of superimposition using the Levy's method with edge moment. So, application of edge moment can give one result and then it can be used with a simply supported plate with uniformly distributed load. Then 2 results can be superimposed, so that the end edge moment value can be found which can make the slope along these in the clamped edge to be 0.

So, that condition when it imposed on the superimposed expression then we can solve a problem of plate which has all the edges clamped. But the method will be very cumbersome and he requires so many steps to complete this problem. So, here with the help of the Rayleigh-Ritz method that is approximate method. We can solve it very easily with the help of strain energy formulations.

Now here you can see the origin is taken at the centre of the plate taking advantage of symmetry, but it is not necessary that one has to take the origin at the centre of the plate. Origin can be taken at one of the corners also. Now here you can see if I take a function w x y = a mn, small a mn is a constant and 4 is taken for just only for convenience, there is no necessity of taking this 4 but I have taken 4.

Because when this function is evaluated, then the quantity becomes 4 at boundary then it is 4 and if we divide this quantity then it becomes only A_{mn}. So, based on that argument it is taken but there is no necessity also. So, if you take say for example at the boundary, at the boundary what happens is the deflection is 0, so boundary is ±a or ±b. So, if you take say x at a, so here you will find that $\cos m\pi$, if m is 1 for example you will get if this factor is 1, so 1 - 1 will get 0.

So, again at y = b you will get 1 - 1 = 0, so deflection condition is satisfied at x = a or x = b or x = -a or x = -b. Similarly, if you compute the slope of this expression that if you calculate the first derivative of this expression $\frac{\partial w}{\partial x}$ then you will get here this a_{11} and multiplied by of course $\frac{m\pi}{a}$ and the sign will be minus because we are the differentiating a cos function.

So, what happens? Because of sine function appearing here as a product term when you put x = a or y = b then you will get $\sin m\pi$ or $\sin n\pi$. So, sine of any integer value $m\pi$ will be 0 or sine of $n\pi$ where n is any integer will be 0. So, therefore slope is also satisfied at the boundary, deflection is also satisfied at the boundary.

So, this function $w(x,y)=\sum_{m}^{\infty}\frac{a_{mn}}{4}\left[1-(-1)^{m}\cos\frac{m\pi x}{a}\right]\left[1-(-1)^{n}\cos\frac{n\pi y}{b}\right]$ can be selected as a shape function to be used in Rayleigh-Ritz method.

The justification you have understood, that these functions satisfy the boundary condition, this slope and deflection boundary condition at the edges. Now edge is here located by the distance +a or -a or +b or –b. So, these conditions are listed here and the plate is subjected to uniformly distributed load. Let us illustrate this problem with a single term expression, so that how the plate problem is used?

Rayleigh-Ritz method is used for solving the problem; you will be able to realize it. Unnecessary complicated with the help of different terms in the series, first let us investigate the solution procedure with the help of only one term of the series. Now, to use the Rayleigh-Ritz method, we first have to select a deflection function; deflection function is selected, so this process is over. Next let us see what is the step? Next step is to calculate the stain energy of the plate and work done by the external load.

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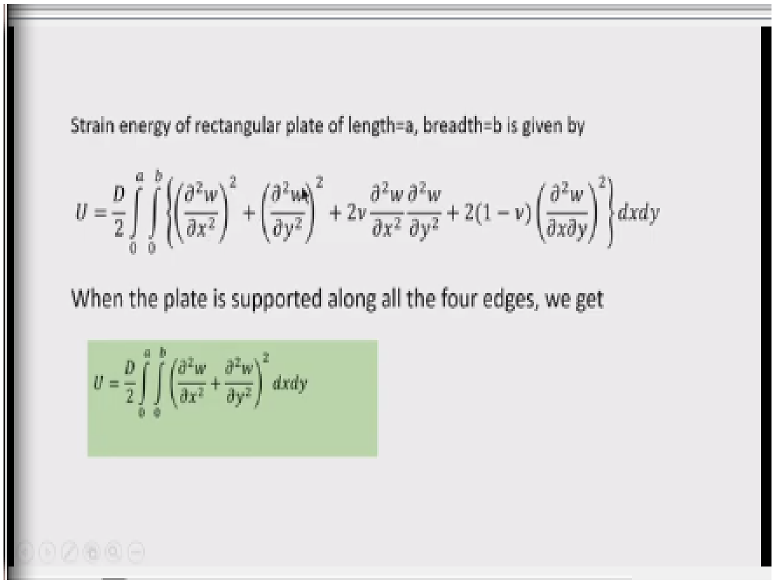


Now, you can see here for rectangular plate the strain energy expressions are given as

$$U = \frac{D}{2}\int_0^a\int_0^b \left\{\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right\} dxdy$$

Now, for the plate which has all the edges supported, then we get a special result that this expression becomes simplified to this

$$U = \frac{D}{2}\int_0^a\int_0^b \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 dxdy$$

So how it becomes? Because we can now write this expression in different way by adding a term

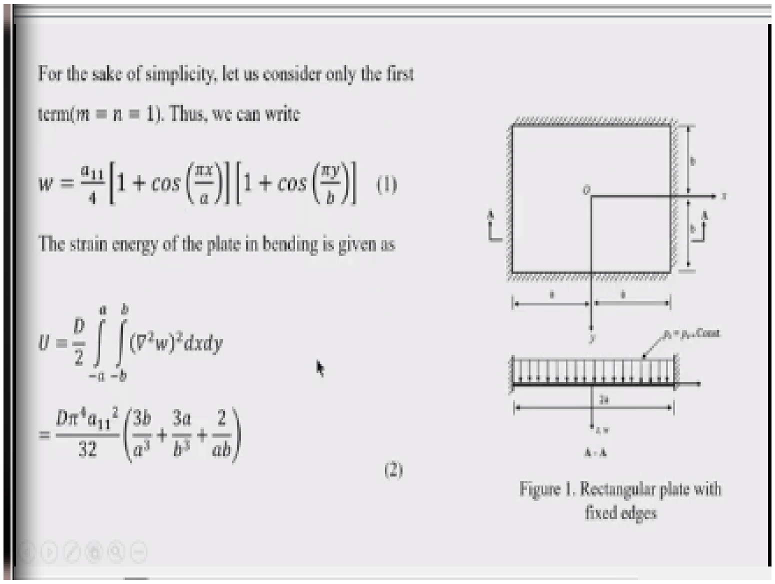
$$2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$$

Then we can write this expression in this form $\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2$ minus some term will come which contains the product of $\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$ minus $\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2$, that I have discussed in the last class.

And the rectangular plate which has all the edges supported whether it is a clamped support or a simple support means simply support simply supported plate. The second term containing the Poisson's ratio will get vanished. So, that means the expression can be simplified in that case into a simple form

$$U = \frac{D}{2}\int_0^a\int_0^b \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 dxdy$$

This expression inside the bracket you can see this, this expression can be written as Laplacian operator $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$ **(Refer Slide Time: 23:24)**



Using this expression now, we calculate the strain energy of the plate taking only one term of the series. Now, go back to the deflected expression that we have assumed if we take m = 1 and n = 1, then this term becomes $(1 + \cos\frac{\pi x}{a})$. And this term, second term becomes $(1 + \cos\frac{\pi y}{b})$ and it will become $\frac{a_{11}}{4}$. So, then this strain energy expression is calculated by taking this function as the deflected shape of the plate.

So, substituting $w = \frac{a_{11}}{4}\left[1 + cos\left(\frac{\pi x}{a}\right)\right]\left[1 + cos\left(\frac{\pi y}{b}\right)\right]$ in the equation

$$U = \frac{D}{2}\int_{-a}^{a}\int_{-b}^{b}\left(\nabla^2 w\right)^2 dxdy$$

we can get this final result by double integration. The limit of the integration of course here it will be -a to +a and -b to +b. So, integrating this expression we get the total strain energy of the plate $\frac{D\pi^4 a_{11}^2}{32}\left(\frac{3b}{a^3} + \frac{3a}{b^3} + \frac{2}{ab}\right)$

Now you can see here in this expression, a $_{11}$ square term is appearing. This is because when we differentiate this expression 2 times $\frac{a_{11}}{4}$, this constant will appear as it is. And when we square it in for the calculation of strain energy, then we naturally get a_{11}^{2} . So, that a_{11}^{2} term is appearing in this strain energy expression which is required because we will differentiate this strain energy expression that is after finding the total potential.

