Plates and Shells
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Module-04
Lecture-13
Examples in Axisymmetrical Bending of Solid and Annular Plate

Hello everybody, so, today I am starting the lecture 3 of module 4 that I was continuing this discussion of circular plate formulated in polar coordinate system. And in that case we have derived the differential equation of equilibrium involving the Rn theta. And then we have seen that if the plate possesses rotational symmetry in respect of support and loading condition then we can convert the partial differential equation into ordinary differential equation since the deflection and this load and other stress resultant will not be a function of theta.

So, in that case we have seen that equation can be arranged in such a manner that successive integration can yield the desired deflection. So, after the deflection is obtained then we can calculate the bending moment and shearing force. Because in that case, the radial moment and tangential or circumstantial moment these 2 moments are of important in axisymmetric problem but this MR theta that you call the twisting moment is 0, because of symmetry and Q theta is also 0.

So, now, let us show further application of this axisymmetrical formulation by giving more examples in practical cases. So, we will today discuss the bending of solid plate as well as annular plate supported in different manners and carrying the load in symmetrical loading. Under symmetrical loading condition only we can discuss this formulation because it is a axisymmetric problem.

So, sometimes the plate may not be fully solid, means there may be a hole in the plate. So, if the hole is symmetrically placed on the circular plate then again the problem falls into category of axisymmetrical bending of the circular plate. And in that case, we can find the solution of the annular plate which has various practical applications like this annular raft foundation for circular water tack. The column of a circular water tank which are arranged in a perimeter of a circle, then these type of column can be founded on an annular raft.

So, there are other practical applications of the annular plate or plate with hole. So, that will be discussed today's lecture.

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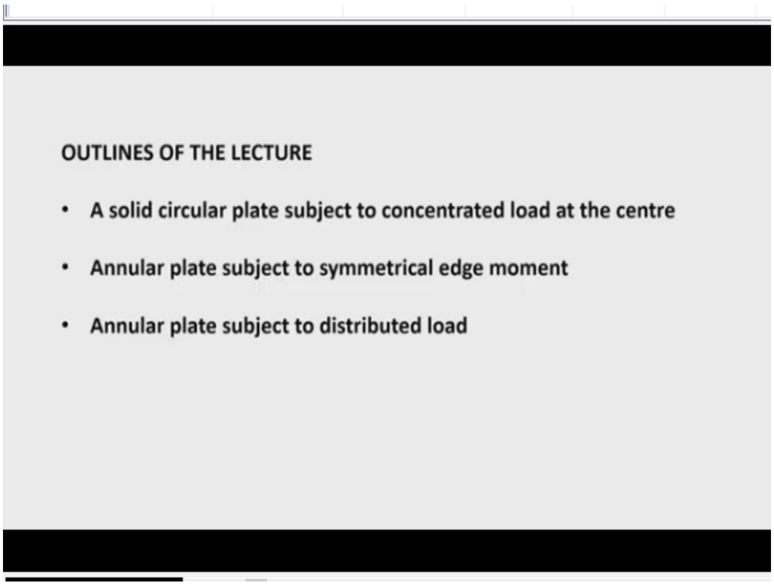

So, outlines of the lecture I am now giving you point by point; solid circular plate subject to concentrated load at the center, annular plate subject to symmetrical edge moment and annular plate subjected to distributed load. So, 3 types of problem we will discuss now and you will see the problem have the different characteristics. In some cases, you will find that boundary condition has to be satisfied in the inner region as well as outer region especially in case of this annular plate.

Or in some cases you will find that especially for concentrated load that I have given in the first item there you will find that discontinuity arises at the application of the point of concentrated load in the expression of slope and curvature. So, that means, when the concentrated load is acting just at the below the point of application of the concentrated load, the slope and deflection cannot be found out.

But in the vicinity of the load this can be found out in fact, there is no load which is concentrated; actually any load has some distributed area. So, practically if such a load occurs in the place especially the concentrated load at the center, then we can find in the vicinity of the load the bending moment and then we take to the right and left of the load and we can take the average of these 2 quantities to find the bending moment at the center. So, let us go to the topic.

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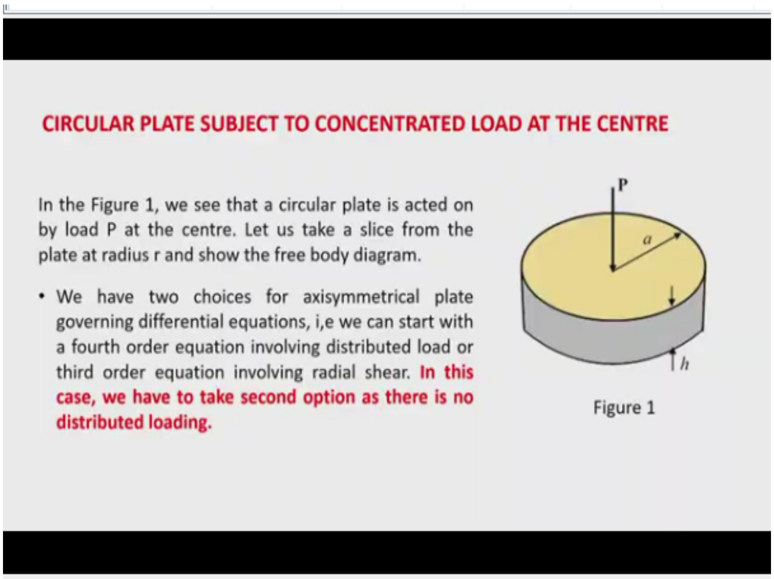

So, here a plate is shown of circular shape, thickness is h and the radius is a, it is subjected to a concentrated load P. Now, this plate may be supported in manner say that it may be clamped along the periphery or it may be simply supported along the periphery. But since it is axisymmetrical problem, this support condition must be also symmetrical about the rotational axis. Now, here rotational axis is the vertical axis that is the line of application of the load.

