Plates and Shells
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Module-05
Lecture-14
Variational Principle in Plate Problem

Hello everybody, today I am starting module 5 of the course plates and shells. So, today is the first lecture of the module 5. And in this module actually, we will start the formulation of the plate problem using some approximate method, approximate methods gives satisfactory result if it is used with caution and this result can be acceptable in practical application. Now, let us see what are the different approximate methods that we want to use in plate problem.

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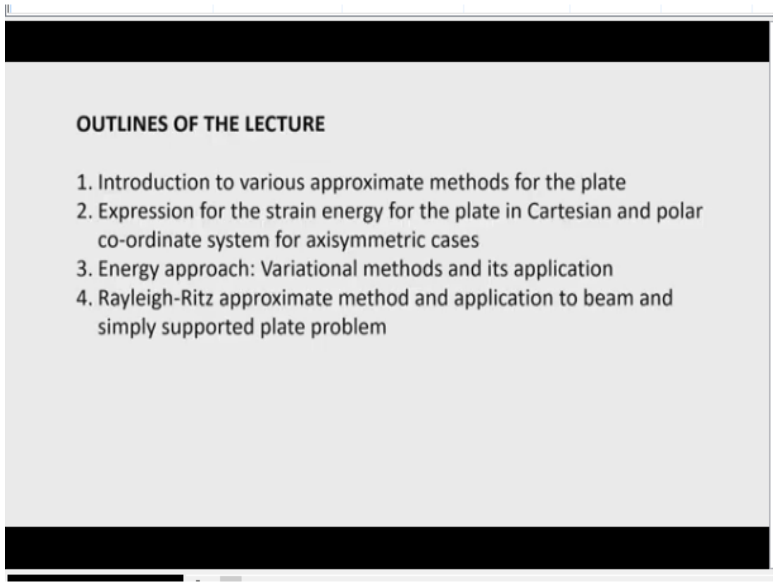

So, today we will discuss the first introduction to various approximate methods for the plate, expression for the strain energy for the plate in Cartesian and polar coordinate system for axisymmetrical cases. Now, first, I will enlist different types of approximate method, and then I will focus on the first one, which required this energy expression of the plate. Then I will discuss the energy approach that is called the variational methods and its application.

Now, based on the energy principle Rayleigh-Ritz approximate method is very popular in case of the mechanics' problem encountered not only in plate in other fields also. So, that Rayleigh's method will be discussed with the necessary derivations and then applications to beam and simply supported plate problem. First, I will show how this method can be applied to a beam because plate is an extension of the beam theory. So, if you know the application of few understand the Rayleigh's method and its use in the beam problems, then it will be easier for you or the users to apply this in a two-dimensional problem like plate.

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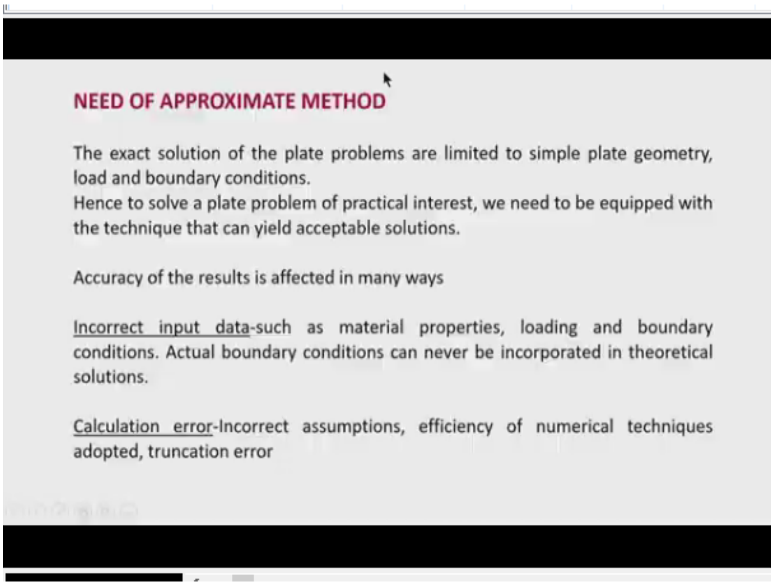

So, first, let us see why we need the approximate method? The exact solution of the plate problems are limited, why it is limited because the deflection that we have found is valid for some cases simple boundary condition, that is a simply supported condition; you have seen that the plate deflections were expressed in terms of double trigonometrical series, which satisfies the simply supported boundary condition along the 4 edges of the rectangular plate.

Or in case of a rectangular plate where the two opposite edges are simply supported, then we use the Levy's solution, Levy's series that is a single trigonometrical series and then we have could find the exact solution of the differential equation, but this type of problem is not always common. For example, we may have a plate or, say, for example, a slab on which three edges are fixed, and 1 edge is free.

So, in that case, it neither falls into the Navier's category or Levy's boundary category. So, in that case, we have to take some other methods. So, therefore, the need of approximate method arises, and also you have seen that the loading condition also plays an important role because if the loading is unsymmetrical, then we could not find the solutions of the circular plate using the axisymmetrical condition for the differential equation.

So, that things are observed, and therefore, some techniques are required to be developed for solving problems, which is different from the condition that we are encountered in case of exact methods; exact methods, of course, gives you the exact result that means, error is negligible, but again it depends on how you formulate the problem. For example, the truncation error that is one important thing you have seen in case of this series solution, where you can truncate the series, say 3 terms or 5 terms or even in 1 term.

So, in that case you have seen that the solution, the answer differs. Difference may be negligible, but this truncation error is also common. So, therefore, need arises to solve or to find an approximate method which can take the errors in boundary conditions or boundary conditions that are different from the ideal situation, and loading is also different in ideal situation.

So, accuracy of the results in approximate method is very important factor because you have to use the approximate method with caution. So, what are the precautions that you have to take? One precaution is that input data should be as realistic as possible. So, incorrect input data such as material properties that we require in plate problem, which are usually the modulus of elasticity of the material and the Poisson's ratio of the material.

So, for steel plate, the modulus of elasticity or Poisson's ratios are almost homogeneous quantity, but concrete is a heterogeneous material, and when this concrete slab is cast, and the gate of concrete may be specified something, but in real situation when the concrete strength is tested, it may not reflect the actual theoretical strength that is assumed in case of mix-design.

So, therefore, the need arises to develop approximate method which can take care of this incorrect input data and minimize the error. The actual boundary conditions can never be incorporated in the theoretical solutions. So, that is also another source of error in the approximate method, or you can tell it this is a major source in case of these analytical problems.

Then calculation errors are also involved; these others are actually due to incorrect assumptions that certain assumptions are made that deflection is small and then say, for example, these in-plane forces are neglected in case of thin plate theory, but and shear deformation is neglected, but even says some cases the thickness is required to be increased. So, therefore, the error kept up because the thin plate theory does not include the shear deformation in the equilibrium equations. So, therefore, the necessity is to use approximate method to handle the complicated situations.

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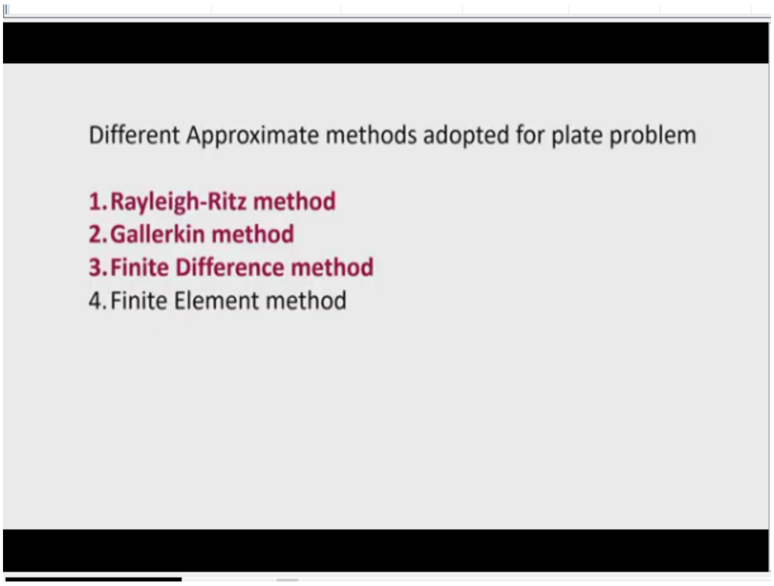

The different approximate methods that are used in plate problems are Rayleigh-Ritz method, Gallerkin method and finite difference method and lastly, finite element method. In this course, we will touch upon these first 3 methods that is Rayleigh-Ritz method, Gallerkin method and finite difference method. Finite element method is outside the purview of this course. So, first 2 methods that you are seeing, that Rayleigh-Ritz method and Gallerkin method, has had its origin from the energy principles.

Then third method is finite difference method. In finite difference method, the main thing is that we have the differential equation for the physical problem whether it is plate or beam whatever maybe beam bending or buckling up column or for example, plate bending, buckling or plate, vibration, many physical problems are there which are represented by differential equation.

Differential equations of equilibrium, static equilibrium or differential equations of equation of motion. So, in that case, when the boundary condition and loading is such that exact solution of the differential equation is not possible, then we express the differential equation in finite difference form. So, we write these in finite difference form. As a result differential equation could be converted in a algebraic equation.

So, you will appreciate that it is easier to solve a algebraic equation compared to differential equation. So, therefore, finite difference method ultimately requires the solution of algebraic equation instead of differential equation. The other 2 methods the 1 and 2 that I have listed here 1 is Rayleigh-Ritz method and another is Gallerkin method. These also, in the final step of formulation you will get that these equations to be solved for unknown parameter are all linear simultaneous equation.

So, in all the cases, we required to solve only the linear simultaneous equation for the plate problem. So, therefore, it becomes easier for the analyst to use this method for such problems which could not be handled by exact solution method approach. Finite element method of course, it is a different approach, and it also has its source from the energy. Energy principle can be used to formulate the finite element method also.

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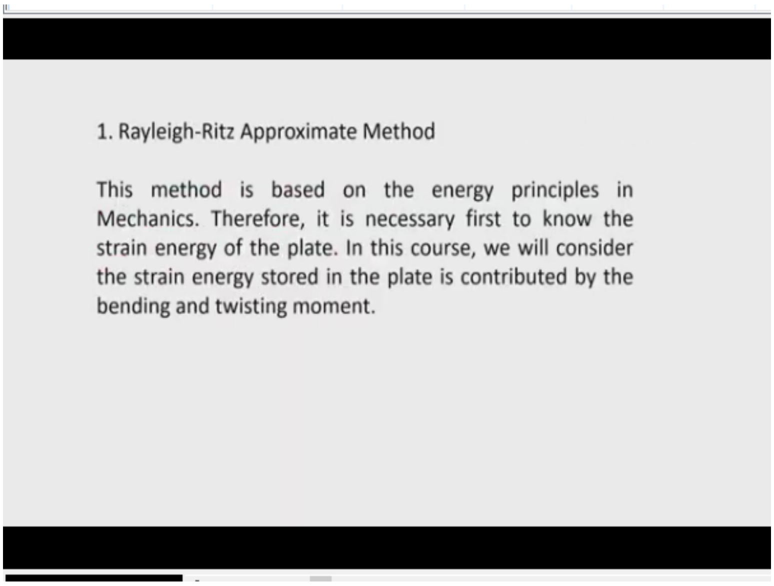

Now, let us see the first one, the Rayleigh-Ritz method, in detail. This method is based on the energy principle in mechanics. Therefore, it is necessary first to know the strain energy of the plate. In this course, we will consider this strain energy stored in the plate is mainly contributed by the bending and twisting moment. That means the contribution of in-plane forces and contribution of shear forces towards the bending strain energy are neglected.

So, that is very important thing only strain energy due to bending moment and twisting moment has to be considered in the formulation of the problem in this course. However, the Rayleigh-Ritz method does not restrict that one has to use only this strain energy for expression for bending moment and twisting moment. One can include this sharing information as well as in-plane forces also in the strain energy expressions.