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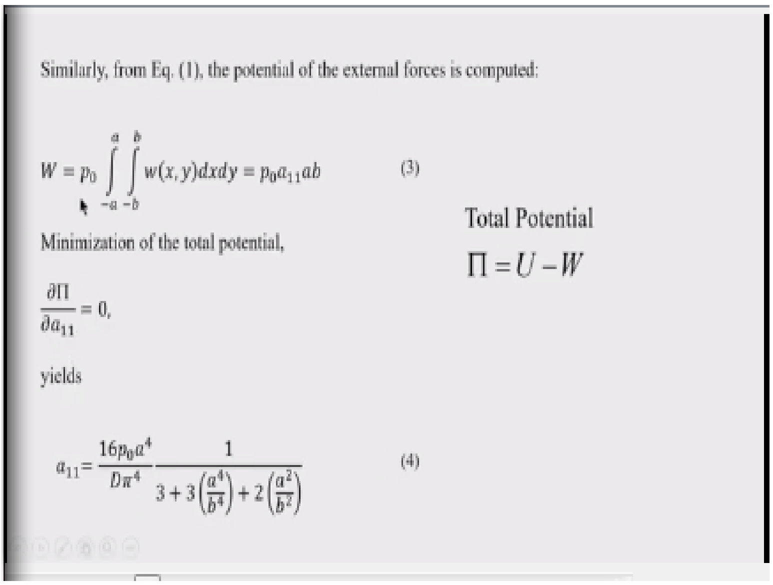

So, now let us calculate the work done by the external load. So, work done by the external load is suppose the load is P_0 which has the uniform intensity over the plate area. So, P_0 I am taking outside the integral sign and then integration is carried out with the product of w (x, y). w (x,y) is the deflected shape and dx dy. So, P_0 is there for calculating the work done, but it is taken outside the integral sign because the load is constant.

So, after integration you can see w is very simple expression. Now if you integrate it with respect to x and y double integration, then you will get that this term and this term can be easily integrated and this is the integration of dx and integration of dy. So when it is integrated, the final results become $p_o a_{11} ab$ $p_0 a_{11} ab$

. So, now total potential is calculated as pi = U – W, what is U? U is strain energy of the plate which is equal to

$$\frac{D\pi^4 a_{11}^2}{32}\left(\frac{3b}{a^3}+\frac{3a}{b^3}+\frac{2}{ab}\right)$$

So, after calculating this total potential, we now use the Rayleigh-Ritz principle. That means we take the derivative of total potential first derivative with respect to a_{11}. So, now you can see the total potential ∏ will contain the previous term

$\frac{D\pi^4 a_{11}^2}{32}\left(\frac{3b}{a^3}+\frac{3a}{b^3}+\frac{2}{ab}\right)$ subtracted by $p_o a_{11} ab$

So, that term will be used to compose the total potential Π .

You know after taking the derivative of total potential, we see that from the previous expression that will be 2 a_{11} . So, the square term of the coefficient now reduced to a linear term quadratic term is reduced to a linear term. So, therefore we get the a_{11} coefficient which contains the load intensity and dimension of the plate and other parameters that contains the material constants of the plate that is the value of E Young's modulus of elasticity Poisson's ratio that forms the flexural rigidity of the plate.

That is $D=\frac{Eh^3}{12(1-\upsilon^2)}$. So, one is the coefficient a_{11} is found, deflected shape that we take for solving the problem can be found out. So, a_{11} is this expression and if I now want to calculate for example if I take a rectangular plate which is the aspect ratio 1.5, what is the meaning of aspect ratio? Aspect ratio is nothing but ratio of the length to the breadth of the plate.

So, if a is a length of the plate and b is a breadth of the plate, then a/b is the aspect ratio. So, if I take a by b **is** as 1.5 and Poisson's ratio as 0.3, then we can calculate the maximum deflection at x = y = 0, why I have taken this as the point of maximum deflection? Because for a symmetrically loaded plate which has symmetrical boundary and loading is also symmetrical, the maximum deflection is going to take place at the centre of the plate. So, centre of the plate has coordinate x = 0 and y = 0, because we have taken origin at the centre of the plate.

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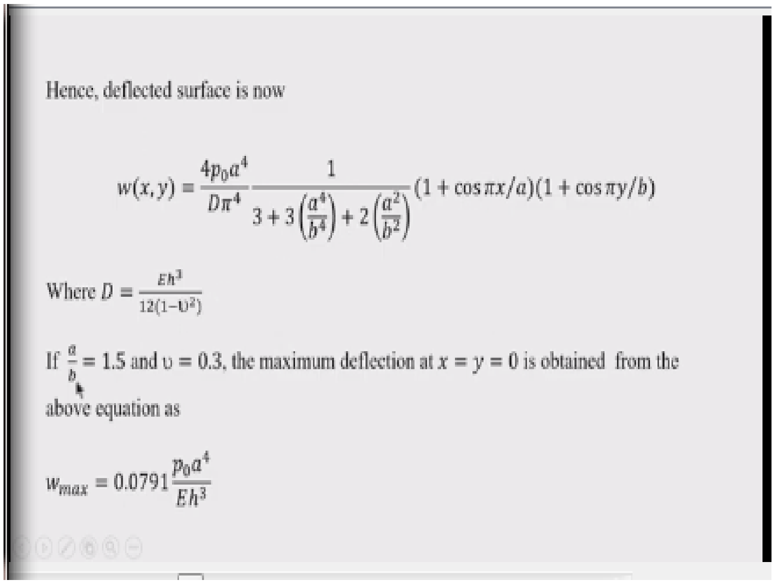

So, by taking this x = 0 as and also y = 0 and substituting these here and a/b = 1.5, that means a = 1.5 b or b = a /1.5. We can express now

$$w_{max} = 0.0791\frac{p_0a^4}{Eh^3}$$

Of course the D term has to be substituted by $D = \frac{Eh^3}{12(1-\upsilon^2)}$ and in place of mu you substitute 0.3. So, this is the result that has been found for as a deflection of the clamped plate subjected to uniformly distributed load.

The boundary of the plate are fixed and for that condition the direct use of Levy's method is not possible. You can obtain using the Levy's method by several steps, superimposing the results of the deflection of edge moment as well as superimposing the deflection of the simply supported plate with the uniformly distributed load. And then imposing the condition that the slope will be

0 for certain value of edge moment then you can find out substitute this edge moment in the expression and then you can find the solution for the plate which has all the edges clamped.

So, this contains several steps if you adopt the Levy's method and the expression will be very lengthy. But here using the approximate method, you can calculate a deflection of this order and deflection will be actually within the reasonable limit if you see with the exact results now because the deflection is actually small in case of the plate because we are using the small deflection theory.

So, the slight variation of the deflection will not cause any difficulty for practical applications. But we have to see whether these 2nd derivative and 3rd derivatives which are used to find the bending moment as well as shear force the accuracy is retained or not, that comparison should be made. And it is obvious that with the use of only single term in the series, the accuracy in the bending moment and shear force will be lost. So, if you want to accuracy of these 2 parameter, then you have to increase the number of terms in the series.

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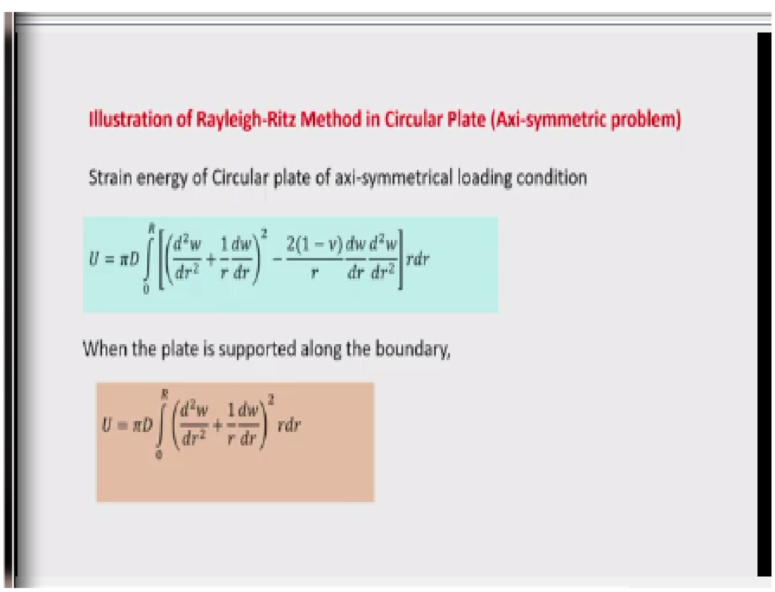

Next let us illustrate a problem of circular plate. Of course, we are taking here axi-symmetrical problem, because so far we discussed only axi-symmetrical problem of the circular plate. The plate which has a axis of rotation, rotational symmetry and loading and boundary conditions symmetrical with respect to these axis of rotation. Now, if I want to use the Rayleigh-Ritz method for the circular plate, then again we need a strain expression.

So, strain energy expression for the circular plate is given as

$$U = \pi D \int_0^R \left[\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} \right)^2 - \frac{2(1-\nu)}{r}\frac{dw}{dr}\frac{d^2w}{dr^2} \right] rdr$$

Now you can see this is the strain energy expression of the plate which is in circular shape.
The boundary may be anything, now when the boundary of the plate is supported whether it is simply supported or clamped, then the second expression can be neglected. So, in that case we get

$$U = \pi D \int_0^R \left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} \right)^2 rdr$$

So, if I want to solve the problem using the Rayleigh-Ritz method in case of a plate which is of circular shape, then we can take this function if the boundary of the plate is supported. Supported means here it may be fixed along the boundary or it may be simply supported.