Now, we can solve such type of problem by taking the differential equation of the equilibrium that we develop for axisymmetrical plate, but we can adopt to any of these 2 equations, one equation is that relating the fourth derivative of deflection to the distributed load and the second one is relating the shear forces to the externally applied load. So, since there is no distributed load, the second option will be better in this case, because in absence of distributed load the forcing function of the differential equation cannot be written here.

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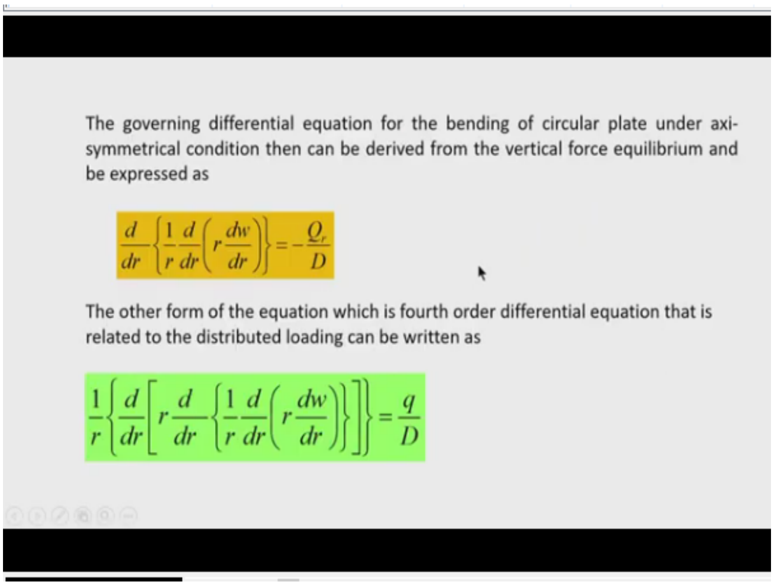

So, let us go to this equation that this second equation which relates this third derivative of deflection with the shear force that can be taken here with convenience, but another form is also existing relating the fourth derivative of the deflection to the distributed load, but this form cannot be used here because of concentrated load action. So, what do we do actually?

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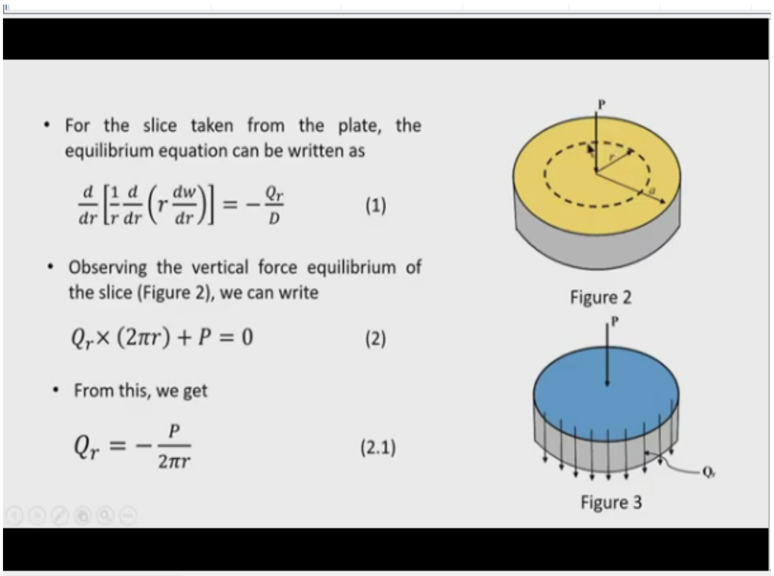

We take a slice of the plate at a radial distance r and consider the equilibrium of this slice. So, these slice is considered here and the equilibrium of vertical forces especially to relate the shear force to the externally applied load, we will consider the equilibrium of the vertical forces. So, free body diagram showing the vertical forces as shown here in this figure and now we can write the differential equation of the plate relating the third derivative of the deflection to the shear force per unit length.

So, the equilibrium equation now takes the form

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]=-\frac{Q_r}{D}$$

So, this equation you can see, this quantity is the slope dw/dr and then differentiation is taken 2 times to obtain the third derivative and to relate this to the shear load/shearing force. Now, observing the vertical force equilibrium of this slice in figure 2 or figure 3 whatever you call then we can write that any radial distance the total vertical shear that is Q_r multiplied by the length of the slice.

Now, this slice is also circular in shape. So, the perimeter will be $2\pi r$. So, $Q_r 2\pi r$ is that total shear force, this must be balanced or sum of the all vertical forces should be equal to 0. So, if I take the shearing force in the same direction of the load, then we can write this with a positive sign and equate to 0. From this equation; equation 2 we get

$$Q_r=-\frac{P}{2\pi r}$$

So, this is the equation for shearing forces per unit length subjected to a concentrated load at the center at a radial distance r. So, now Q_r can be substituted in the equation 1 and then we can get the differential equation relating the concentrated load with the third derivative.