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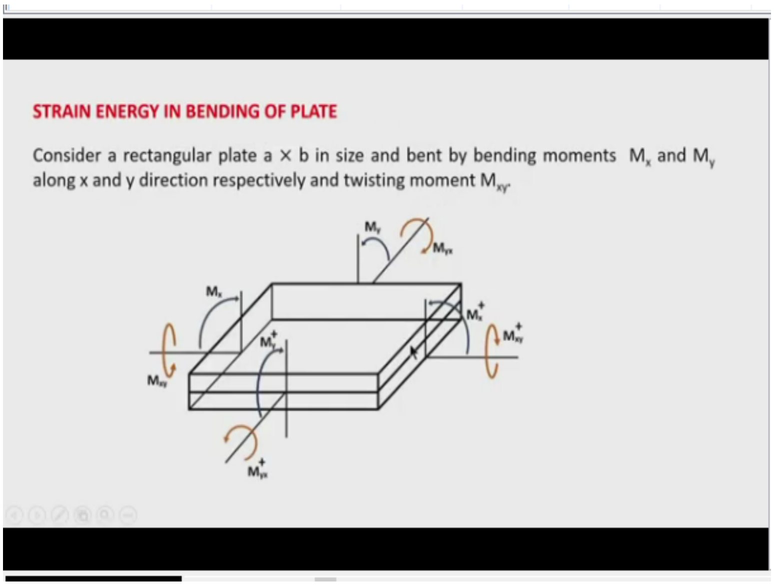

Now, let us see strain energy expression for the bending of plate. So, here you are seeing that a plate which is a dimension A, B and in that case, we take an element of the plate whose length is dx along the x-direction and dy along the y-direction. So, moment you are seeing M_x is the bending moment along the x-axis, and on the opposite side, the M_x+ M_x + increment of the moment, increment of the moment is $\frac{\partial M_x}{\partial x}dx$ along the x-direction.

Similarly, in that case, M_y is the bending moment along the y direction, in the opposite edges the bending moment will be M_y +, what does it mean M_y + means $M_y + \frac{\partial M_y}{\partial y}dy$. So, these are the increments. So, similar meaning can be given on the M_{xy} and M_{xy} +. So, plus quantity indicates the incremental quantity.

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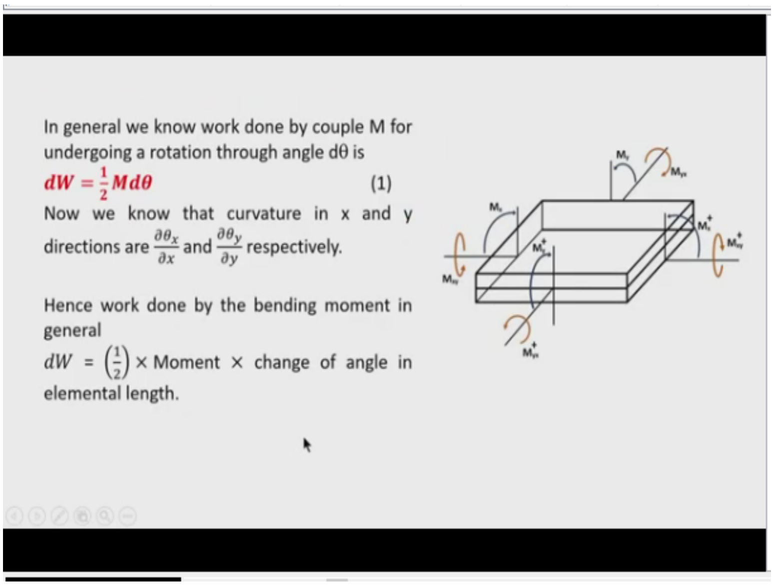

Now, let us see in general, we know that work done by the couple M. M is any couple for undergoing a rotation to an angle θ. You can see that the strain energy is nothing but by the work done by the external force, which is stored as a strain energy. So, in that case, suppose a body is deformed by a force; here, of course, the body is undergoing the rotation due to application of bending moment.

So, due to resistance of these applied bending moment that is we call it moment of resistance, the energy stored in the plate. So, this energy, if we want to formulate it, will assume that the load is applied gradually, and therefore, the work done by the couple will be $\frac{M}{2}\, d\theta$. That is the final initially say; for example, it is a 0 couple and gradually increased.

And then, when the average value is taken $\frac{M}{2}$, and within angle rotational increment of $d\theta$, we take this strain energy expression as $dW = \frac{M}{2} d\theta$, dW is nothing but the work done by the couple, which is stored as strain energy. Now, let us see the $d\theta$, here; if you see in this figure that M_x is the bending moment along the x-direction and there you will find that M_x + , that is the bending moment with increment.

So, $M_x + \frac{\partial M_x}{\partial x} dx$. So, that is the bending moment on the opposite edges. Now, similar thing happens; suppose at this point the slope is $\frac{\partial w}{\partial x}$, in the opposite edges, the slope will be $\frac{\partial w}{\partial x}$ +

increment of slope. So, increment of slope is $\frac{\partial w}{\partial x} + \frac{\partial}{\partial x}\frac{\partial w}{\partial x}dx$. So, increment of slope will be $\frac{\partial^2 w}{\partial x^2}dx$. So, with that condition, we can now say the work done by the bending moment or couple is half the moment into change of angle in elemental length.

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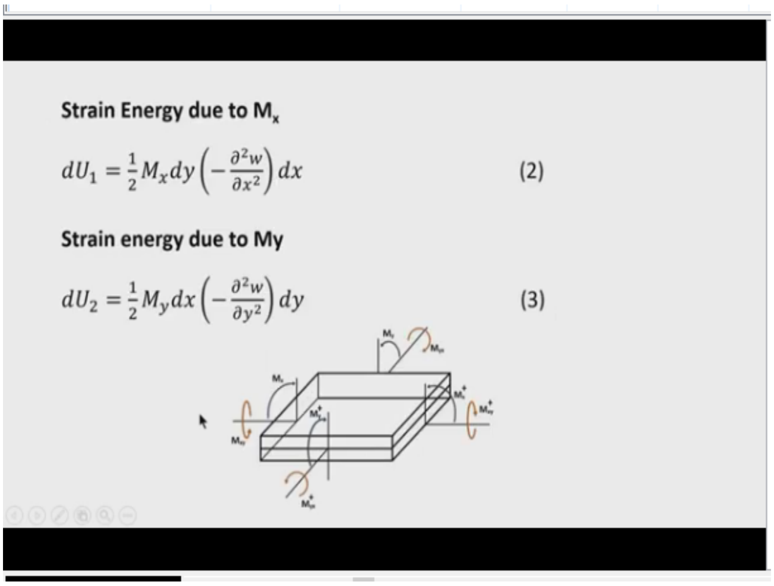

So, knowing this fact now, we can write the strain energy due to M_x is dU_1, say this part is U_1, contribution towards the strain energy due to bending moment along the x-axis is U_1. So, dU_1, we can write $\frac{1}{2}M_x dy$, why it is $M_x dy$ because M_x is acting along the edge whose length is dy. So, therefore, $M_x dy$ is the total bending moment along this edge because M_x, M_y, etcetera are considered to be quantity per unit length.

So, within this small element, we assume that the moment is uniformly distributed. So, this $M_x dy$ is the bending moment along this edge and therefore, $\frac{M}{2}$, this is the M into the change of the slope. So, change of this slope here we have seen that change of the slope if it is $\frac{\partial w}{\partial x}$ here. Here it will be $\frac{\partial w}{\partial x} + \frac{\partial}{\partial x}\frac{\partial w}{\partial x}$. So, naturally, this quantity comes here.

So, the minus sign is taken because, with the increase of deflection, the slope value decreases; therefore, minus sign is taken in this expression. Strain energy due to M_y is also written in a similar fashion $\frac{1}{2}M_y dx$ because M_y is acting along the edge whose length is dx. So, $\frac{1}{2}M_y dx$ is the average moment acting on this edge multiplied by the change of the slope. So, that quantity is taken because here now slope is along the y-direction. So, this is strain energy due to M_y.

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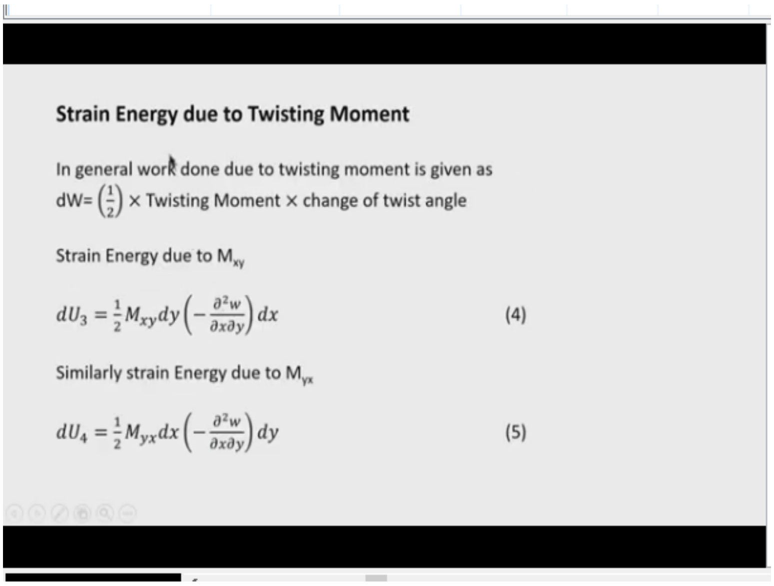

Next, we consider this strain energy due to twisting moment. So, in general, work done due to twisting moment is given by dW= (1/2)× Twisting Moment × change of twist angle. So, strain energy due to M_{xy} now can be written with the above analogy that we have used for M_x and M_y, $\frac{M_{xy}}{2}dy$ is the twisting moment acting along the edge which is parallel to y-axis. So, then this change of twist angle that is taken here.

So, this gives the strain energy due to M_{xy}. Similarly, strain energy due to M_{yx}, on the other edge M_{yx} can also be calculated using this relationship. So, then we have taken all the components of moment for calculating the strain energy, what are the components of moment? The M_x in the x-direction M_y in the y-direction and the twisting moment M_{xy}. $M_{xy} = M_{yx}$, and for this, this expression can be simplified. Strain energy for the twisting moment expression can be simplified by taking $M_{xy} = M_{yx}$.

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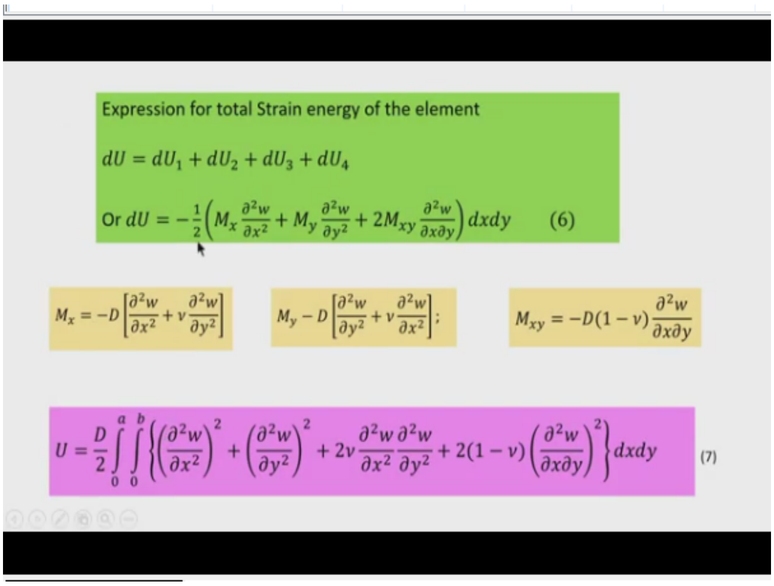

So, expression for total strain energy now becomes the summation of all components. So, what are the components let us see dU_1 is one component that is contributed by M_x, dU_2 is another component that is contributed by M_y, then dU_3 is another component contributed by M_{xy}, and lastly, dU_4 is the part of the strain energy that is produced due to M_{yx}. So, adding all this component $dU = dU_1 + dU_2 + dU_3 + dU_4$.