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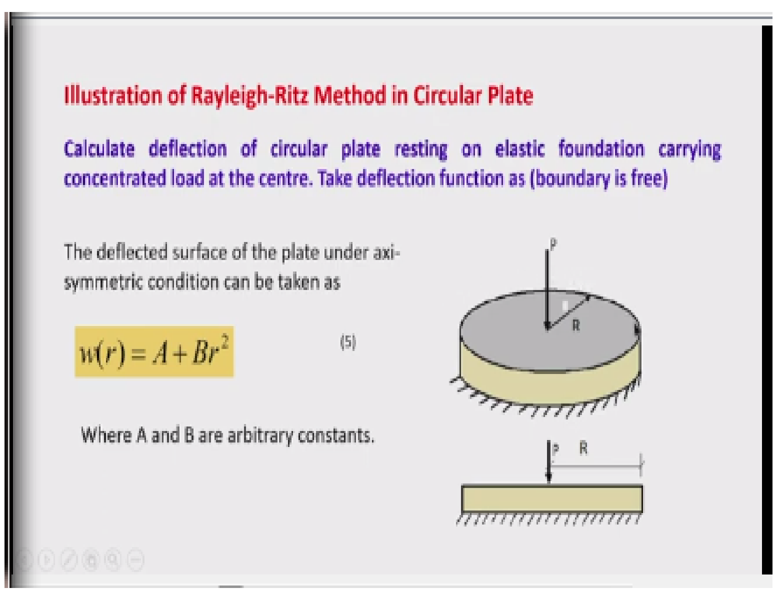

But consider here a problem which a circular plate is resting on elastic sub grade, just you consider imagine it or idealize a foundation slab that is circular footing. For example a circular footing for a circular column if somebody wants to construct it, then you will get this footing this foundation slab, maybe idealized or modeled as a plate which is resting on the elastic sub grade. So, elastic sub grade here acts like a spring that it will offer resistance to the downward deflection of the plate subjected to vertical loading.

So, due to the upward soil pressure you will know that the deflection will be reduced. So, here the sub grade will act like a spring and this kind of model is generally known as this winkler plate model or in case of beam this is winkler beam model. Now here the boundary is not clamped, not fixed. So, here the boundary is taken as a free boundary. So, in case of free boundary you know that there is a possibility that the rigid body deformation may take place.

So, in that case we have introduced that term say w r = A, this constant A represents a rigid body displacement. But you also mind that rigid body displacement will not contribute to the strain and hence there will be no contribution of strain energy due to rigid body motion.

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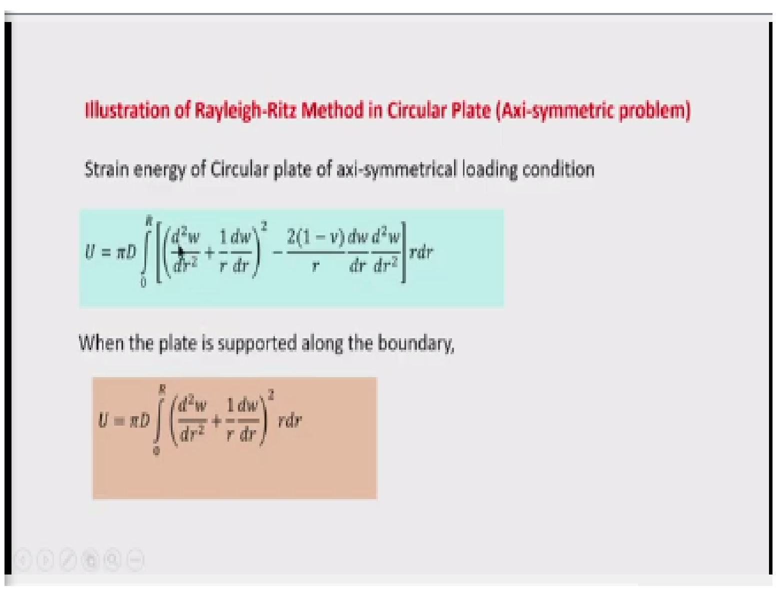

This is obvious from that expression that the strain energy expression that you have see here contains the second derivative as well as first derivative. So, if I take a term like that deflection as $w(r) = A + Br^2$ then first derivative as well as secondary derivative of the constant A will get 0. So, the rigid body displacement if it is there it will not contribute to the strain energy of the expression.

However, due to free boundary condition, we take the rigid body displacement term as in the deflection surface expression. So, we assume the deflection surface expression with 2 term series that is 2 term polynomial you can tell in R. Because in axi-symmetrical problem the deflections are functions of only radial distance, so it will be independent of the angular position theta. So, if it is taken the deflection function and the loading in the plate is concentrated load at the centre.

So, out of the 2 choices of the strain energy we cannot adopt this, this is not possible because the plate is not supported along the edges. So, we have to go to the first one, so first expression will be used to compute the strain energy of the plate.

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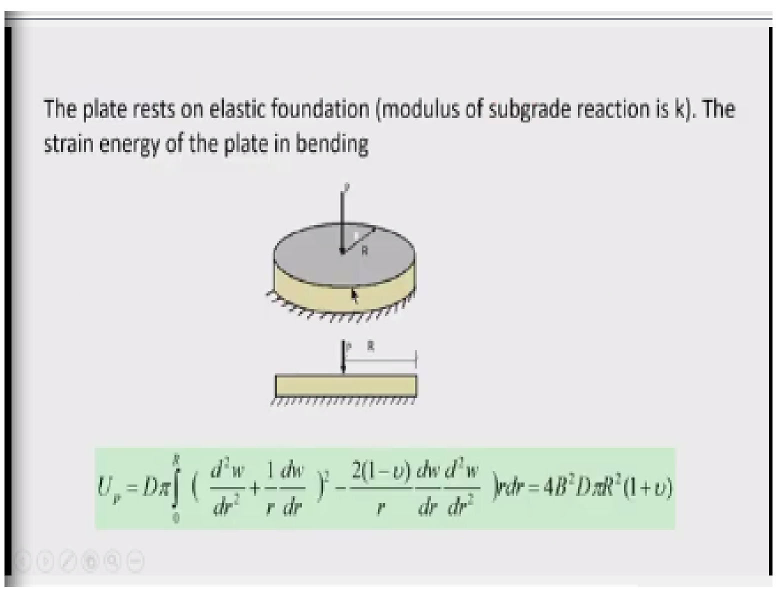

So, taking this expression and this first expression the strain energy is calculated for the plate and it is found as $4B^2 D\pi R^2 (1+\upsilon)$. Now here also you can note that the deflection term function that contains the rigid body displacement A which appears as a constant. It does not make it is presence here, so the strain energy due to bending of plate contains only the expression that is the second constant B.

So, strain energy expression of the plate is found by integration of this expression with respect to R. Now, next question comes whether there will be any contribution of the strain energy of the elastic sub grade? Yes, because the elastic sub grade acts like a spring and the work done in the spring will be stored as a strain energy.

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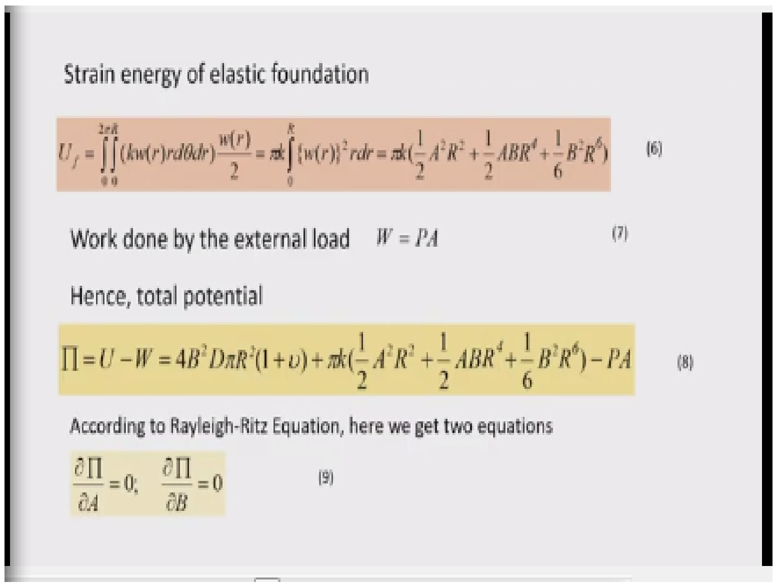

So, strain energy of this spring foundation is calculated. That is if you can see here the area of the slab on which the spring force acts is nothing but $rd\theta dr$. So, this is the area on which the spring force act. And spring force is nothing but this $kw(r)$. So, $kw(r)rd\theta dr$ is the spring force. So, spring force multiplied by deflection and it is that is half we take a factor half.

And then integrating with respect to dr and dΘ, dΘ integration is from limit is 0 to 2π and the integration with respect to r, radial distance has limit the lower limit is 0 and upper limit is R.