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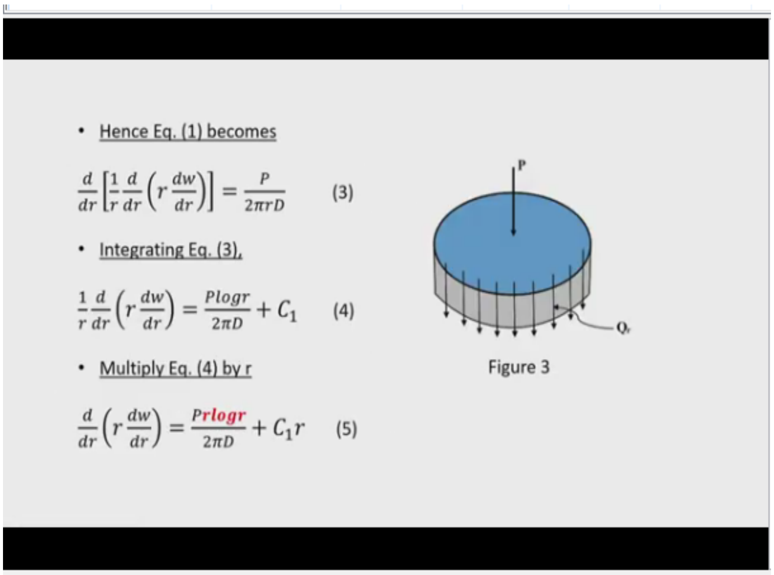

Now, in order to do this, I substitute this Q_r with this - $P/2\pi r$. So, this equation now becomes this,

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]=\frac{P}{2\pi rD}$$

So, this equation is actually the shearing force equation and that has to be integrated to extract the deflected surface w. Now, the steps are underlined.

So, integrating equation 3. So, if I integrate the equation 3, then I get the expression inside the third bracket directly. So, we get-

$$\backslash\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)=\frac{Plogr}{2\pi D}+C_1$$

In the right hand side, I get P by 2pi D is a constant. So, $P/2\pi D$, which is a constant, but 1 by r integration will be log r. So, P logr is given in the numerator. So, P log r divided by $2\pi D$ and then constant of integration appears and we name it as C_1.

Now, you can see that log r term is coming because of reciprocal of r that is appearing in the equation 3 and when we integrated dr/r, the log r terms appear, but mind that these log r has to be evaluated with respect to the base e. So, this is a natural logarithm that we have to evaluate. Now, let us proceed go ahead. So, what we will do here again you see we have to integrate, but before integration, you are seeing that 1/r term is associated in the left hand side.

So, if I integrate this equation, then integration by parts will give you a complicated expression and you will not be able to get this w so easily. So, what can I do? I eliminate 1/r that means I multiply both sides by r then the right hand side there will be no differential coefficient involved. So, what about the order of r or degree of r appears in the right hand side, it will not give trouble for integration.

So, multiplying both sides by r, we now write $\frac{d}{dr}\left(r\frac{dw}{dr}\right) = \frac{Prlogr}{2\pi D} + C_1 r$

That is just after multiplying with r we get this expression equation number 5. Now, you can see I have marked it with the red colour r log r because these appears as a product of 2 functions of r that means when again I integrate I required to integrate this expression rlogr with respect to r using the integration by parts rule.

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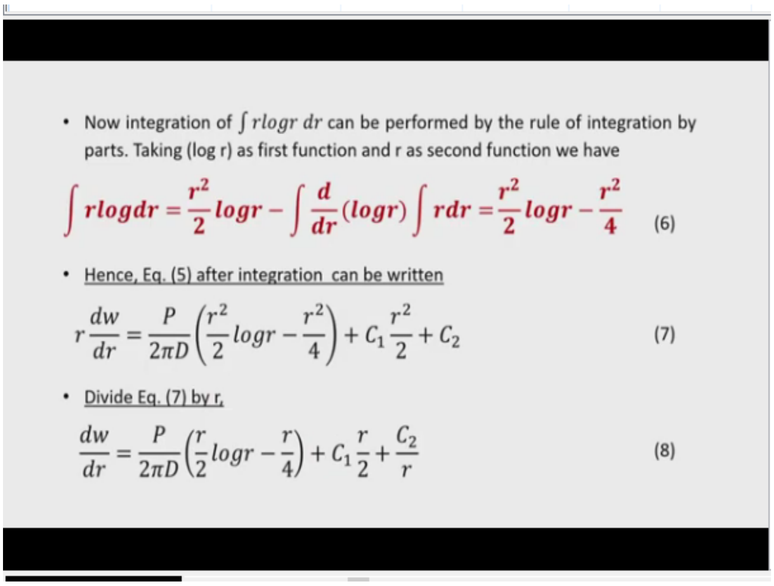

So, next integration of r log dr can be performed by the rule of integration by parts. Now, you have to note here very clearly, that when I integrate this r log r, then "log r" has to be taken as the first function and "r" has to be taken as the second function. So, after integrating we get in the right hand side that r log r integration

$$\int \boldsymbol{rlogdr} = \frac{\boldsymbol{r^2}}{\boldsymbol{2}}\boldsymbol{logr} - \int \frac{\boldsymbol{d}}{\boldsymbol{dr}}(\boldsymbol{logr}) \int \boldsymbol{rdr} = \frac{\boldsymbol{r^2}}{\boldsymbol{2}}\boldsymbol{logr} - \frac{\boldsymbol{r^2}}{\boldsymbol{4}}$$

$$\int rlogdr = \frac{r^2}{2}logr - \int \frac{d}{dr}(logr)\int rdr = \frac{r^2}{2}logr - \frac{r^2}{4}$$

So, hence, equation 5 after integration becomes this r dw/dr = P/2piD and this result is substituted here.

So $r\frac{dw}{dr} = \frac{P}{2\pi D}\left(\frac{r^2}{2}logr - \frac{r^2}{4}\right) + C_1\frac{r^2}{2} + C_2$

Now, here we are integrating a third order equation. So, naturally 3 constants of integration will appear. Now, we are reaching our target that means we have to obtain now the w the deflected surface. So, the one more step is required. So, divide equation 7 by r.