We get ultimately this $-\frac{1}{2}\left(M_x \frac{\partial^2 w}{\partial x^2} + M_y \frac{\partial^2 w}{\partial y^2} + 2M_{xy} \frac{\partial^2 w}{\partial x \partial y}\right) dxdy$ is coming. So, $dxdy$ represents the area of the elemental length. So, actually, this is if you see these quantities that strain energy density, and when we want to find the total strain energy, we must integrate it about the domain of the plate.

Now, let us express the strain energy in terms of deflection. Knowing that strain energy $M_x M_y M_{xy}$ can be related to the deflection that is the curvature in the x-direction as well as y-direction and also M_{xy} can be related to the twisting curvature then we have this expression already derived in my earlier classes $M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right]$.

So, this is the bending moment expression along the x-direction. Then $M_y - D\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right]$, $M_{xy} =- D(1 - \nu)\frac{\partial^2 w}{\partial x \partial y}$. So, substituting these values here M_x, M_y, M_{xy} whatever is there, we ultimately get this expression for the strain energy of an element in terms of deflection and its derivative, mainly the second derivative of the deflections.

So, now, you can see after arranging this, this strain energy can be now written in terms of the derivative of the deflection. So, here you will get $\frac{\partial^2 w}{\partial x^2}$ is directly coming from here, and half term will be there. So, $\frac{D}{2}$ will be there and minus all these bending moment expressions have negative sign. So, ultimately this strain energy will be a positive quantity.

So, you are getting here first term is the curvature square, its curvature in the x-direction, and this is square up. So, this is the first term, then second term is the curvature in the y-direction and also it is square up. Then you are getting $\nu\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial y^2}$ and then other term which are related to the twisting curvature is here $2(1 - \nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2$.

So, you get this expression for this strain energy after substituting this in the strain energy expression, and this is the most useful expression for solving the pate problem because the plate problem using the Rayleigh-Ritz's method approximate or any other Gallerkin method requires the first approximation of the deflection. So, therefore, this form of the strain energy is very convenient to be used for plate problem. Now, this expression can be written in another way.

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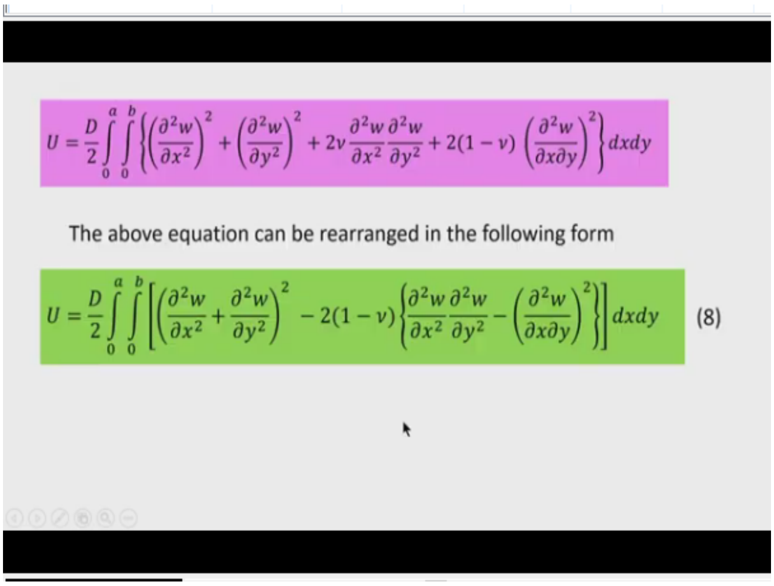

This expression now can be written in a very convenient form and from where some physical interpretation can be done, and the strain energy expression can be simplified for some cases. Now, here you can see if we add $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$ without Poisson's ratio here in this expression if I add this and then again subtract it to keep the expression same without affecting the expression.

So, we add a quantity; what is the quantity 2? $\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$. So, that quantity is added in this expression, and then same quantity is subtracted. So, ultimately the equalities not affected. So, the same quantity it is up here $-\ 2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$ is also added. So, then you can see if I take 3 terms $\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2$ + another term that I added $2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$.

These 3 terms can be combined and can be written in this form say $\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2$. So, this is written in a compact form, then the other term that remains that is we subtracted one term we added, and another term is subtracted; the same term is subtracted. So, the term that is substituted is again $2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$.

So, combining this term with this expression, now, we can express the strain energy in this form. So, equation 8 is very useful for the problems that we will discuss that we can neglect some terms to simplify the calculations. So, after knowing the strain energy expression in general for the plate in Cartesian coordinate system.

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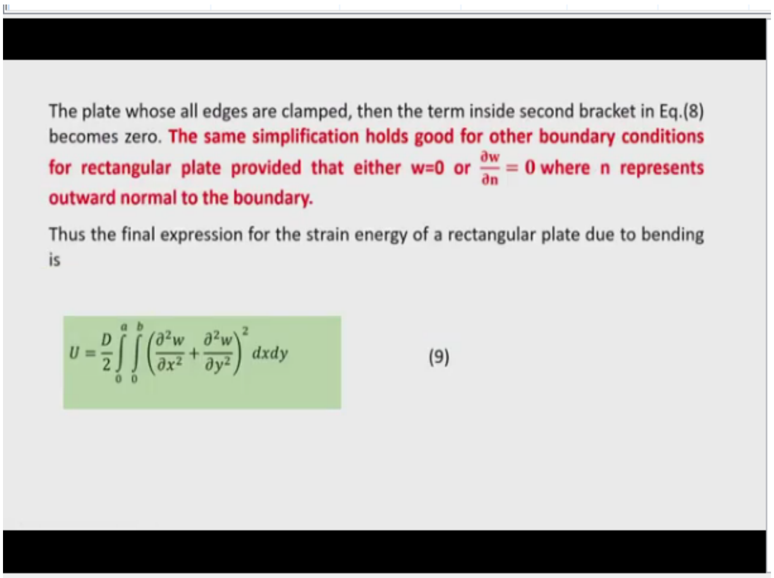

Now, we can see that some plate has edges which are clamped, there are many plates which all edges are clamped, it is argued like that the plate whose all edges are claimed then the term inside the second bracket in equation 8, actually the second term the term inside the second bracket, this is the term inside the second bracket can be ignored or it vanishes.

The same simplification holds good for other boundary conditions for rectangular plate provided that either w is 0 or slope is 0, where n represents outward normal to the boundary. That means slope along the normal direction is 0, and w along the boundary is 0. So, that conditions are satisfied in case of plates whose all edges are clamped and also other for simply-supported edges.

This condition also holds good because curvature is 0. So, therefore, we can take this expression as a simplified form neglecting this quantity inside the second bracket, so strain energy for the plate, which has all the edges supported, can be written as $U = \frac{D}{2}$ integration, why the integration is required because this expression was only for 1 element. So, after integrating, we get the strain energy for the entire plate that is the entire domain we use to integrate 0 to a, 0 to b are the limits of the plate.

So, after simplification, you can see the strain energy can be written as $\frac{D}{2}\int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 dxdy$. So, this is the expression of strain energy for the plate. Now, in case of, say, for example, a beam problem where D is flexural rigidity in case of the plate, but in case of beam D may be replaced by EI, and the deflection in the other direction was ignored.

In case of the curvature in the other direction, these width direction is ignored in case of beam or also in case of plate which are very long that means, plate has length, breadth ratio is very high that means width is very small compared to length, in that case, the curvature in other directions there is if the x-direction is the longitudinal direction curvature in the y direction can be neglected. So, in that case, the bending moment the stain energy expression is also used after neglecting this curvature in the y-direction.

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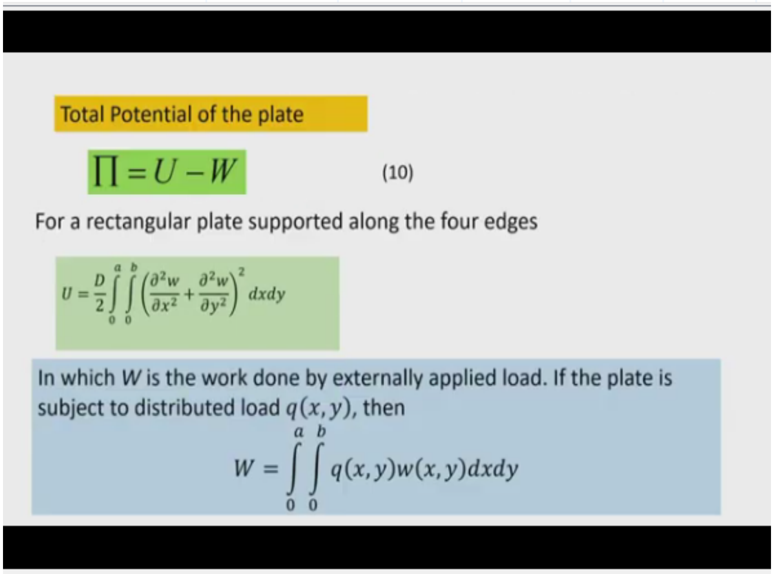

So, knowing the strain energy expression for the plate or any structural element whether it is beam or other elements then we can find the total potential of the system. Here I am mentioning plate because the course is focused on the plate only, but this term that I have written $\Pi = U - W$ can be used for any type of structural element also. So, Π is the symbol that is used here to denote the total potential of the plate.

So, what is total potential that is strain energy minus work done W is the work done due to external loading, we are considering only static problems. So, the kinetic energy is not coming here, but when you consider the kinetic energy for analysis of vibration, then the total potential expression will also include the kinetic energy term. Now, for a rectangular plate that is supported along the 4 edges, the strain energy expression is this.

If it is not supported along the 4 edges that means, say for example, 1 edge is free then you cannot use this expression. Then you have to use the general expression with this term containing the Poisson's ratio $(1 - \nu)$ is the factor, but in case of plates where all edges are claimed, or the plates are supported along all 4 edges rectangular plate such that w is 0 or the slope normal to the boundary and represents the outward normal to the boundary 0 then we can use the simplified form of the strain energy.

W is the work done by externally applied load; if the plate is subjected to distributed loading, say $q(x, y)$, then W is nothing but the load multiplied by the deformation, and it is integrated over the domain. Now, here we have not used the term half, based on the reason that the deformation does not change during the application of the load or the load does not change during the where the plate is undergoing the deformation. So, therefore, in case of W, we have not used the factor half.