After integration of this expression we get the strain energy stored in the elastic foundation that is elastic sub grade = $\pi k(\frac{1}{2}A^2R^2 + \frac{1}{2}ABa^4 + \frac{1}{6}B^2R^6)$

So, here the strain energy expression U_f will contains this A square term A^2 R^2, where A^2 is coming from with deflection surface that we have introduced the one rigid body displacement that is a constant A. So, the strain energy of the foundation contains this term $\frac{1}{2}ABa^4 + \frac{1}{6}B^2R^6$. So, that expression now have to be added to the strain energy of the plate.

So, strain energy of the plate is this, next we had the strain energy of the foundation to get the total strain energy of the plate and foundation. Now we calculate the work done by the external load, you can see that the load is applied at the centre, so deflection at the centre is only to be used for calculating the load. So, deflection at the centre according to our assumed expression if we put r = 0 here deflection at the centre is A. so, now using the expression that W is work done = load into displacement and the displacement the load does not change during the deformation. So, therefore the work done = P × A.

So, then total potential will be Π = U - W and U = $4B^2D\pi a^2(1+\upsilon)$ this term is the contribution of the strain energy of the plate plus this $\pi k(\frac{1}{2}A^2R^2 + \frac{1}{2}ABa^4 + \frac{1}{6}B^2R^6) - PA$ is the contribution of the elastic foundation. You can see the first term this $4B^2D\pi a^2(1+\upsilon)$, υ is the Poisson's ratio is the contribution of the strain energy of the plate.

And the second term $\pi k(\frac{1}{2}A^2R^2 + \frac{1}{2}ABa^4 + \frac{1}{6}B^2R^6$ is the contribution of the elastic foundation minus work done P×A. Now apply Rayleigh-Ritz method, so if I apply Rayleigh-Ritz method there are 2 constants involved, one is capital A and another is capital B.

By differentiating this total potential with respect to A and with respect to B separately, So,

$$\frac{\partial \Pi}{\partial A} = 0; \quad \frac{\partial \Pi}{\partial B} = 0$$

we get 2 equations. And this will be linear equations simultaneous equation, so 2 linear simultaneous equation have to be solved for finding the A and B which requires to completely know the deflection function.

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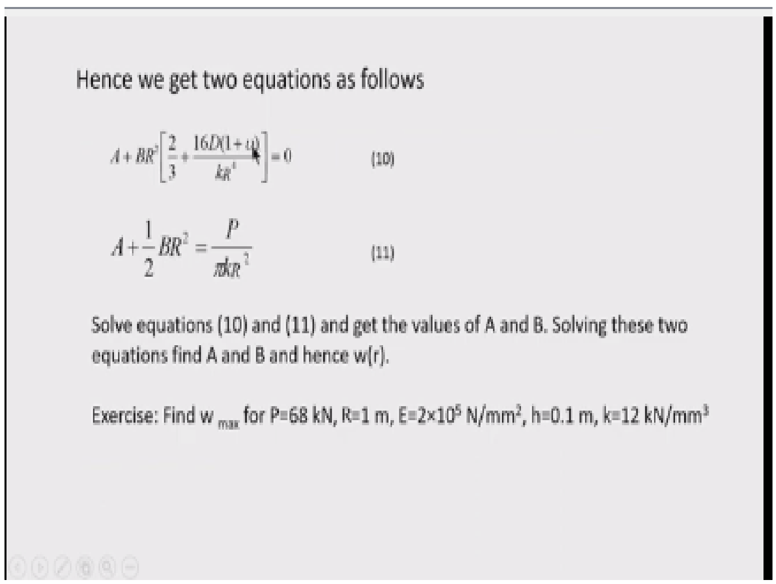

So, now after doing this operation the 2 linear equation that we are arriving here is

$$A + Ba^2\left[\frac{2}{3} + \frac{16D(1+\upsilon)}{kR^4}\right] = 0$$

Then second equation is

$$A + \frac{1}{2}Ba^2 = \frac{P}{\pi kR^2}$$

So, these 2 equation can be solved and A and B are found out which can be substituted in the expression for the deflection to calculate the deflection any point on the plate.

So, this will be approximate deflection of course, because it may not satisfy the boundary conditions at the edges which are very difficult boundary condition that is the edge moment as well as edge shear, both has to be 0 because it is the free end. Now as a exercise you can calculate the numerical value of this plate problem, if you for calculating the maximum deflection.

Maximum deflection of course will be at the centre because it is symmetrically loaded and there is no support actually, so it is a symmetrically loaded and it is resting on the elastic sub grade. So, it is supported by the elastic sub grade but at the edge there is no support. So, here if you take say P = 68 kN and the radius of the plate as 1 meter, modulus of elasticity say it is a steel plate is 2 into 10 to the power 5 Newton per mm square, thickness of the plate can be taken as say 0.1 meter, that is the 100 mm.

And then get k the modulus of sub grade reaction is taken as 12 kilo Newton per millimeter cube. So, by adopting these data you can find the deflection of the plate.

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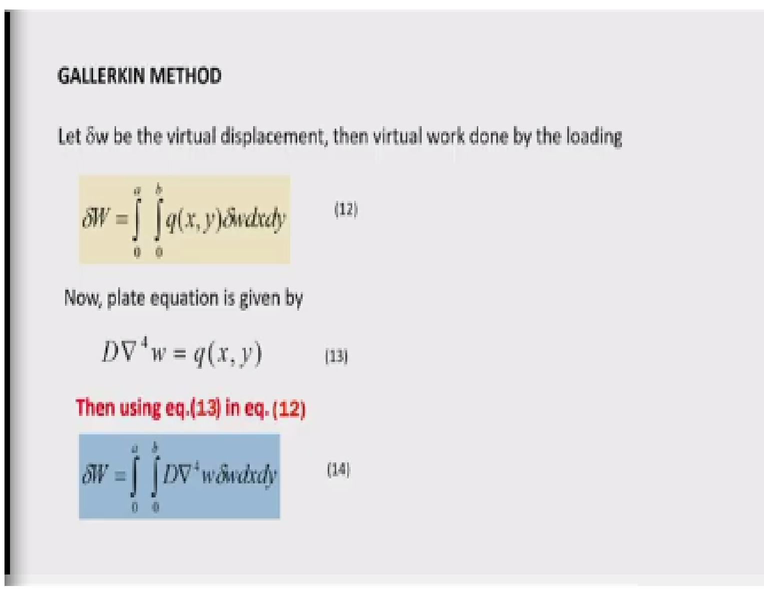

Now let us illustrate another method which is also derived from the work energy principle and that is due to Gallerkin and therefore this method is popularly known as Gallerkin method. Now in this method let δW , where w is the deflection of the plate and del w is the variation of the displacement or you can call it virtual displacement. So, this quantity is the virtual displacement δW is the virtual displacement of the plate.

Then virtual work done by the loading is δW = the double integration, double integration is required because the plate is extended in x as well as in y direction. And $q(x, y)$ is the load applied on the plate and δW is the virtual displacement. So, load into virtual displacement and then integrated in the limit 0 to n 0 to b will give you the variation of the work, that is you can call it the virtual world.

Now the plate equation is known to us, plate equation is D, here D is the flexural rigidity of the plate, $D\nabla^4 = q(x, y)$. Now here you can see that ∇^4 is the differential operator, which contains 3 terms, that is $\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$ Now you see this expression that is $D\nabla^4 w = q(x, y)$. So, in place of q x y we now put D del 4 w and we get again the virtual work expression as $\delta W = \int_0^a \int_0^b D\nabla^4 w \delta w dx dy$ ……..(14). Here you can see equation 12 and 14 are same because both has the expression for the virtual work δW.

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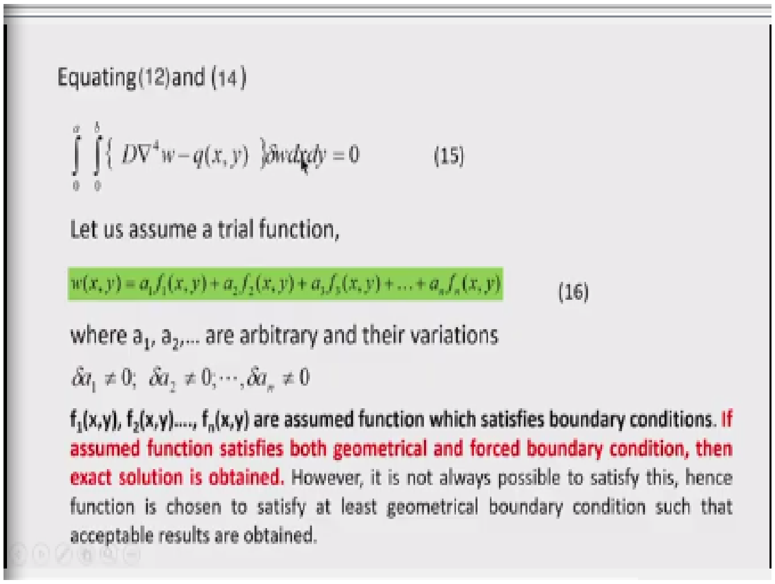

So, now equating 12 and 4, we can now write this expression $\int_0^a \int_0^b \{ D\nabla^4 w - q(x,y) \} \delta w dxdy$

That means this expression and that expressions are now equated, so after equating we can write this δW equal to this equal to this. So, we can now take this common term, common term you can see $\delta w dxdy$ and other terms will be this your $D\nabla^4 w - q(x,y)$ and this is the common term, so it is equated to 0.