So, we have done this, so that we get shown expression of dw/dr that after this final integration we will be able to get w. So, dividing this both sides by r, we get in the left hand side dw/dr that is the slope of the plate equal to

$$\frac{dw}{dr} = \frac{P}{2\pi D}\left(\frac{r}{2}logr - \frac{r}{4}\right) + C_1\frac{r}{2} + \frac{C_2}{r}$$

Now, this expression represents the slope along any radial distance, because it is axisymmetrical problem. So, at any angle angular reaction it will be same.

Now, you can see the features of this expression, at the point of application of the load that is load is applied at the center because it is a concentrated load. So, r is equal to 0 you will get that dw/dr is undefined quantity. So, that means, in the vicinity of center only this stroke can be evaluated, but not exactly at the center, this is because of concentrated load acting at a single point. In fact, there is no such load acting at a single point load as some distributed area, whatever small maybe.

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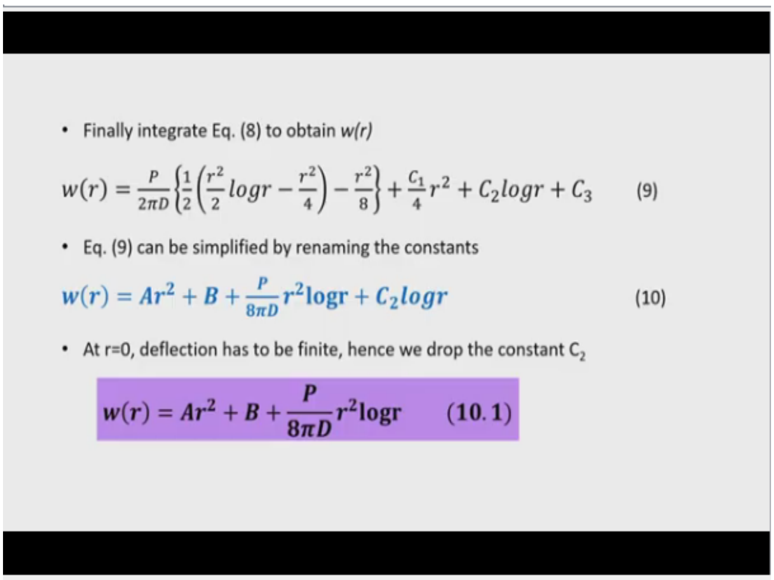

So, finally, integrate equation 8, so after final integration again you see inside the first bracket r log r term appears. So, this expression can be used again here for integration. So, after final integration, we get w(r) that is w is a function of r, i.e. deflected surface is the function of r, because it is axisymmetrical problem. So, there will be no variable theta in the deflection expression.

Because at any orientation in the plate the deflection value will be same along the radial direction. So,

$$w(r) = \frac{P}{2\pi D}\left\{\frac{1}{2}\left(\frac{r^2}{2} logr - \frac{r^2}{4}\right) - \frac{r^2}{8}\right\} + \frac{C_1}{4} r^2 + C_2 logr + C_3$$

After this step no more integration is required, because we obtained w(r) that is the deflected surface, but still some tasks are remaining.

So, what happens here you can see that there are terms which contains P, that is the load. There are terms which contains r square log r, there are terms which contains only r square and there is also term which contains only log r and only a constant. So, the constant can be arranged or clubbed together and we can write.

$$\boldsymbol{w(r) = Ar^2 + B + \frac{P}{8\pi D} r^2 \mathbf{logr} + C_2 logr}$$

Where some r^2 term are clubbed together and the constant is named as A. C_3 we have named fresh as B. Then the term with P is very important because the deflection bending moment whatever we get is due to load P only, if P does not exist in the expression then there is no value of this solution. So, here we get this term with P as $P/8\pi D$ you can see the product of this $r^2 \log r$. And other terms we have just clubbed together with a constant B, suppose $P/2\pi D \times r^2/4$ that constant term is there with r^2 so it is clubbed together with the constant A.

So, constant of integration are A, B and C_2; that have to be evaluated applying the boundary condition at the edges. Now, you can examine the nature of the expression you will be interested because at the point of application of the load will get the maximum deflection as well as you will get the maximum bending moment, it is obvious, but see the expression.

If you put r = 0, then deflection is not finite here, because this term is becoming unbounded and this is $r^2 \log r$. So, 0 multiplied by any high number will be 0. So, this term will give trouble in the expression. So, therefore, for finite deflection, we drop the constant C_2 and therefore, final expression for the deflection can be written as

$$w(r) = Ar^2 + B + \frac{P}{8\pi D} r^2 logr$$

So, this is the final expression for the deflection.

Now, with this expression that is evaluated for a concentrated load acting at the center, we can now proceed to evaluate the value of deflection, bending moment etcetera based on the boundary condition.

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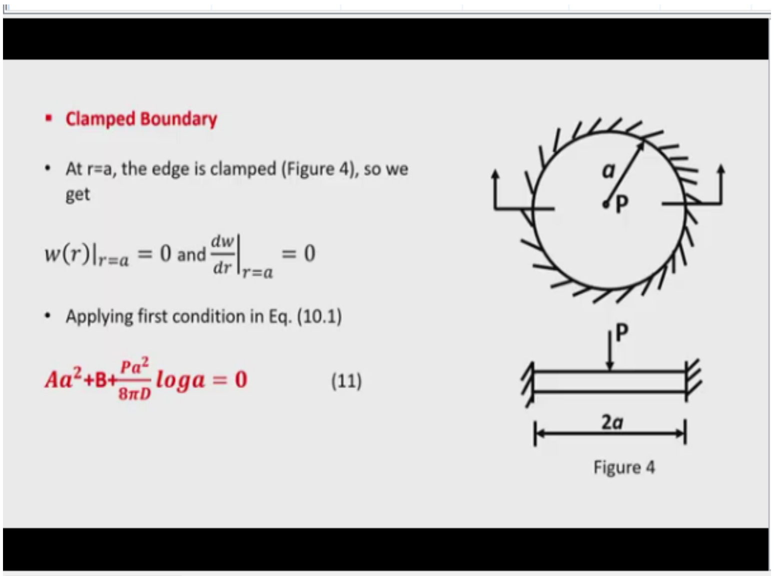

Now, first let us consider a clamp boundary that is plate is clamed at the periphery a circular plate clamped or welded, a plate is welded at the periphery or say for example, a circular slab in a assembly hall or somewhere in a temple it is supported on a ring beam. So, this type of application is there and it is subjected to a concentrated load, there may be concentrated load or load which is distributed over a very small area.