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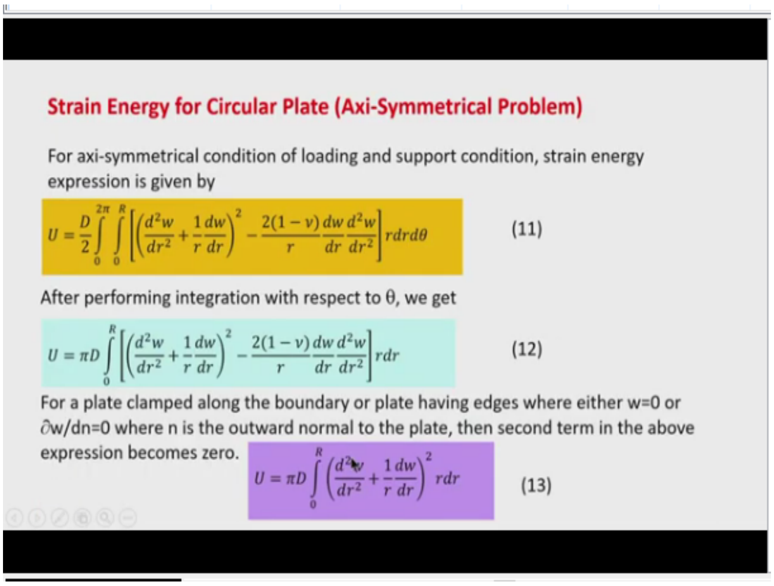

So, on the similar argument from the basic principle considering the radial moment and circumferential moment that the bending moment and twisting moment are used to derive the strain energy of the rectangular plate. Now, for circular plate in axisymmetrical cases, there will be no twisting moment. So, in that case, this strain energy expression can be written after integrating over the domain writing the strain energy expression for the element and then integrating over the domain the strain energy expression becomes $U = \frac{D}{2}\int_0^{2\pi}\int_0^R \left[\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right)^2 - \frac{2(1-\nu)}{r}\frac{dw}{dr}\frac{d^2w}{dr^2}\right] r dr d\theta$, why $r dr d\theta$. The $r d\theta$ is the length of a sector or $r d\theta$ at a distance r, and dr is the width of the sector. So, area of the sector or element of the circular plate is $r d\theta \times dr$. So, this is the area of the element, and this is the strain energy of the element. So, where it is multiplied by the area, and it is integrated, we get the total strain energy of the circular plate.

Now, you can see here that these functions w are its derivatives, in case of axisymmetrical problem I am not dependent on θ that means, this w does not contain any theta. So, therefore, we can integrate this expression independently with respect to theta. As a result the $d\theta$ *integration from* 0 *to* 2π *is nothing but* 2π. So, after that, only integration with respect to r remains, and R is the radius of this circular plate.

So, after performing integration with respect to θ, we get ultimately, the final expression for the strain energy of the circular plate in axisymmetrical cases as

$= \pi D \int_0^R \left[\left(\frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right)^2 - \frac{2(1-\nu)}{r} \frac{dw}{dr} \frac{d^2 w}{dr^2} \right] rdr.$ Again this simplification is possible when the boundary of the plate as either w = 0 or $\frac{dw}{dn} = 0$ where n is the outward normal to the plate.

Then second term in the above expression vanishes. So, therefore, in case of plate which has supported boundary, then we can write the strain energy expression for the plate as $\pi D \int_0^R \left(\frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right)^2 rdr.$ Now, suppose we want to solve a problem of a plate which is having a fixed boundary that it is fixed along the all edges. So, in that case, it is not necessary to consider the second term because the $\frac{dw}{dr}$ slope becomes 0. So, in that case, the second term can be used; the first term that is the square term will be used to calculate the strain energy of the plate.

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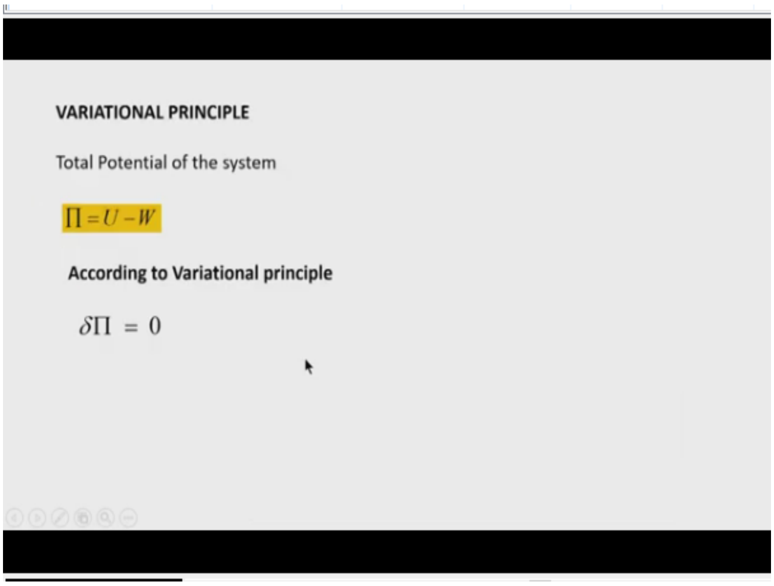

Now, based on the strain energy strain energy and total potential of the system, we now write that total potential $= U - W$. Now, based on the energy principle and minimization of the energy total potential, it can be written that $\delta\Pi = 0$. That has immense significance in case of physical problems that I encounter in our engineering applications. Because this single equation that is, this δ represent the variation of the total potential is 0.

This is possible because of minimum potential energy and the stable equilibrium configuration. So, if we write that $\delta\Pi = 0$ that means, Π involve your strain energy and work done, which involves also the deflection expression. So, in case of formulating a problem, when we want to establish the equilibrium equations, and we are not able to formulate the equilibrium equation from the force balance or Newton's law, then we can take the help of variational principle, which will give the same differential equation with ease in some cases, in some cases it may be complicated also.

But problems which has complicated boundary and complicated loading conditions in that case, the variational principle that is $\delta\Pi = 0$, that equation gives the yields the differential equation of the equilibrium as well as boundary condition. So, these equations simultaneously gives the differential equations of the equilibrium as well as boundary condition. So, I will explain one problem, the case how the differential equation is obtained using the variational principle and how the boundary conditions are known using the variational principle. So, let us see an application of the variational principle.

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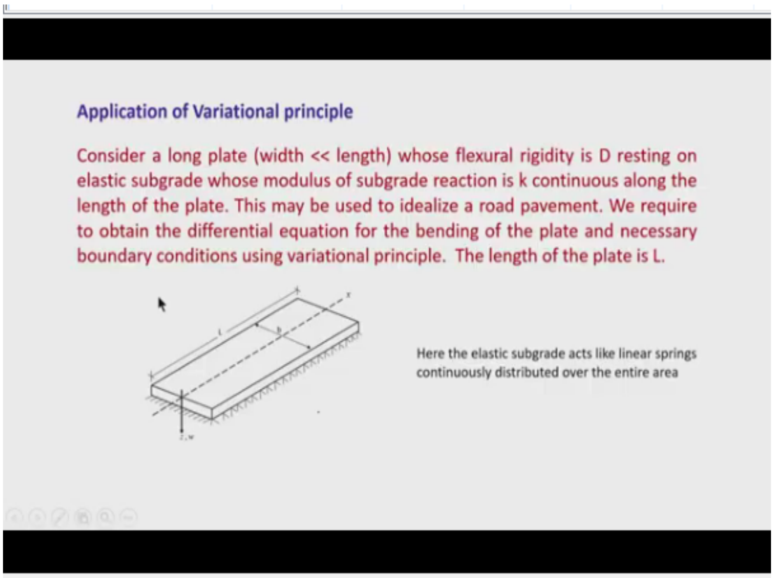

Consider a long plate, I am taking a long plate width is very, very less than length whose flexural rigidity is D, resting on elastic subgrade whose modulus of subgrade reaction is k and the subgrade is continuous, the plate is supported over the subgrade continuously, there was no break of this support. So, this condition may be used to idealize a road pavement because road pavement is using rested on the subgrade, and you know that pavement adding a concrete slab in many cases, so, concrete slab acts like a plate very long plate resting on the elastic subgrade.

Now, elastic subgrade are acting also like a spring, that it will produce an upward reaction to the deflection of the plate. So, the spring force that will be produced in the subgrade will be upward, and it will resist the downward load or the load acting on the plate. That means it will reduce the deflection. So, here the elastic subgrade acts like a linear spring continuously distributed over the entire area.

So what do we require? We require to find the differential equation of the system of this plate, as well as boundary condition using the variational principle.

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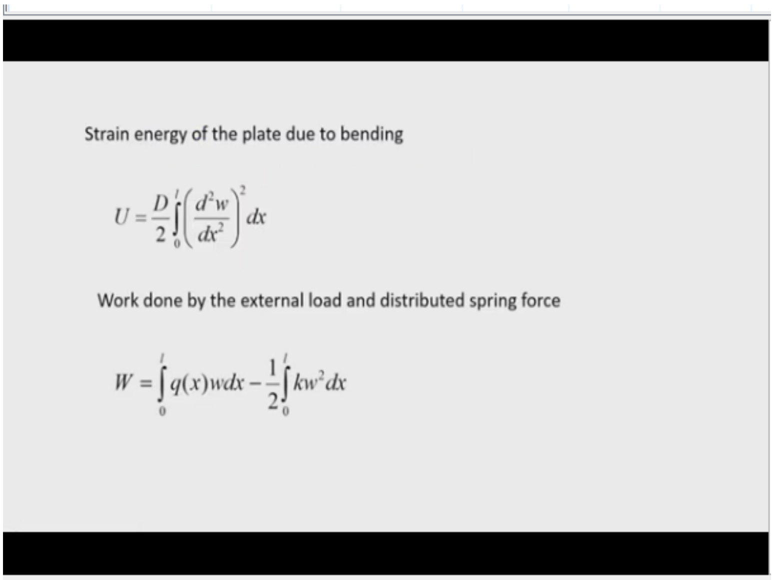

So, to use the variational principle, the first thing is that we must know the strain energy expression of the plate without knowing the strain expression; you cannot proceed to use this variational principle. So, strain energy expression of the plate, because of the long plate, the curvature in the other direction is neglected, and then we write it $U = \frac{D}{2}\int_0^l \left(\frac{d^2w}{dx^2}\right)^2 dx$.

So this is the strain energy of the long plate, which is actually behaving like a Beam, then what done by the external load, what are the load on the plate that is suppose some distributed load is acting on the plate. Here we have taken a distributed load acting on the plate continuously along this plate, or it may be discontinued; partial distribution also does not matter. But in that case, we are taking that load is continuously distributed along the length of the plate.

So, with the work done by the external load is $q(x)wdx$, then the work done by the distributed spring force due to subgrade reaction, the spring force is opposite to the displacement. So, therefore, work done by the spring force will be negative. And we take that work done by the spring force as $kw\times\frac{w}{2}\times dx$. So, half is taken outside and then we have integrated over the length of the plate. So, these represent the work done by the spring force.

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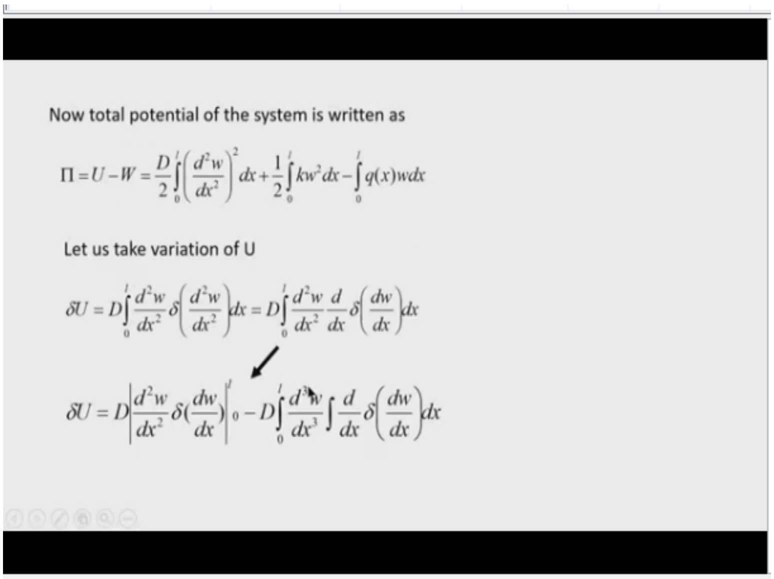

Now we write the total potential Π equal to $U - W$. So, $(U = \frac{D}{2}\int_0^l \left(\frac{d^2w}{dx^2}\right)^2 dx) - W$, and what is W? W we have got 2 components. So, first component is due to this distributed external load and second component is due to spring reaction. So we have written this first component was negative there, and it is now becoming positive because -W is there and work done due to distributed loading.