So, now let us assume a trial function

$w(x,y) = a_1 f_1(x,y) + a_2 f_2(x,y) + a_3 f_3(x,y) + \ldots + a_n f_n(x,y)$.

So, this is the assumed function for the deflection of the plate. So, this should be taken in such a way that the boundary conditions are satisfied. It is not necessary that differential equations should be satisfied by the trail function but at least the geometrical boundary condition must be satisfied.

If it satisfies both geometrical and force boundary condition, then exact solution can be obtained. Now in this expression, a_1, a_2, a_3 etc., are the arbitrary constants and $\delta a_1;\ \delta a_2; \cdots, \delta a_n$ are their arbitrary variation and these are nonzero quantities. So, now substituting this, then we can now write the expression of δW variation of the displacement δW will be $f_1(x, y)$ δa_1 + $f_2(x, y)$ δa_2 + ………like that $f_n(x, y)$ δa_n. So, if I substitute this expression here after taking the variation of this displacement.

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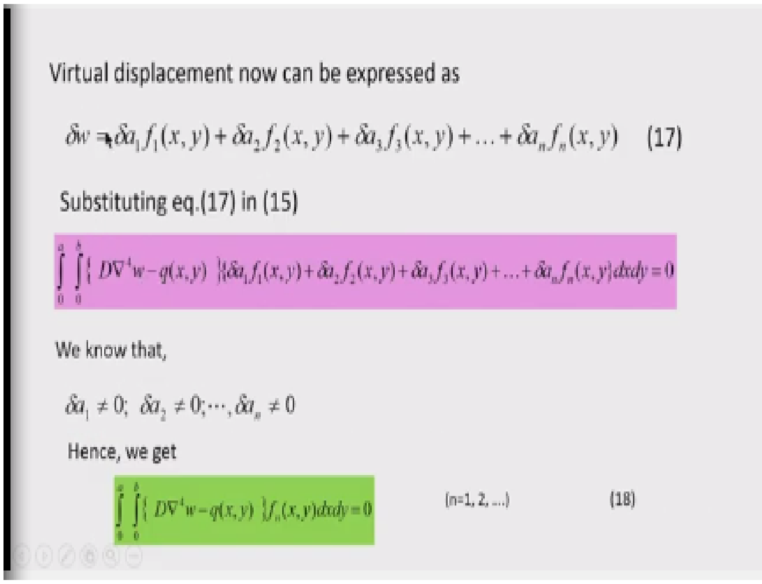

Then we write this expression in this form, that in place of del w in the original expression this is now substituted and then integration is done with respect to dxdy. Now you can see that if I break the terms that means if I multiply term by term and then interpret the results of the multiplication of the left hand side. Then we can get there since δa_1 δa_2 etc., are nonzero quantities. So, therefore we must get $D\nabla^4 w - q(x, y) f_1(x, y) = 0$ Then again we will get $D\nabla^4 w - q(x, y) f_2(x, y) = 0$, then again we will get $D\nabla^4 w - q(x, y) f_3(x, y) = 0$, of course integration have to be done all other expressions integration = 0, like that will get.

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Since w(x,y) is approximate which is chosen as trial function, $\{\nabla^4 w - q(x,y\}$ will not be equal to zero, instead we can write

$$\{\nabla^4 w - q(x,y\} = \epsilon_r \qquad (19)$$

where right hand side of eq.(19) represent error in solution due to assumed trial function. Hence equation (18) can be written as

$$\int_0^a \int_0^b \epsilon_r f_n(x,y)dxdy = 0 \qquad (20)$$

Writing eq.(20) for each n, i,e n=1, 2,...etc.

$$\int_0^a \int_0^b \epsilon_r f_1(x,y)dxdy = 0$$
$$\int_0^a \int_0^b \epsilon_r f_2(x,y)dxdy = 0$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \qquad (21)$$
$$\int_0^a \int_0^b \epsilon_r f_n(x,y)dxdy = 0$$

So, for any term a general term say n, n number of terms we get this expression

$$\int_0^a \int_0^b \{ \ D\nabla^4 w - q(x,y) \ \} f_n(x,y)dxdy = 0$$

So, n here varies from 1 to up to infinity but infinite number of terms cannot be taken in practical applications. So, 1 has to truncate the series up to the limited number of terms. Since w (x,y) is approximate which is chosen as the trail function. So, $\{\nabla^4 w - q(x,y\}$ will not be 0.

Instead, it will show an error because the w is not a exact function. It cannot satisfy the differential equation. So, therefore

$$.\{\nabla^4 w - q(x,y\} = \epsilon_r$$

Some error term is given here. So, Gallerkin equation now can be written as error (ε_r) into assumed function and it is integration over the area of the plate is equal to 0. So, that is the final expression for the Gallerkin method and this integral has to be found for all the function in the series, all the terms in the series.

So, suppose we have taken a term, say polynomial term, say 1st term is say x, 2nd term is a x^2, 3rd term we say x^3 like that. So, that means f_1 is x, f_2 is x^2, f_3 is x^3 like that. So, this Gallerkin equation is from like that

$$\int_0^a \int_0^b \in_r f_1(x, y)dxdy = 0$$

$$\int_0^a \int_0^b \in_r f_2(x, y)dxdy = 0$$

So, n number of equation can be formed using this the Gallerkin principle and you will get in each equation. These will be a linear equation, simultaneous equation with unknowns coefficient a_1, a_2, a_3 etc.. By solving these equations for unknown coefficients a_1, a_2, a_3 we can now obtain the deflection surface completely, so this is the Gallerkin method.

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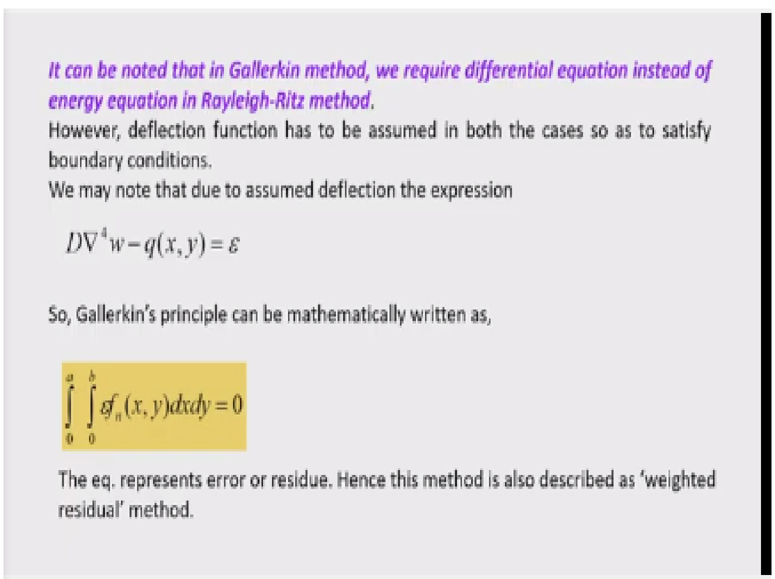

So, it can be noted that the difference between the Gallerkin method and Rayleigh-Ritz method is that. In Gallerkin method we require differential equation instead of strain energy expression. Now 2 methods have their origin from the strain energy the principle, but the procedure of the method will differ. Because in one case, say for example Rayleigh-Ritz method, we start with the strain energy expression of the plate or beam whatever may be?

But here, instead of strain energy expression we required this differential equation of the plate. And we substitute a assumed trial function which is not the actual deflected surface of the plate. So, therefore we get an error, this error multiplied by the assumed function and integrated over the domain of the plate equated to 0 will give the Gallerkin equation

$$\int_0^a \int_0^b \varepsilon f_n(x,y)\,dxdy = 0$$

which can be solved to find the unknown value of the constants that is used to form the deflected surface.

But one common thing is that in Gallerkin and Rayleigh-Ritz method the deflection function that we have to assume should satisfy the boundary conditions, geometrical boundary condition as well as force boundary condition. If both the boundary conditions are satisfied the results will be accurate. But if not, at least geometrical boundary condition must be satisfied otherwise the results will be erroneous and may not be acceptable in practical applications.