So, in that case, let us find out the maximum deflection bending moment etc. So, at r = a, the edge is clamped, so, the limit of the plate is from 0 to a, a is the radius of the plate, see this figure and here therefore we get the deflection at the boundary is 0 and slope at the boundary is 0 along the radial reaction because of the clamp condition. Now, applying the first condition in equation **10.1**.

What is 10.1? Let us see this 10.1. If I put r = a then w(a) = 0, So

$$Aa^2 + B + \frac{Pa^2}{8\pi D}\log a = 0$$

So, this is the first equation for finding the constant of integration A and B. Second equation has to be obtained by applying the condition on slope at the clamped and the slope along the radial direction is 0. So, the first derivative of w is 0. So, we have to take the first derivative of w, see the expression for w is now known in terms of constant.

So, we can obtain the first derivative of w, you can obtain the first term derivative with respect to r will be twice Ar derivative of B will be 0 because it is a constant, derivative of third term that $P/8\pi D$ will be constant term and here product of 2 terms are there. So, therefore, you have to differentiate r square first and coefficient will be remaining as log r plus you differentiate log r and with that multiplication of r^2 will be there.

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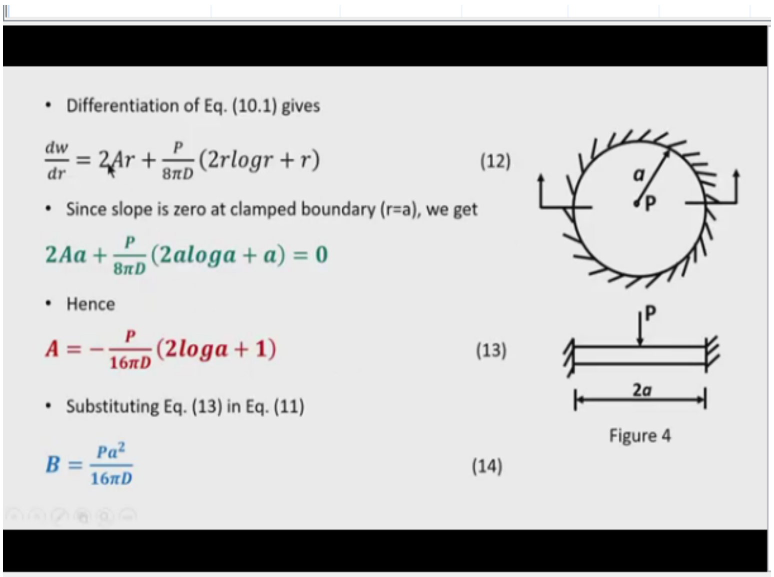

So, using this we obtain the first derivative of this expression as

$$\frac{dw}{dr} = 2Ar + \frac{P}{8\pi D}(2rlogr + r)$$

So, you can see here this expression is the result of the differentiation of r square log r. So, after differentiation of r square log r it has decomposed into 2 terms. Now, see the slope is 0 at the clamped boundary r = a. Now, we apply the clamp boundary condition. Clamp boundary condition is

$$2Aa + \frac{P}{8\pi D}(2aloga + a) = 0$$

$$aAa + \frac{P}{8\pi D}(2a\log a + a) = 0$$ equation (11)

This is the condition.

So, now, we get 2 equations, one is this equation relating A and B here and here also another equation, but here B term does not appear because B is a constant which appearing as a sole term and when we differentiate it vanishes. So, in this expression we get $aAa + \frac{P}{8\pi D}(2a\log a + a) = 0$. From this expression one can see that the constant A can be obtained as minus, I take this term in the right hand side. So, it becomes

$$A = -\frac{P}{16\pi D}(2loga + 1)$$

Now, if I substitute this equation 13 in equation 11, equation 11 is this, because the equation 13 gives the value of A, it gives the value of A in terms of known parameters, because P is known, the flexural rigidity of the plate D which depends on the material constant E, then mu and also the thickness of the plate is known, radius of the plate is known, so, the constant A is fully evaluated.

So, after knowing the constant A, we can find the constant B from the expression 11. So, A is substituted from the previous equation here and we can now evaluate B. After evaluating B this B expression comes as

$$B = \frac{Pa^2}{16\pi D}$$

So, we get the desired constants of integration that is required to completely know the deflected surface. Now; substituting A and B in the deflected surface that we have found from the solution of the differential equation that is 10.1.