In order to find out the differential equation and boundary condition, I told you that we have to take the variation of U. So, take the variation of U, δU variation is a operator just like differentiation. So, these δU is that means the quantity inside, whatever U is there, U is

$\frac{D}{2}\int_0^l \left(\frac{d^2w}{dx^2}\right)^2 dx$. That has to be operated with the variational operator, δ so δ, δ operation is done here.

You can see that U quantity is this. So, if I operate this quantity with δ that means, like a differentiation, we take this, this will be $2\frac{d^2w}{dx^2}$ and then the variation of this quantity. So, it is written based on this similar to the differentiation process. So, $\frac{d^2w}{dx^2}\delta\left(\frac{d^2w}{dx^2}\right)$, that is the variation of this quantity into dx is of course there, and it is integration is there, 0 to l.

Now you can see this quantity has to be integrated, so how it will be integrated. You see, there, this is one function, and that is another function. So, we have to use the integration by product rule. So, if we use the integration by product rule, it is further written in this form before starting the process of integration. So, this the differential operator and this variational operator interchanged.

So, it is written as $\frac{d}{dx}\delta\left(\frac{dw}{dx}\right)$. So, after writing this, the integration process has been started. So, the first integration is yielding this result that is if I follow the integration of products rule, then this first function into integration of the second function. So, integration of the second function is $\delta\left(\frac{dw}{dx}\right)$ and then I put the limit, 0 to l. That means this quantity has to be found out at the boundaries 0 to l minus derivative of the first integral $\frac{d^3w}{dx^3}$ into integral of the second function. So, integral of the second function, that is this.

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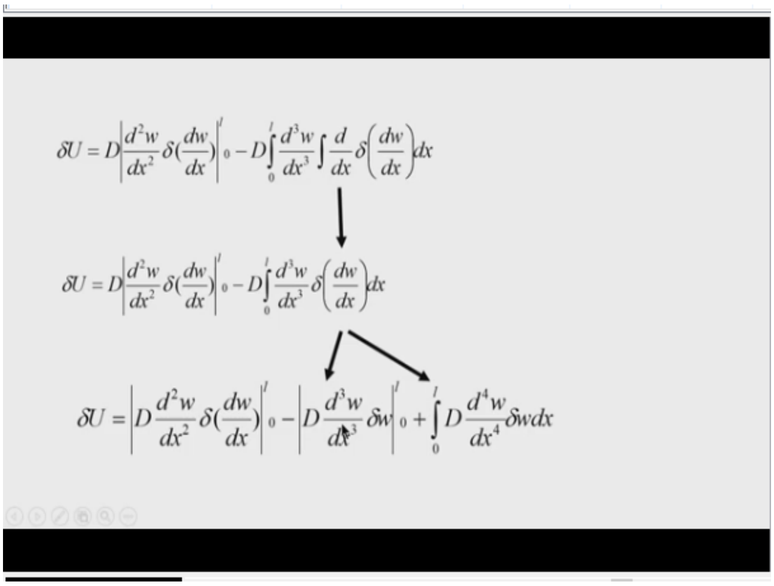

And in the next slide, you will find that I have integrated this second function, and again, we are getting this. So, this is remaining as it is. Now we are getting this term. And again, this is integrated. Having observed that, this is again a product of 2 functions. So, the integration by product rule is again followed, and we got here, say D, the first function into integration of the second function.

So, how it is done, that differential operator $\frac{d}{dx}$ and variation operator interchange and therefore, we are getting after integration, delta w and we put the limit to evaluate this quantity at the boundary, then minus and minus, minus, plus sign will be there. Then, D is there already derivative of this term. So, $\frac{d^4w}{dx^4}$ the fourth derivative is coming and then integration of this $\frac{\delta w}{\delta x}$. So, we arrived up to this. So, dU is completed, dU is completed.

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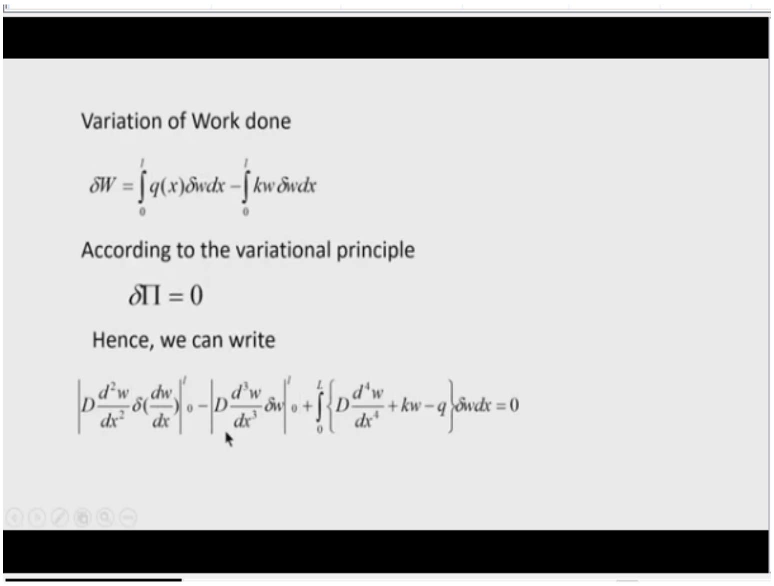

Then variation of work done is calculated $\delta W = q(x)\delta w dx$, and it is a very simple thing you can see a $\frac{1}{2}kw^2$, so it is operated with δ, it becomes $kw\delta w$ and dx is already there in the integration. So, according to the variational principle, we now write $\delta\Pi = 0$. Now substituting the component of δΠ, δΠ is obtained as $\delta U - \delta W$. So, substituting all these components that were evaluated in the earlier process, we now write δΠ equal to this quantity equal to 0.

So, you are getting here the quantity that have to be evaluated at the boundaries. Here also the quantity that have to be evaluated in the boundary. And here, there is one integral. Now the right-hand side is 0, that means, here, we now know that δW is not 0; it is arbitrary quantity. It is arbitrary, virtual displacement, which is not 0. So, it indicates that the term inside the second bracket should be 0 and other the boundary quantities should be 0.

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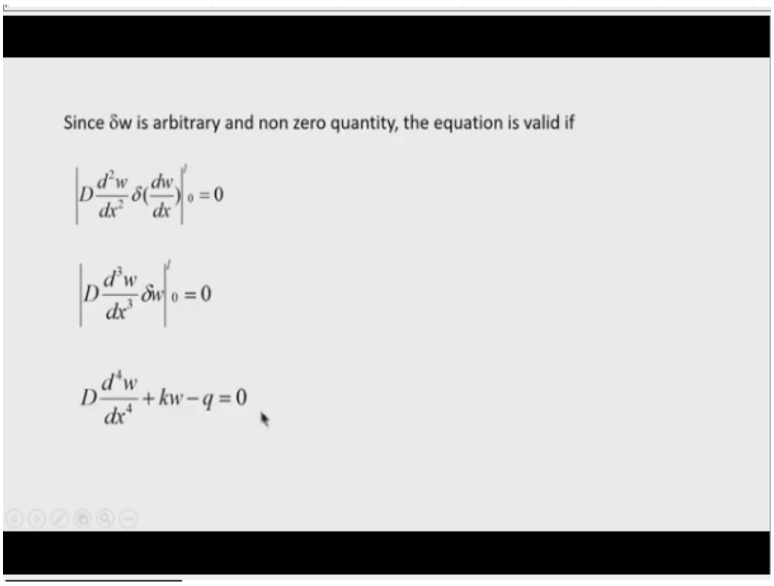

So, based on that, we can now write this first quantity is this, we can write, this second quantity is this, so we write this is equal to 0. And third is the quantity inside the integral, and this is also equal to 0. So, the one that is the last expression that I have written it gives the differential equations of equilibrium of the problem and other 2 equations that I have written above is nothing but quantities related to the boundary values. Now several combinations of boundaries values are possible. Let us see.

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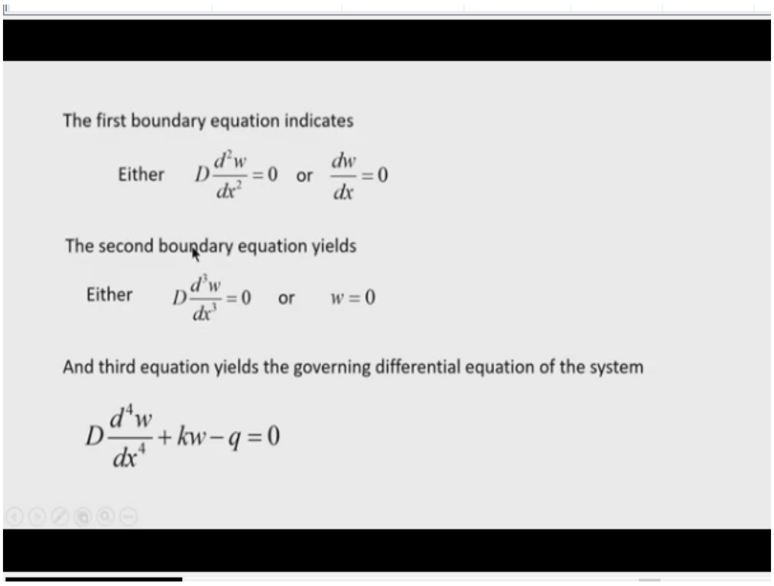

So, from the first equation, we can now see that either this is 0 or this is 0. Product of 2 quantities are 0 there, so, therefore, this is 0, or this is 0, second boundary condition also yields like that. And third equation yields the governing differential equation. So, taking different combinations of these boundary values, we can now interpret physically the boundary conditions.

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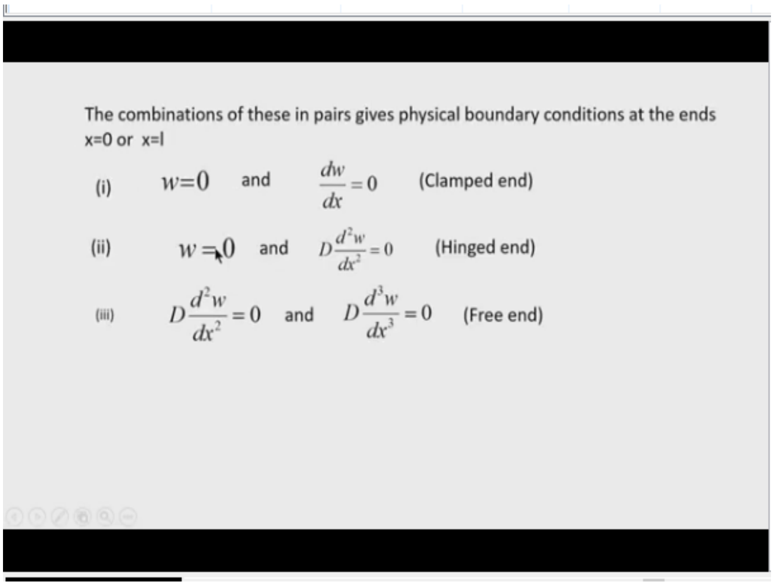

The combinations of these in pairs give the physical boundary conditions at the ends, x = 0 and x is = l. So, x = 0. Now, one combination is w = 0, $\frac{dw}{dx} = 0$. So, that combination gives clamped end, and another combination may be w = 0, $D\frac{d^2w}{dx^2} = 0$. This combination gives his hinged end or pined end, or simply supported end. Then the third combination, the $D\frac{d^2w}{dx^2} = 0$ and $D\frac{d^3w}{dx^3} = 0$.

So, that indicates this is the bending moment, and this is the shear is 0. So, it is for the free end. So, in this way, we have seen that use of variational principle with the knowledge of strain energy expressions and the work done by the external load, we can arrive at the differential equations and boundary conditions for the physical problem.