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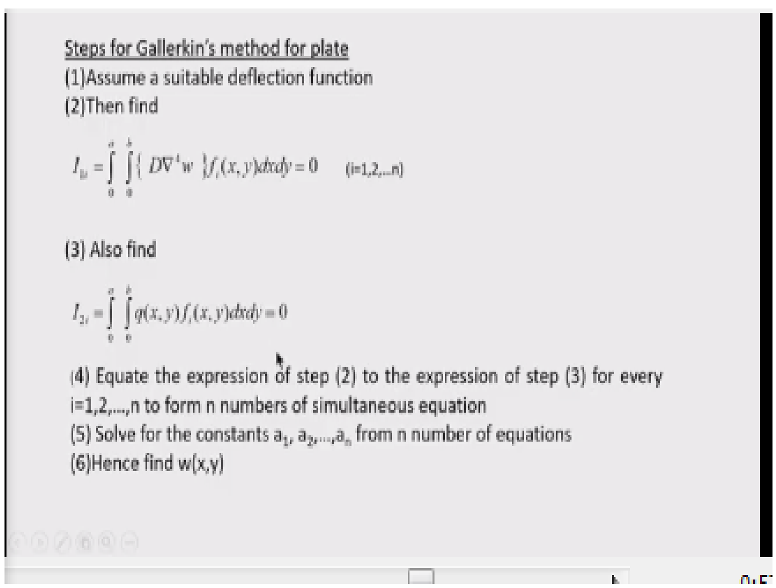

So, let us discuss the steps of the Gallerkin method. I will give you the systematic procedure how to be adopted in Gallerkin method for solving the plate problem? So, first step is assume a suitable deflection function. Deflection function can be assumed with a trigonometrical series, deflection function can be assumed with a polynomial function. It depends on the choice of the analyst or it can be assumed as a combination of this.

$$I_{1i} = \int_0^a \int_0^b \{ \ D\nabla^4 w \ \} f_i(x,y) dxdy = 0$$

D is the flexural rigidity of the plate into $\nabla^4 w$, ∇^4 is the operator differential bi harmonic differential operator, w is the assumed deflection function multiplied by $f_i(x,y)$. So, f_i is one of the function that is used to represent the deflected surface in this series. So, i here varies from 1, 2, 3 etc. up to n. So, for example if we are solving a simply supported plate, so we are generally taking a function for plate which has all the edges simply supported as double trigonometrical series.

For example, first term is say $a_{11} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$, second term will be a_{11} a_{12} $\sin\frac{\pi x}{a} \times \sin\frac{2\pi y}{b}$, like that the terms can be composed. So, here you can see when I use the Gallerkin's equation, then here $f_i(x,y)$, that functions of x y, we will use only the terms which does not contain a constant. That means, in the first term say $a_{11} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$, a_{11} should not be used here.

In this function only the function that is used to compose the deflected surface has to be used. And of course here in place of w we use the full expression of the deflected surface. Then this integral is calculated and it is denoted as I_{1i}. And these small i varies from 1 to n, that is number of terms that we have taken in the deflected surface. In the third step we find

$I_{2i} = \int_0^a \int_0^b q(x,y) f_i(x,y)\,dxdy = 0$, this integration is carried out, and it is denoted as I_{2i} equate the expression of step 2 to the expression of step 3.

For every i want to varying up to n, to form n number of simultaneous equation and this equation again it will be a linear equation. And solving these linear equations we will now get the constants a_1, a_2, etc. and then deflected surface can be composed of.

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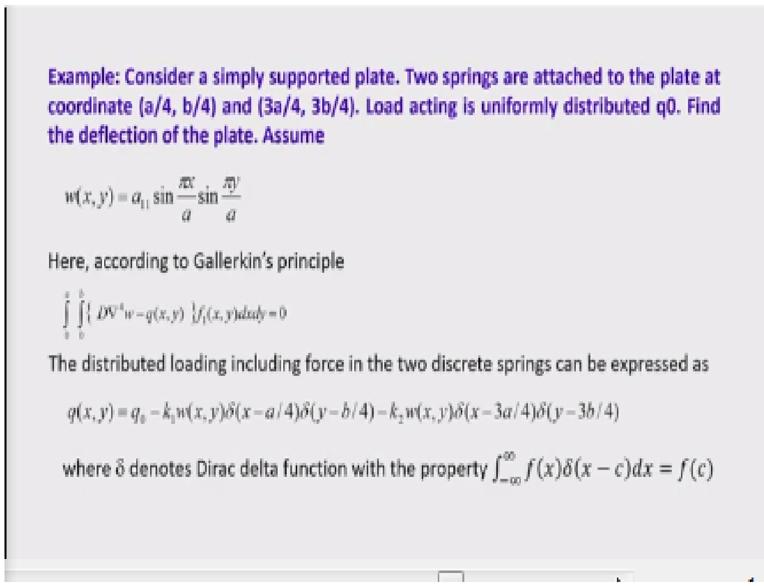

Let us give one example of plate with the help of Gallerkin method. Considered a simply supported plate, simply supported plate means here I mean that all the 4 edges are simply supported, rectangular plate. The length is a and the breadth is b, length is in the x direction and breadth is in the y direction. Now the plate is subjected to uniformly distributed load q_o, load acting in the plate is uniformly distributed q_o.

In addition 2 springs are attached at the discrete point, that points are denoted by the coordinate $a/4, b/4$. So, there is 1 spring attached to the plate at a point x coordinate $a/4$ and y coordinate $b/4$. And another spring is attached to a point which has coordinate $3a/4, 3b/4$. The coordinates are measured with respect to origin which is taken at the top left hand corner of the plate.

Now, let us for simplicity, we assume a one term deflected series.

$$w(x,y) = a_{11} \sin\frac{\pi x}{a} \sin\frac{\pi y}{a}$$

So, for simply supported plate we can see that Navier series can represent the deflection surface truly, provided the a_{11} is correctly evaluated. But these functions $\sin\frac{\pi x}{a} \sin\frac{\pi y}{a}$ completely satisfy the boundary condition of the plate. That is the deflection at the edges is 0 and the bending moment at the edges is also 0.

Now here according to Gallerkin principle, we can now write

$$\int_0^a \int_0^b \{ \ D\nabla^4 w - q(x,y) \ \} f_1(x,y) dxdy = 0$$

Now here we have this first integration that will carry out $k_1 w(x,y)$ that integration is the first integration I_1. Now before that let us show what is $q_{(x,y)}$? $q_{(x,y)}$ is the distributed loading on the plate, now distributed loading on the plate is given as q_o, there is ok uniformly distributed.

But there is also 2 spring setters at certain points. Now spring offers a upward reaction to the downward deflection of the plate and that reactive force of the spring is given by the stiffness of the spring multiplied by the deflection. Say first spring that is attached at the point $a/4, b/4$ as a spring constant k_1 and the second spring which is attached at a point $3a/4, 3b/4$ has a spring constant k_2.

$$q(x,y) = q_0 - k_1 w(x,y)\delta(x-a/4)\delta(y-b/4) - k_2 w(x,y)\delta(x-3a/4)\delta(y-3b/4)$$

So, resistance offered by the spring at this point $a/4, b/4$ will be

$k_1 w(x,y)\delta(x-a/4)\delta(y-b/4)$. Now the discrete force I have represented here with the help of direct delta function. Because the location of the spring is at $a/4, b/4$, so I use the direct delta function with arguments $x-a/4$ for x coordinate and another location for Y coordinate, I use the direct delta function with the argument $y-b/4$

Similarly for the second spring that is attached at the point $3a/4, 3b/4$, the upward force or upward resistance, resisting force offered by the spring is $k_2 w(x,y)$, that is the deflected surface. But it is evaluated at this point and it is presented now mathematically with the help of direct delta function. So, direct delta function with arguments $x-3a/4$ into direct delta function with argument $y-3b/4$. Now, so this is the loading that is expressed on the spring.

The property of direct delta function here can be used to simplify the calculation. The property of direct delta function is known to us that if suppose a function f_x is multiplied by a direct delta function x - c with argument x - c. And it is integrated within a limit say -infinity to +infinity, any limit. So, the value of the integral will be only the function evaluated at which the direct delta function is defined. So, direct delta function is defined at the point only at c that is the point. So therefore, the value of the integral will be the function evaluated at c.

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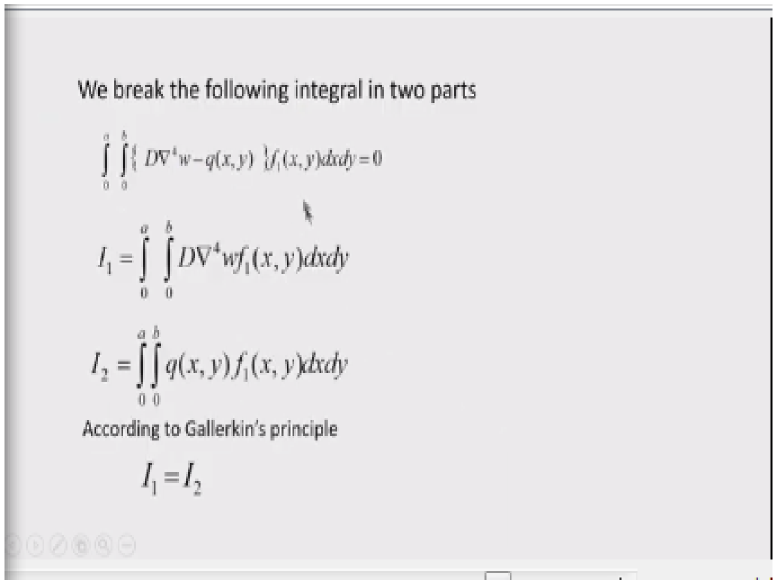

So, we break the following integral into parts. First part is this, this is the integral. So, first part is

$$I_1 = \int_0^a \int_0^b D\nabla^4 w f_1(x,y)dxdy$$

Second integral is

$$I_2 = \int_0^a \int_0^b q(x,y) f_1(x,y)dxdy$$

Because there is only one function is chosen, so I write as a function as f_1. According to Gallerkin principle, now I equate $I_1 = I_2$.