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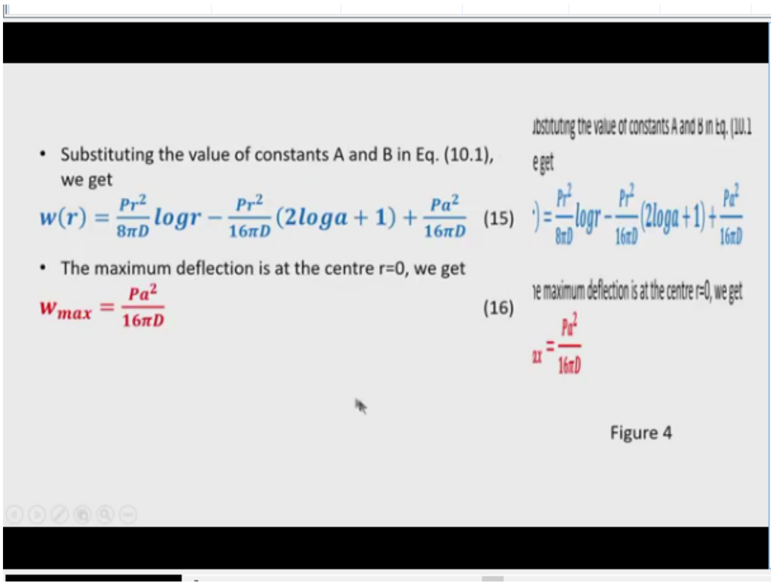

We will find that deflection equation now becomes

$$w(r) = \frac{Pr^2}{8\pi D} logr - \frac{Pr^2}{16\pi D}(2loga + 1) + \frac{Pa^2}{16\pi D}$$

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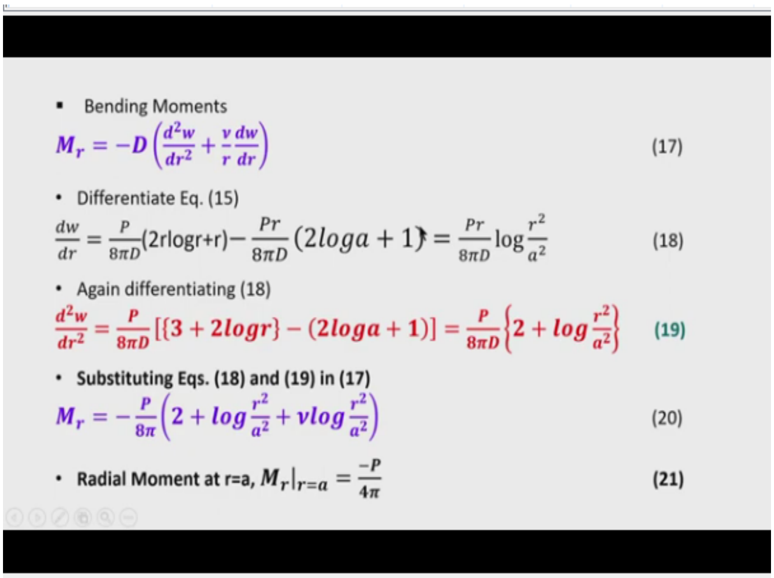

So, bending movement expression becomes the

$$M_r = - D\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right)$$

where, υ is the Poisson's ratio.

Now, after differentiating the w two times we can get the slope as well as the bending moment.

Slope,

$$\frac{dw}{dr} = \frac{P}{8\pi D}(2r\log r + r) - \frac{Pr}{8\pi D}(2\log a + 1) = \frac{Pr}{8\pi D}\log\frac{r^2}{a^2}$$

Bending moment,

$$\boldsymbol{\frac{d^2w}{dr^2} = \frac{P}{8\pi D}[\{3 + 2\log r\} - (2\log a + 1)] = \frac{P}{8\pi D}\left\{2 + \log\frac{r^2}{a^2}\right\}}$$

So, bending moment expression is now written as

$$M_r = - \frac{P}{8\pi}\left(2 + \log\frac{r^2}{a^2} + \nu\log\frac{r^2}{a^2}\right)$$

You can see the expression for radial moment. Examine the expression for radial moment, the radial moment at the center does not exist because the slope is discontinuous, the first derivative is discontinuous, so the second derivative also. So, therefore, you have to find this the radial moment in the vicinity of the application of the load, but at the edges, the clamped bending moment that is fixed and bending moment can be evaluated.

So, by putting 'a' in the expression we can get the maximum bending moment. So, when you put this r^2 by a^2, r is substituted as a. So, this becomes 1. So, log 1 is 0, similarly, this is also becoming 1. So, therefore, the maximum radial moment that is occurring at the clamped edge is equal to

Radial Moment at r=a,

$$M_r\Big|_{r=a} = \frac{-P}{4\pi}$$

So, this is the value of maximum radial bending moment and you can see that this quantity does not depend on the position ratio.

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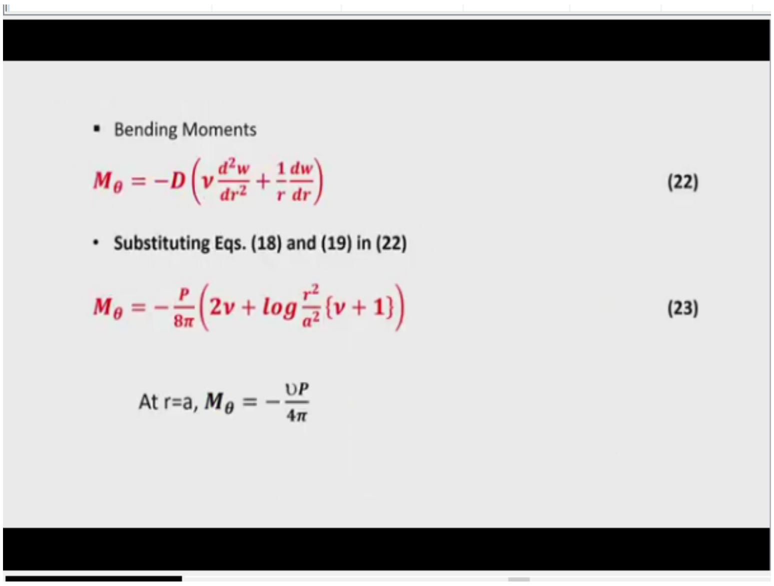

Now, let us come to the circumferential moment. The circumferential bending moment can be obtained by this expression

$$M_\theta = -D\left(\nu\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right)$$

Now, again substituting the second derivative and first derivative of the deflected surface, we get,

$$M_\theta = -\frac{P}{8\pi}\left(2\nu + log\frac{r^2}{a^2}\{\nu + 1\}\right)$$

Now, here you can see this expression M_θ not only dependent on the value of the load and the ratio of the r / a, but also it depends on the material property mu especially. So, at r equal to a again this term becomes 0 because log 1 is 0. So, we get at the edges the tangential moment is

$$M_\theta = -\frac{P}{4\pi}$$

Now, compared to the earlier value of the moment $-P/4\pi$ if you calculate these below, then it will find this moment is reduced because of this poisson ratio effect.