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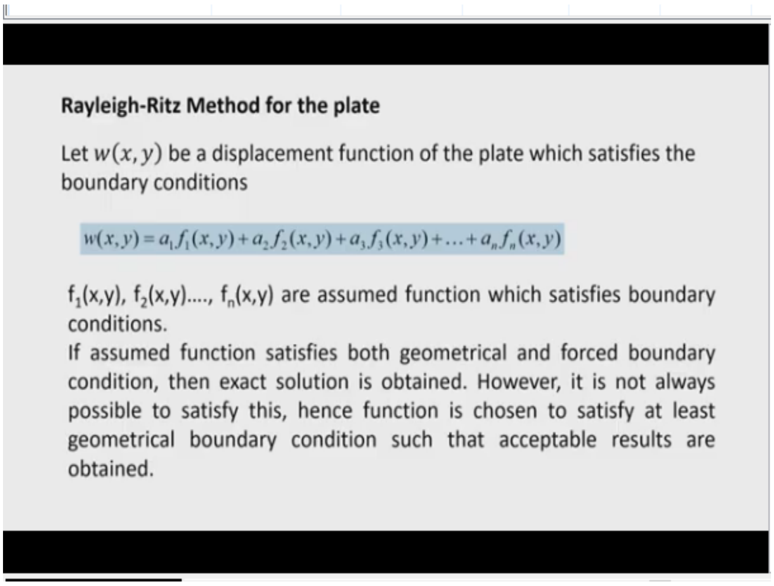

Now let us derive the Rayleigh-Ritz method. In the Rayleigh-Ritz method, we shall assume that w the deflection is a continuous function of x and y. So, we shall assume that $w(x, y) = a_1 f_1(x, y) + a_2 f_2(x, y) + a_3 f_3(x, y) + ... + a_n f_n(x, y)$, like that we can take n number of terms. So, when f_1, f_2, f_n, I assume function with satisfy boundary condition that is most important thing.

So, these functions are taken in such a way that boundary conditions are satisfied that there may be displacement boundary condition, which are imposed on displacement and slopes, but there may be also the force boundary condition, which may be applied to the shear force and bending moment. So, that means or twisting moments. So, in that case, the f_1, f_2, f_n are set to be chosen in such a way that the boundary conditions are satisfied.

Now in many problems, it is difficult to satisfy the force boundary condition. So, we get, naturally, the errors in the final result. But deflection boundary conditions are, in most cases it is satisfied. And if both the conditions are satisfied, then we get the exact results for the problem.

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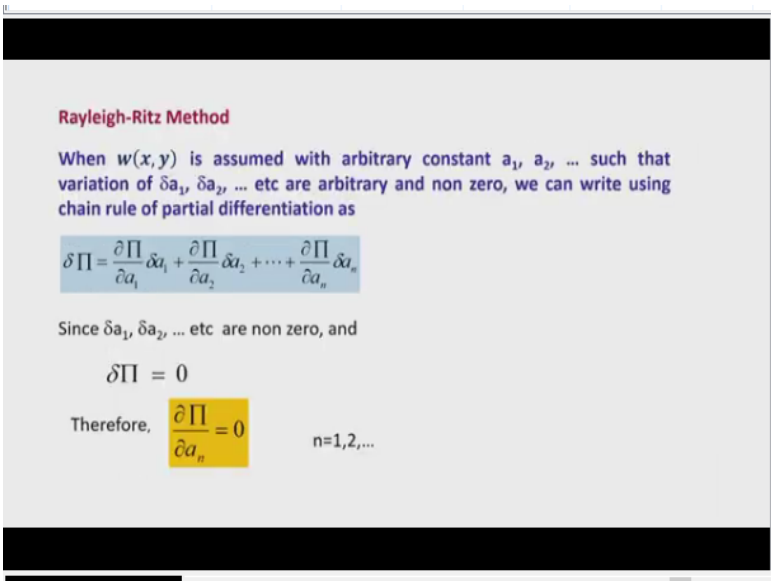

So, how it is derived Rayleigh-Ritz methods, we know the total potential Π and total potential when you know that total potential contains the strain energy, as well as work done, which are also dependent on the displacement, as well as its derivatives. So, therefore, this total energy, total potential pi are also functions of the unknown constant a_1, a_2, a_3, etcetera. So, therefore using the chain rule of partial differentiation, we can now express $\delta\Pi = \frac{\delta\Pi}{\delta a_1}$.

Mind that this is the partial differentiation, but this is the variational quantity into $\delta a_1 + \frac{\partial\Pi}{\partial a_2}\delta a_2 +$ like that it will go up to n^{th} terms. Now where δa_1, δa_2, $etcetera$ are arbitrary and non-zero constants. So, we can now say that when $\delta\Pi = 0$, then these are not 0, δa_1, δa_2, δa_n are not 0. That means for any general term said $\frac{\partial\Pi}{\partial a_n}$ should be 0. So, this is the equation to be used in Rayleigh-Ritz formulation.

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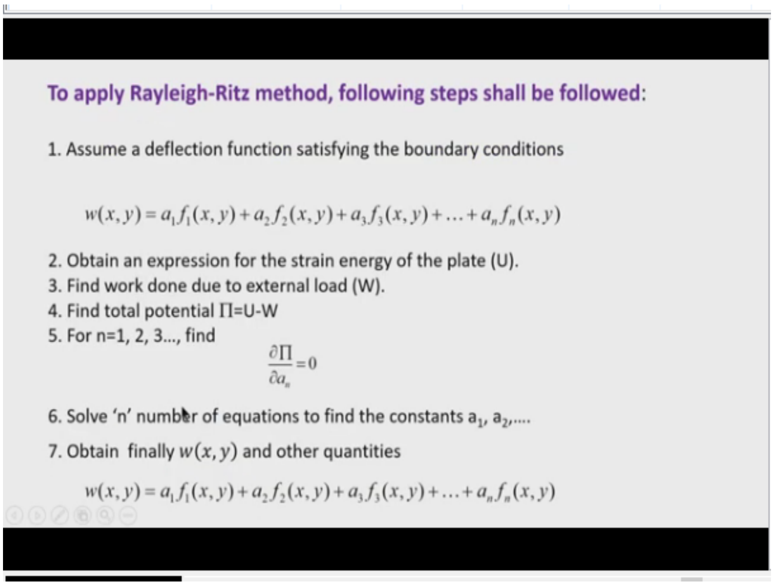

So, what are the steps? Let us see. First step is assume deflection function, we have assumed the deflection function like that, then obtain the expression for the strain energy of the plate U, find the work done due to external load W, find the total potential $\Pi = U - W$, then you see that $\Pi = U - W$ contains this a_1, a_2, a_3 in square form because it is a energy expression, it is coming from energy expression pi EI.

So, therefore it will contain the square of this quantity. Now, we are showing it for any term and we have to carry out for all number of terms this derivative that $\partial\Pi/\partial a_1 = 0$, $\partial\Pi/\partial a_2 = 0$ like that. Now, the beauty of this method is that since the energy expressions are all square terms quadratic terms and then when we differentiate with respect to say a_1, a_2, which appears as a square quantity in the total potential.

Then after differential equation, it becomes a linear equations. So, that means we have to solve n number of linear simultaneous equation involving the unknown parameters, a_1, a_2, etcetera. So, when a_1, a_2, a_n, etcetera are known, we now know the deflection equation completely. That means deflector surface is now given as $w = a_1 f_1(x, y) + a_2 f_2(x, y) + a_3 f_3(x, y)$ like that up to nth term, if you consider. Now accuracy of this method can be increased. If we properly select the displacement function such that it satisfy the boundary condition as close as possible. And also, if we increase the number of terms, the accuracy, the truncation error will be reduced.

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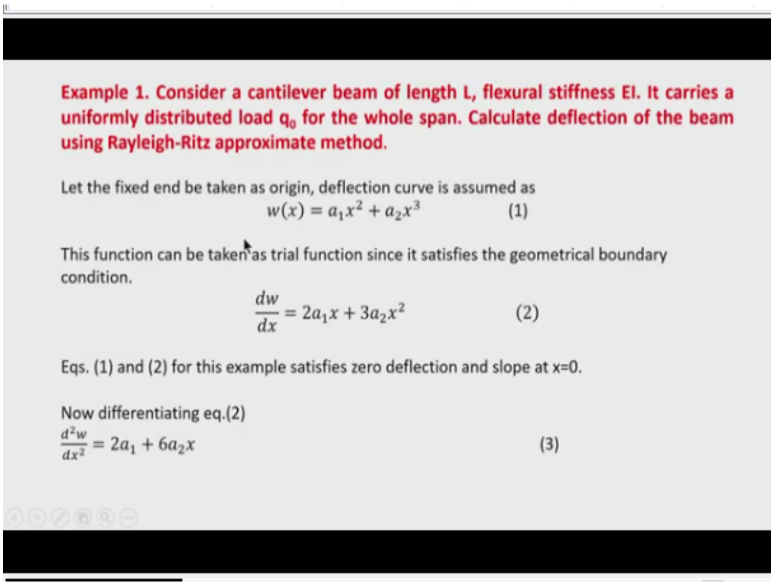

So, let us see an example, first consider a cantilever beam of length L, flexural rigidity EI, I have selected a problem of beams, so that you can understand the concept of this problem and then you can apply to the plate problem. So, it carries a uniformly distributed load q_0 for the whole span. Now we required to find the deflection of the beam using Rayleigh-Ritz method.

So, first, we use the 2 terms of this series Rayleigh-Ritz deflected series, or we use a polynomial expression with 2 terms, the quadratic term and cubic term, to approximate the deflected elastic line of the beam. Now, once this is known, then we can calculate the slope of this curve as well as the curvature of the curve. So, after getting this quantity, that is the slope and curvature of this.

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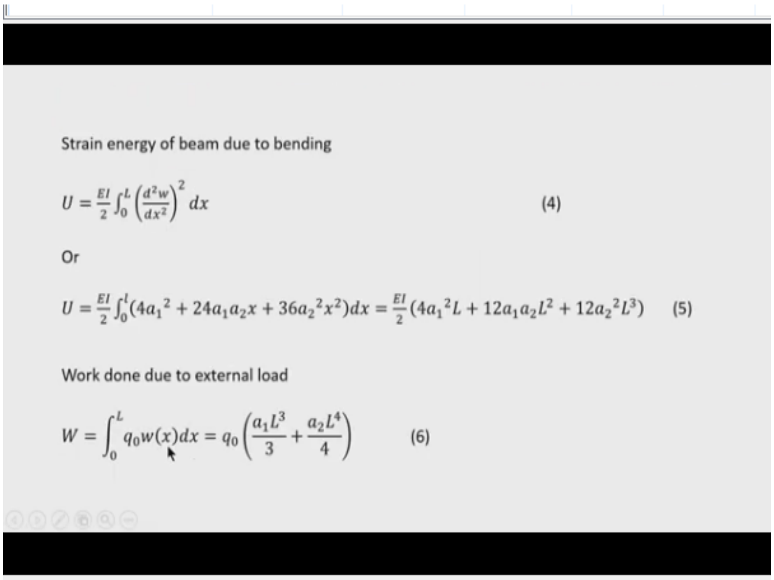

We now calculate the strain energy due to bending of the beam. So, strain energy due to bending of the beam now, it will be $U = \frac{EI}{2}\int_0^L \left(\frac{d^2w}{dx^2}\right)^2 dx$. So, after substituting the value of $\frac{d^2w}{dx^2}$ from the previous expression and then performing the integration, we arrive at the expression for U as this $\frac{EI}{2}\left(4a_1^2L + 12a_1a_2L^2 + 12a_2^2L^3\right)$.

What done by the external load is simply the q_0 is the uniformly distributed load into w, x, because w is a function of x displacement into dx, integrated over the length of the beam. And then we get $q_0\left(\frac{a_1L^3}{3} + \frac{a_2L^4}{4}\right)$. So, this is the work done expression, and this is the strain energy expression.