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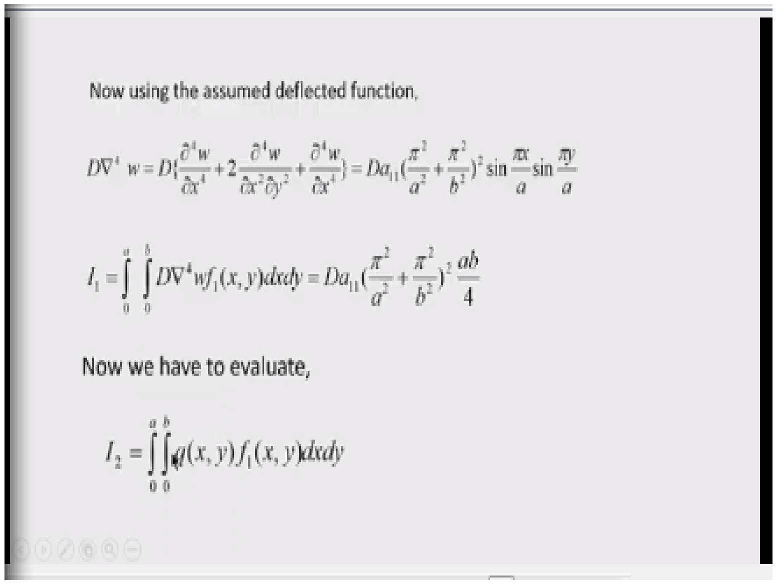

And then the first integral becomes this, $I_1 = \int_0^a \int_0^b D\nabla^4 w f_1(x,y)dxdy = Da_{11}(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2})^2 \frac{ab}{4}$

This is the standard expression that we know. The second integral now has to be evaluated with the help of direct delta function; the evaluation will be very simplified.

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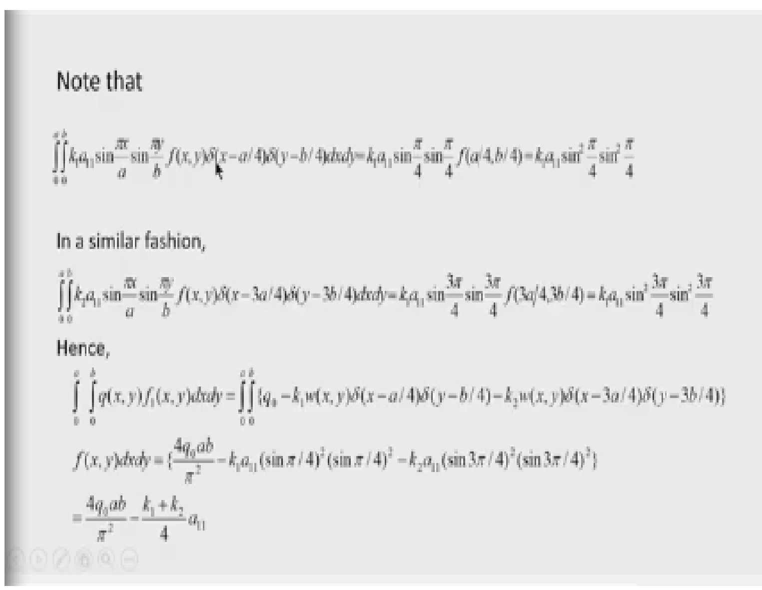

So, with the help of direct delta function, you can see the integration is carried out very conveniently

$$\int_0^a\int_0^b k_1 a_{11}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}f(x,y)\delta(x-a/4)\delta(y-b/4)dxdy = k_1 a_{11}\sin\frac{\pi}{4}\sin\frac{\pi}{4}f(a/4,b/4) = k_1 a_{11}\sin^2\frac{\pi}{4}\sin^2\frac{\pi}{4}$$

I is seen in the equation that $k_1 a_{11}\sin^2\frac{\pi}{4}\sin^2\frac{\pi}{4}$ because the x y that has to be substituted as $a/4$, so it becomes $\sin\frac{\pi}{4}$. And here also y is also $b/4$, so it is again $\sin\frac{\pi}{4}$. Similarly, the other function will also be calculated with the quantity $3a/4$ and $3b/4$.

So, therefore we get this in a similar fashion for 2 springs k_1 and k_2. First we have evaluated for K_1, then in the second case the evaluation is done for the second spring force k_2. So, we get this term and then using in the Gallerkin final equation $I_1 = I_2$, we now substitute all these quantities.

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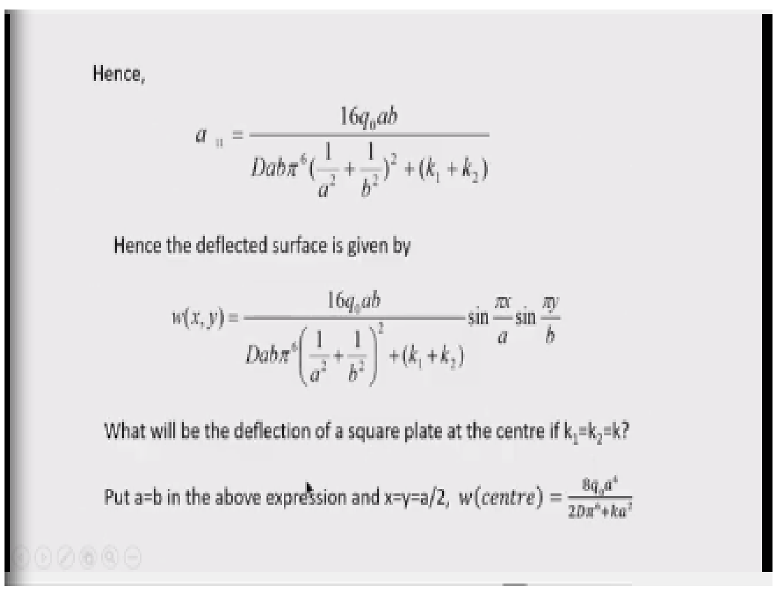

And then we get

$$a_{11} = \frac{16q_0ab}{Dab\pi^6(\frac{1}{a^2}+\frac{1}{b^2})^2+(k_1+k_2)}$$

Now you can see how this quantity a_{11} comes? After evaluating the I_2, we get this term

$$\frac{4q_0ab}{\pi^2}-\frac{k_1+k_2}{4}a_{11}$$

Now you can note here that this term, the first term is the contribution due to uniformly distributed load and this term is due to contribution of the spring forces.

And the first term that we got $=\frac{8q_oa^4}{2D\pi^6+ka^2}$

Equating I_1 and I_2 and then finding this a_{11} that is the only unknown in this expression, we get this quantity.

$$a_{11}=\frac{16q_0ab}{Dab\pi^6(\frac{1}{a^2}+\frac{1}{b^2})^2+(k_1+k_2)}$$

Now you can note here, this expression can be verified easily. Because when there was no spring and the plate is like a rectangular plate, simply supported along all edges subjected to only uniformly distributed load. So, in that case results are well known.

So, if these springs are absent then we can take $k_1 = k_2 = 0$. So, the expression coincides with the expression that we have obtained in case of a rectangular plate subjected to uniformly distributed load and has simply supported edges. So, deflected surface of the plate now can be written in this form

$$w(x,y)=\frac{16q_0ab}{Dab\pi^6\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2+(k_1+k_2)}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

with the variable $\text{in}\frac{\pi x}{a}\sin\frac{\pi y}{b}$. Now if these 2 springs are say of same stiffness the spring constants are same. Then at the centre if we are interested to calculate the deflection at the centre, the centre coordinate is x = y = a / 2, and w at the centre will be equal to

$$\frac{8q0a4}{2D\pi6+ka2} = \frac{8q_o a^4}{2D\pi^6+ka^2}$$

So, this is the maximum deflection of the plate in that case.