Then one can find also the shearing force but shearing force can be found from the equilibrium in the vertical direction or from the calculation finding the third derivative and equating to the shearing force, because we know these deflected surface so, we can now obtain the derivative up to third order easily.

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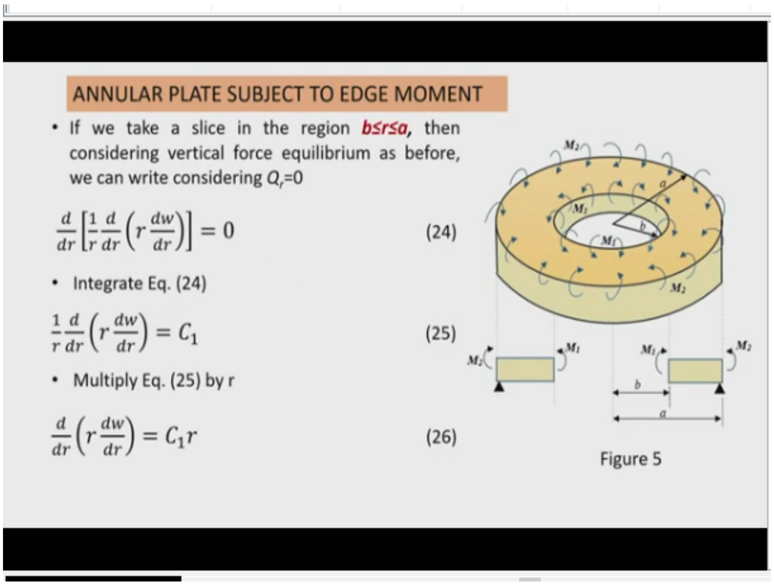

Let us come to a case where the plate has a hole at the center. So, as I told you this type of plate is encountered in practical application in several occasions, for example a circular shaped elevated water tank is to be constructed and then we arrange the column in this circumference of this circle. So, for this column which are arranged in the circumference of the circle, it will be convenient to adopt an annular raft foundation accommodating all the columns.

So, this type of problem is found application in practice and plate theory for finding this stress resultant can be used. Now, here we are examining a case where the annular plate is subjected to a moment at the edges. So, at the outer edge the moment is M_2, at the inner edge moment is M_1 and you can see here that the inner edge has a radius of B that means, the radius of the hole is b and the outer radius is a.

So, total radius of the complete plate is a, but there is a hole at the center concentric hole the radius of the hole is b. So, let us solve this problem. Now, you can see that in the region b are ranging from b to a that is in this region, there is no distributed load. So, that means shearing force will be 0. So, based on that we can write this equation

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]=0$$

So, this is the equation or expression for shear loads and this is equal to 0 because no shearing action is taken place here because no load is there.

So, this indicates that the homogeneous equation has to be solved to find the deflected surface. Now integrate this equation, after integration, we get the quantity inside this third bracket and the quantity is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = C_1$$

So, because this is 0. So, after integration a constant term will appear. Now multiply equation 25 by r. If I multiply equation 25 by r then it becomes

$$\frac{d}{dr}\left(r\frac{dw}{dr}\right) = C_1 r$$

This expression can be again integrated.

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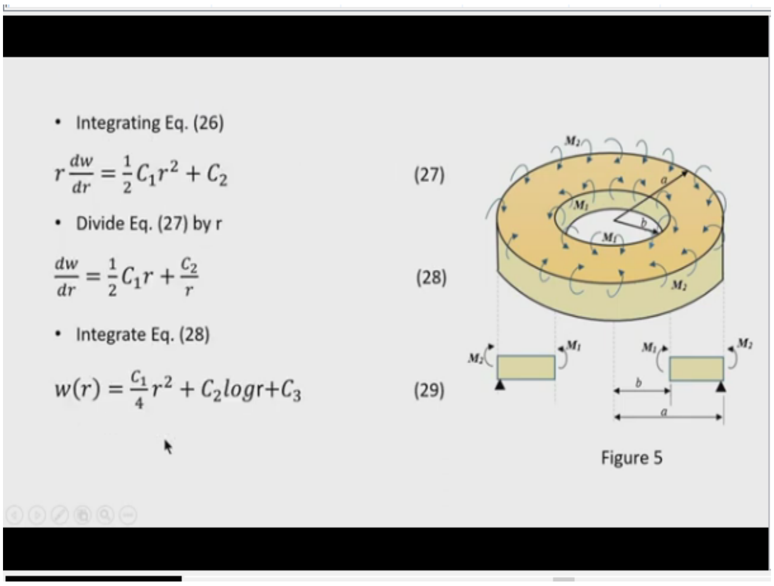

So, after integrating these expressions, we get

$$r\frac{dw}{dr} = \frac{1}{2}C_1 r^2 + C_2$$

Then to obtain w again the tricks have to be applied that we now divide these both sides of the equation with r. So, dividing both sides of the equation by r we

$$\frac{dw}{dr} = \frac{1}{2}C_1 r + \frac{C_2}{r}$$

Now integrate equation 28. After integrating equation 28 we now finally arrive at the desired expression of w (r).