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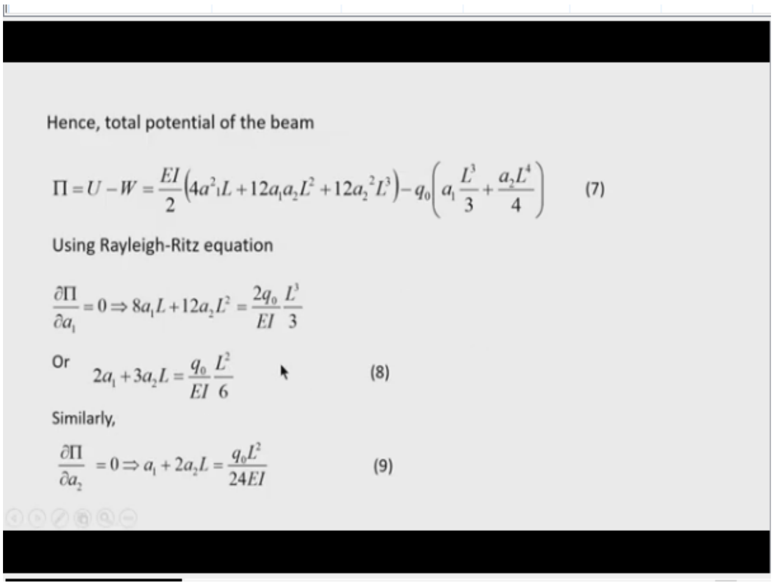

So, total potential now we can calculate $\Pi = U - W$, substitute U from the previous expression, this expression, the equation number 5 and work done from the equation number 6. We now write $\Pi = U - W = \frac{EI}{2}\left(4a_1^2L + 12a_1a_2L^2 + 12a_2^2L^3\right) - q_0\left(a_1\frac{L^3}{3} + \frac{a_2L^4}{4}\right)$. So, this is the Π. Now, we use the Rayleigh-Ritz equation.

So, what is the Rayleigh-Ritz equation? $\frac{\partial \Pi}{\partial a_1} = 0$, $\frac{\partial \Pi}{\partial a_2} = 0$, $\frac{\partial \Pi}{\partial a_3} = 0$, but since we have taken only 2 terms in the deflected line because it is a beam problem, so we call it line. So, we have to differentiate it with respect to a_1 and a_2 separately. So, first to differentiate with a 1 and then we get $\frac{\partial \Pi}{\partial a_1} = 0 \Rightarrow 8a_1L + 12a_2L^2 = \frac{2q_0}{EI}\frac{L^3}{3}$

Then similarly, we differentiate $\frac{\partial \Pi}{\partial a_2} = a_1 + 2a_2L = \frac{q_0L^2}{24EI}$. So, you can see the equation 8 and equation 9 contains 2 unknown quantities, a_1 and a_2 and this can be solved easily.

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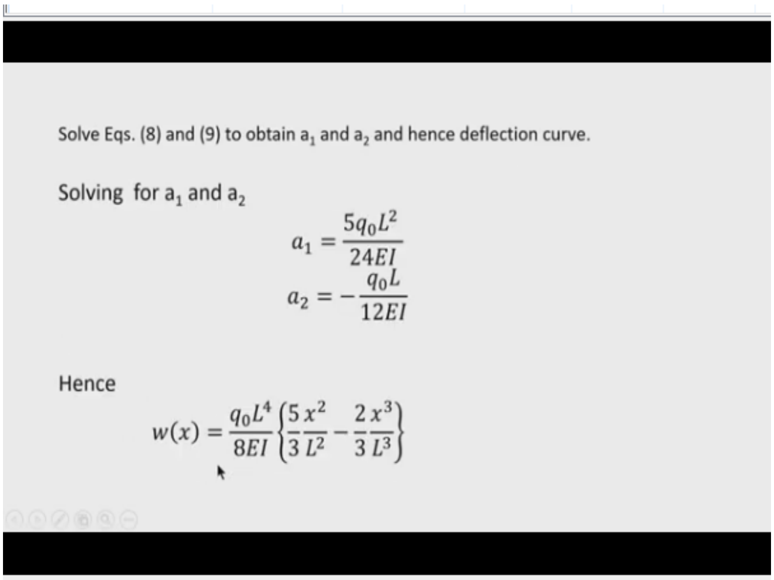

And after solving, we get a_1 as this and a_2 equal to this $a_1 = \frac{5q_0L^2}{24EI}$ and $a_2 =- \frac{q_0L}{12EI}$. Substituting a_1 and a_2 in the expression for deflection that we assumed previously here, equation number 1 in this problem, substituting a_1 in a_2 and arranging this, we can now write the expression of deflection in terms of this $w(x) = \frac{q_0L^4}{8EI}\left\{\frac{5}{3}\frac{x^2}{L^2} - \frac{2}{3}\frac{x^3}{L^3}\right\}$.

Now we have taken the origin at the fixed end of the beam. And $x = L$ end is free. So, for the cantilever problem, we know the deflection in the free end, deflection in the fixed end, of course, it is 0, slope is also 0. So, the deflection at the free end, if we want to calculate, we put $x = L$. And then we can see that exact value is obtained; we know from the strength of material formula that w at the cantilever free end or tip will be $\frac{q_0L^4}{8EI}$ where q_0 is the distributed loading in the cantilever, L is the length and EI is the flexural rigidity.

So, hereby taking 2 terms in the deflection series or deflection polynomial, we have chosen a polynomial function, but it is not necessary to choose a polynomial function; a trigonometrical function can also be chosen. Now, this function is chosen because you can observe that this function satisfy the deflection at the fixed end and slope at the fixed end. Fixed end is 0 ends.

So, put x = 0 deflection is 0. Again, if you differentiate it and put x = 0, the slope is 0. So, slope and deflection at the fixed end is satisfied. So, therefore, this represents a possible deflection function of the equation. By following the Rayleigh-Ritz procedure, we now obtained the deflection surface after evaluating the unknown a_1 and a_2, and this expression shows that deflection is exactly obtained; there is no error in the deflection.

However, if you calculate the bending moment, you will get some error and shear force again you will get error. So, a comparison I will give in the next slide, that what is the error involved in calculation of bending moment and shear force.

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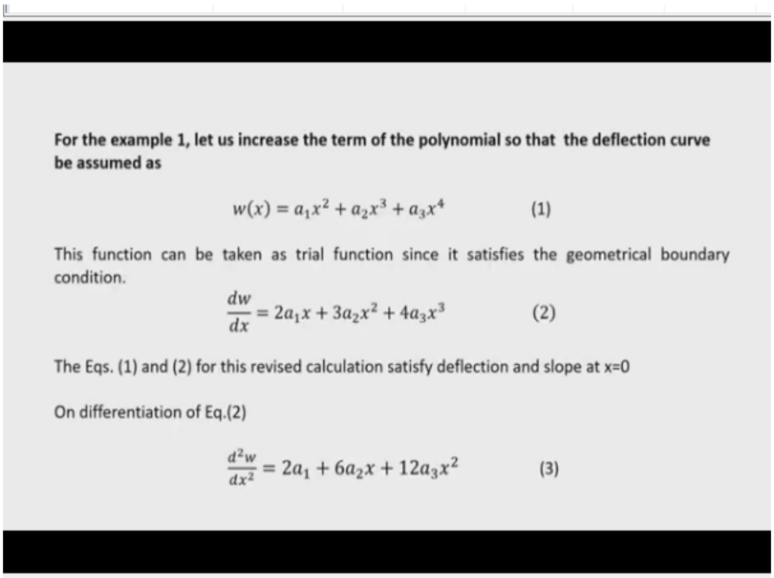

First, let us see the weather accuracies increased by using one more term in the deflection series. So, if I add another term, a_3x^4 in the previous deflected series, deflected expression. This is also admissible function because it satisfies the boundary condition at the fixed end. Force boundary condition, we are not sure whether it will satisfy or not. So, at the first loop, it is observed that the function satisfies the boundary condition at the fixed end because putting $x = 0$ deflection, we are getting 0.

And derivative we evaluate here as $2a_1x + 3a_2x^2 + 4a_3x^3$, again if we put x = 0, we get slope $= 0$. So, both deflection and slope boundary conditions are satisfied at the fixed end. So, this is a admissible function for the deflection.

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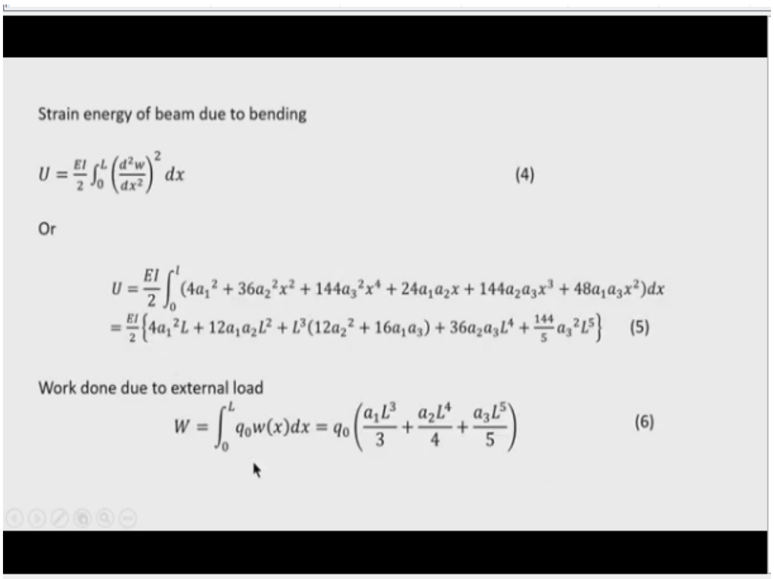

So, as usual, we take the second derivative, and this result is used in the strain energy expression, $U = \frac{EI}{2}\int_0^L \left(\frac{d^2w}{dx^2}\right)^2 dx$. So, the second derivative is taken, and then the strain energy is evaluated. So, after evaluating the strain energy, we get $U = \frac{EI}{2}\int_0^l \left(4a_1^2 + 36a_2^2x^2 + 144a_3^2x^4 + 24a_1a_2x + 144a_2a_3x^3 + 48a_1a_3x^2\right)dx$. What done by the external load is calculated as this because this is a very simple integration, no derivative of the W is involved, so we calculate this quantity.

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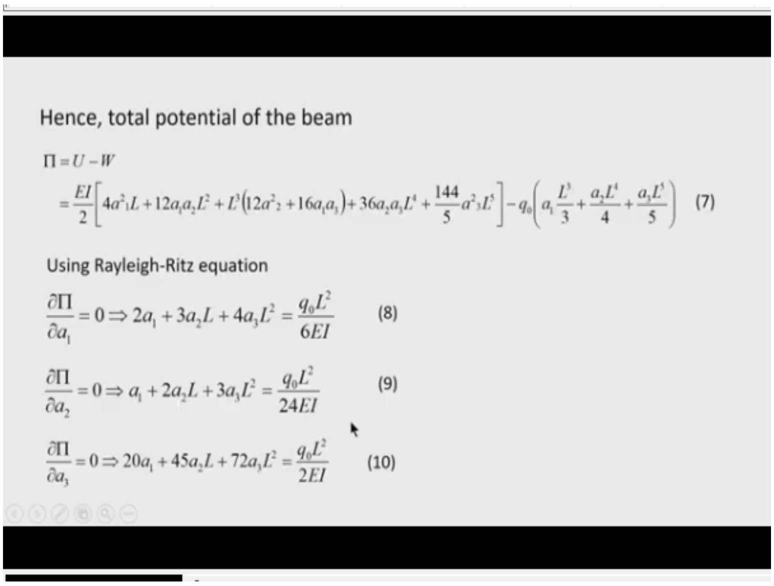

And then, the total potential is obtained and using Rayleigh-Ritz equation because here 3 constants are involved, 3 terms are used in expressing the deflection. So, we use the partial derivative of $\frac{\partial \Pi}{\partial a_1}$. Again, $\frac{\partial \Pi}{\partial a_2}$ and $\frac{\partial \Pi}{\partial a_3}$ and these 3 partial derivatives are evaluated and equated to 0. So, we get 3 linear equations involving a_1, a_2, a_3.