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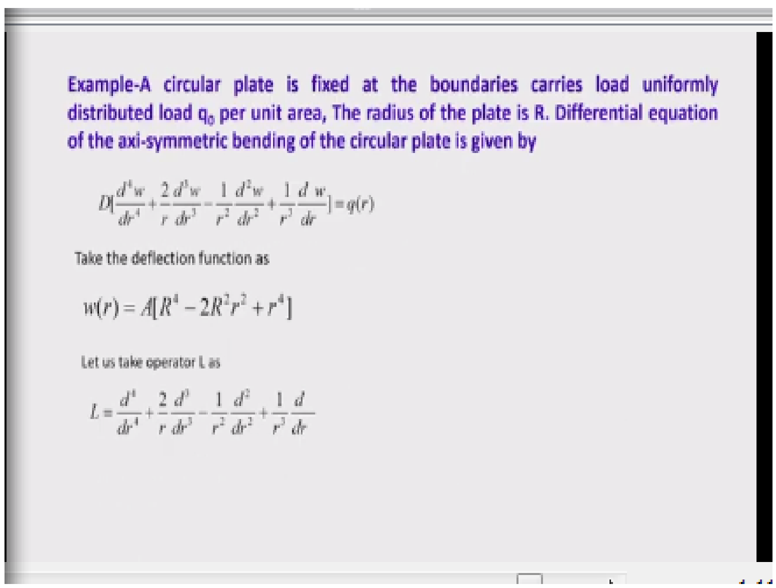

Now a circular plate is fixed at the boundaries carries load uniformly distributed over the area. The radius of the plate is R and differential equation of the axi-symmetrical bending of the circular plate is given by this. So, this expression is found from the differential equation of the circular plate where this operator you can isolate. You can see this differential operator L is

$$L=\frac{d^4}{dr^4}+\frac{2}{r}\frac{d^3}{dr^3}-\frac{1}{r^2}\frac{d^2}{dr^2}+\frac{1}{r^3}\frac{d}{dr}$$

So, once you know this operator and the deflection surface of the fixed plate, circular plate can be assumed like that. You can verify that this expression can represent the deflection surface because it satisfy the clamped edge condition. At the clamped edge the w is 0, that is if R is the radius of the plate, by substituting r as R, you will get w at boundary 0.

Again if you differentiate it and then the substitute r = R that is at the edges the slope will be 0. So, 2 conditions are satisfied easily, so therefore the function $w(r) = A[R^4 - 2R^2r^2 + r^4]$ can represent the deflected surface of the circular plate clamped along the boundaries. So, we take the uniformly distributed load on the plate as q_o .

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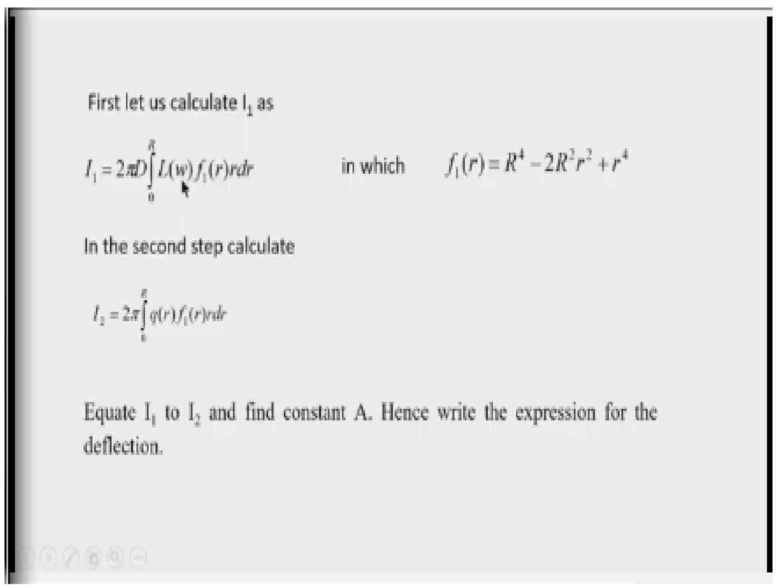

And then I_1 is calculated, $$I_1 = 2\pi D\int_0^R L(w) f_1(r) rdr$$

Here the differential operator L w is taken to write the expression in compact $f_1(r)$ is the deflection function which is $f_1(r) = R^4 - 2R^2r^2 + r^4$

As I have told you earlier the deflection function here you have to take or in other cases also rectangular plate that you have to take into calculation without the constants, arbitrary constant that is associated.

In the second step, calculate I_2 that is with the load,

$$I_2 = 2\pi \int_0^R q(r) f_1(r) r$$

so load is $q(r)$ and deflection function is the $f_1(r)\,\mathrm{dr}$. Of course here $q(r)$ is constant is equal to q_o. So, if 2 integrations are carried out and then equated both the expressions are equated I_1 is equated to I_2, then we can find the unknown constant A. Once the unknown constant is found then deflection is completely known.

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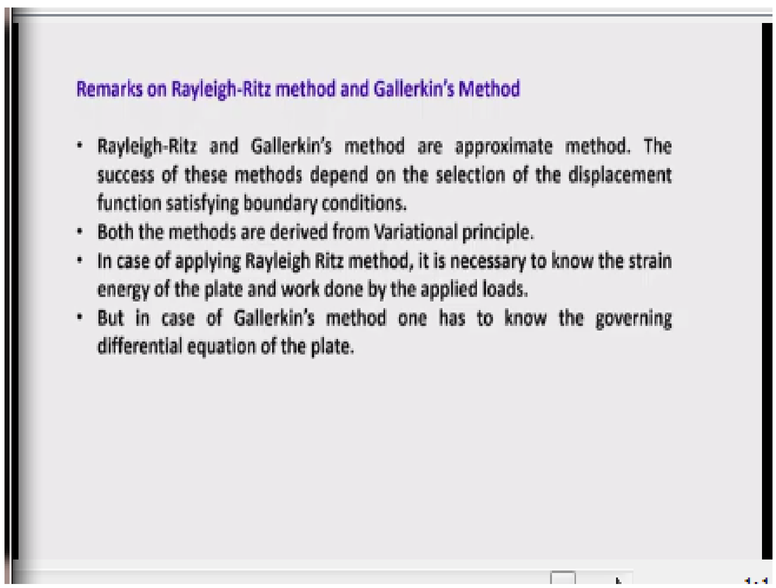

So, let us compare the Rayleigh-Ritz method and Gallerkin method. Rayleigh-Ritz and Gallerkin method are a approximate method, both are approximate) method, it cannot give the exact result. But it can give a result which is very close to the exact result. If the deflection function is chosen accurately or satisfying the boundary condition, the exact solution can be only obtained in some limited cases but with the help of approximate method that is the Rayleigh-Ritz or Gallerkin method.

We can attempt the problem of plate with any type of boundary conditions and loading. So, that is the beauty of the approximate method. For practical purpose the solution can be adopted. The success of these methods, however depend on the selection of the displacement function satisfying the boundary conditions. Both the methods are derived from variational principle, so that is the similarity is there in these 2 methods.

But in case of Rayleigh-Ritz method we required to know the strain energy expression and the work done by the applied load. But in case of Gallerkin method, we must know the governing differential equation of the plate. You can see even both the methods are derived from the fundamental principle of work and energy. But approach becomes different because in one case you require the strain energy expression in Rayleigh-Ritz approach.

In another case, the Gallerkin method you require the governing differential equation. So, if the governing differential equation of the system is known, we can choose the Gallerkin method. But when the strain or energy expression is known, we can choose the Rayleigh-Ritz method. So, let me summarize the lecture that I have delivered today.

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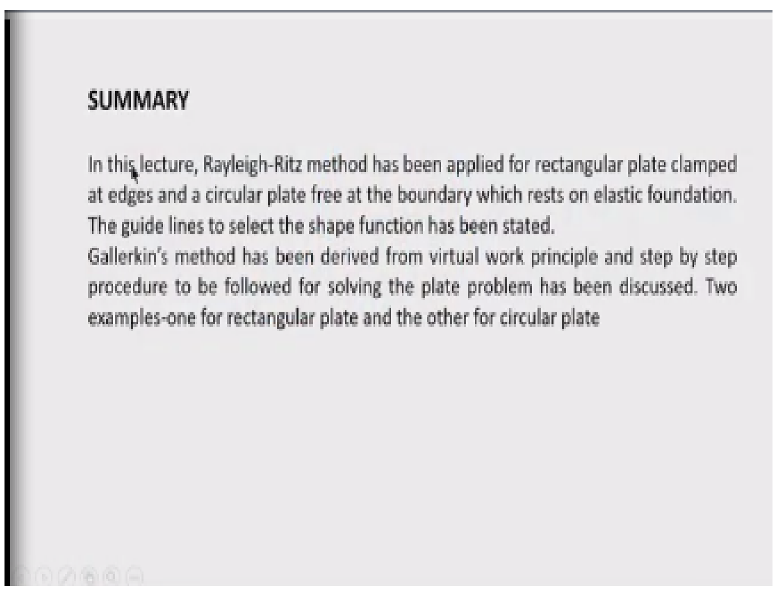

In this lecture, Rayleigh-Ritz method has been applied for rectangular plate clamped at edges and circular plate free at the boundary which rests on elastic foundation. The guidelines to select the shape function has been stated or discussed. Gallerkin method has been derived from virtual work principle. And step by step procedure to be followed for solving the plate problem has been discussed. Two examples, one for rectangular plate and the other for circular plate has been discussed in our lecture. Thank you very much.