You can see these 3 linear equations are obtained. Using this any method, say for example, Kramer's rule is a possible method for solving the linear simultaneous equation.

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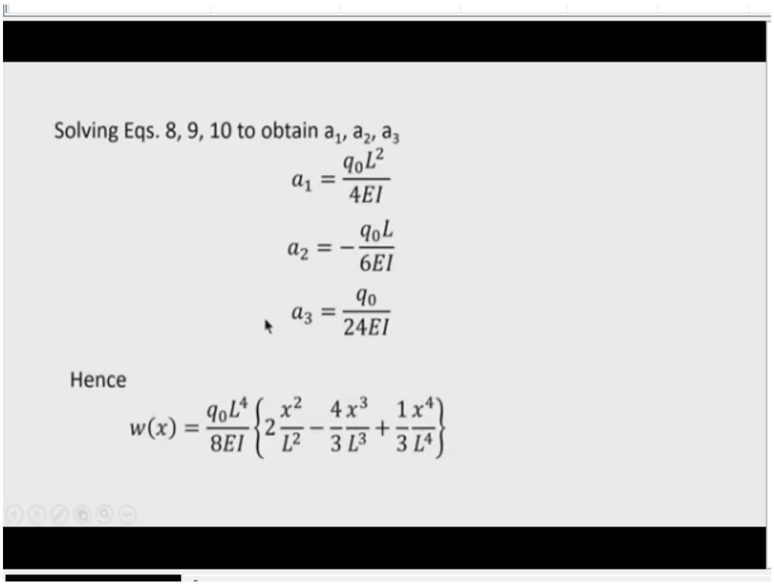

We now obtain a_1, a_2, a_3. So, obtaining this a_1, a_2, a_3, we now express the deflection. So, from this deflection equation, we can now obtain the slope, bending moment and shear force using the derivative of the deflection function.

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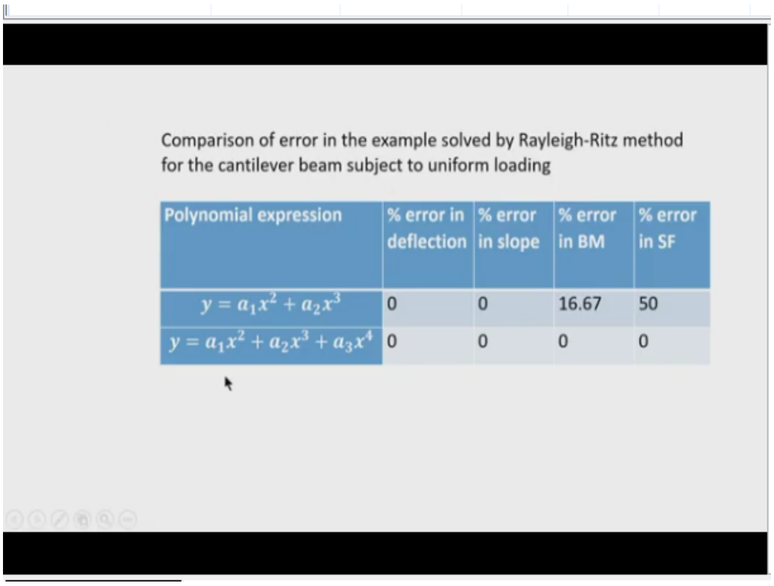

Now, here I have given a comparison for the cantilever problem. First, I have taken that this 2 terms expression in the polynomial of the deflection, $a_1 x^2 + a_2 x^3$. Now percentage error in deflection I have calculated, you can also check it, it will be 0, percentage error in slope is also 0, percentage error in bending moment is 16.67, whereas percentage error in shear force is 50%.

But when we increase one term of the deflected series, we get the percentage error in deflection slope and percentage error in bending moment and shear force. Bending moment and shear force refers to the bending moment and shear force at the fixed end because these fixed end moments are maximum. Whereas these slope and deflection are referred to the free end where the standard values are known.

Standard values of deflection at the fixed free end is $w = \frac{q_0 L^4}{8EI}$, whereas slope at the free end is $\frac{q_0 L^3}{6EI}$. So, comparing this result from the Rayleigh-Ritz method with the standard value, we can now compare the errors. So, we can see that if the deflection function is properly chosen to satisfy the boundary conditions and containing the sufficient number of terms, we can obtain the acceptable result.

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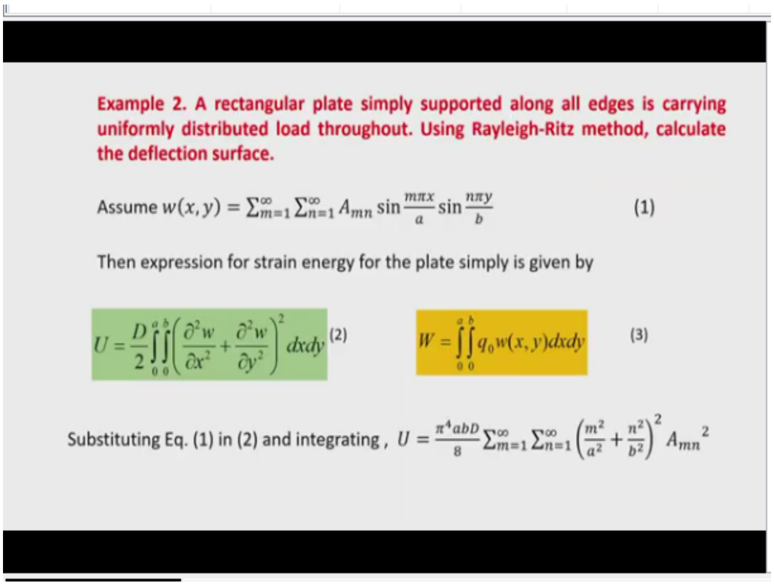

Now we give you a plate problem example. So, take a simply supported plate which conveniently expressed, deflection is explained by the double trigonometrical series because you know that sin $sin\ \frac{m\pi x}{a}$ sin $sin\ \frac{n\pi y}{b}$. If these terms are summed up, up to infinite series, then it represents the deflection of a simply supported plate. That means the plate is having all edges are simply supported.

Now since the plate is supported along the all edges, we now take the expression for strain energy, considering only the first 2 terms, because the terms where the Poison's ratios are involved, that is neglected because the deflection and slope normal to the boundary of the supported edge were taken 0 for the clamped edge and it is also true for this simply supported edge, because the product of curvature is there, so it becomes 0.

So, therefore we take $U = \frac{D}{2}\int_0^a\int_0^b (\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2})^2\ dxdy$ and work done due to external load, if q_0 is the uniformly distributed load over the plate that $q_0\, w(x,\ y)\ dxdy$. Now substituting this expression here and integrating, you have to carry out the double integration that strain energy due to bending of plate now becomes $\frac{\pi^4 abD}{8}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 A_{mn}^{\ 2}$. So, this is the strain energy of the plate.

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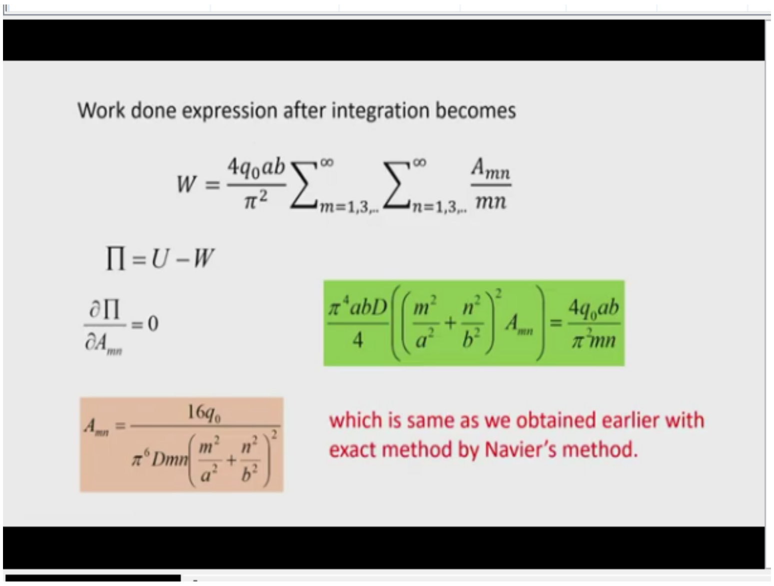

The work done due to external loading is expressed in this form, $\frac{4q_0 ab}{\pi^2}$ summation m = 1, 3, odd number of terms will be there as we have seen in case of Navier's method. And also, for n odd number of terms will be there will be contributing to the work done. So, therefore, we get this work done as some $\sum_{m=1,3,..}^{\infty} \sum_{n=1,3,..}^{\infty} \frac{A_{mn}}{mn}$.

Now, calculate the potential $\Pi = U - W$. Then apply the Rayleigh-Ritz's method $\frac{\partial \Pi}{\partial A_{mn}} = 0$. After applying this, we now arrive, the value of A_{mn} you can see when this strain energy is this expression is differentiated with respect to A_{mn}. We get here $\frac{\pi^4 abD}{4}$ A_{mn} will be there. So, here we get $\frac{\pi^4 abD}{4}$ other terms are there already.

A_{mn}, is there, and if you differentiate this quantity with respect to A_{mn}, then it will be 1. So, we are getting in the right-hand side $\frac{4q_0 ab}{\pi^2}$. And other times,A_{mn} will be there. So, when we compare the coefficient of the like term, this is the result. So, from where A_{mn} is obtained as $\frac{16q_0}{\pi^6 Dmn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}$.

So, that thing is well known to us because, in Navier's method, where do we solve the differential equation of the plate using the series double trigonometrical series, we got the same expression. So, that means, in that case, the force boundary condition, as well as geometrical boundary conditions, are satisfied, using the double trigonometrical series, so, therefore, it yields the exact solution.

Even we are using the Rayleigh-Ritz's method, but we are arriving at the exact solution of the differential equation of the plate. So Rayleigh-Ritz's methods are very powerful technique in solving the boundary value problems in physical in mechanics or mathematical physics. And conveniently, it is used for the plate problem of different nature.

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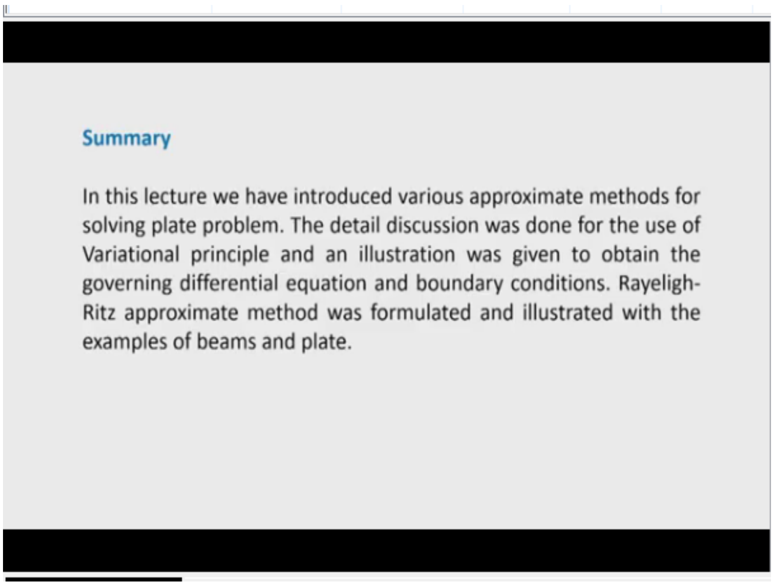

Let us summarize the today's lecture. In this lecture, we have introduced various approximate methods for solving plate problem, that detailed discussions was done for the use of variational principle, and then illustration was given to obtain the governing differential equation and boundary conditions, then Rayleigh-Ritz approximation method was formulated and illustrated with the example of beams and plates. Thank you very much.