So, w(r) is now equal to

$$w(r) = \frac{c_1}{4} r^2 + c_2 \log r + c_3$$

Now here we are getting 3 constants of integration as expected after final integration and now the 3 constants of integration have to be found out applying the boundary condition at the edges.

Let us consider the edges are simply supported. So, that means at this outer edge a moment M_2 is applied and the inner is M_1 is applied and in the region 0 to b there is no material that is hollow portion of the plate. So, therefore, at the r = 0 there is no question of any stress resultant or deflection because no material is there. So, this expression w(r) = $\frac{c_1}{4} r^2 + c_2 \log r + c_3$ is valid only for the region 'r' in between b to a.

So, only in this region it is valid. So, we can keep now all the terms. So, $\frac{c_1}{4} r^2 + c_2 \log r + c_3$ all the terms now are important to find out the constant a constant deflection of the plate.

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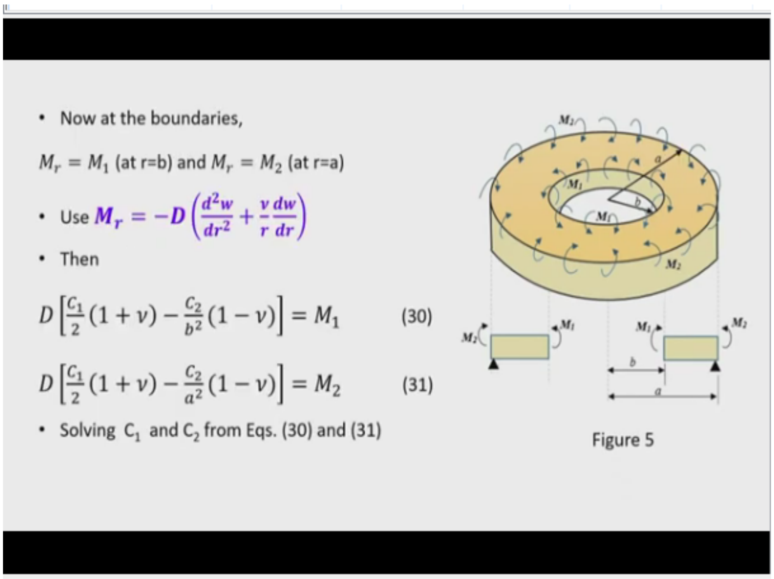

Now, at the boundaries we examine that the radial moment M_1 at r = b. So, at the inner edge the moment that is applied is b and it is a symmetrically applied moment, as moment uniform along the periphery inner periphery and outer periphery. So, that we have to bending under this symmetrical moment, because if the moments are antisymmetric, then axisymmetrical condition the differential equation cannot be used.

So, M_1 is existing at the inner edge. So, therefore, r = b we now equate the bending moment M_r to M_1. Now, expression for M_r is

$$M_r =- D\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right)$$

Now, if you look at this expression that has to be differentiated first derivative and second derivative both have to be found out.

Now, here you can see after differentiation and applying the condition at r = b and r = a that is the inner edge and outer edge we get 2 equations, the first equation is after application of the condition of radial moment at the inner edge at r = b we get

$$D\left[\frac{C_1}{2}(1 + \nu) - \frac{C_2}{b^2}(1 - \nu)\right] = M_1$$

And then after applying the boundary condition at the outer edge that is r = a we get

$$D\left[\frac{C_1}{2}(1 + \nu) - \frac{C_2}{a^2}(1 - \nu)\right] = M_2$$

So, 2 equations now, we get and 2 equations can be solved for C_1 and C_2 and after solving C_1 and C_2 we go for finding these another constant C_2, but application of 2 boundary condition gives only the 2 equations, you see there are 3 constants of integration, but we get only 2 equations. So, let us see how the third constant can be evaluated. First let us obtain the 2 constants of integration C_1 and C_2.

Solving C_1 and C_2 from equation 30 and 31 that can be solved because; this is a linear equation with C_1 and C_2. So, it can be solved by Cramer's rule or by simply this any method you apply from algebra you can get the value of C_1 and C_2.

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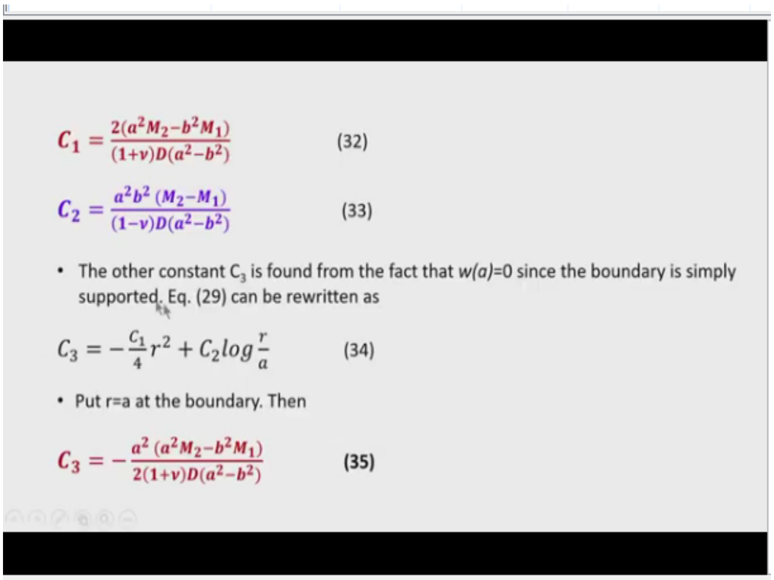

So, obtaining the value of C_1 and C_2 we can write now

$$C_1 = \frac{2(a^2M_2-b^2M_1)}{(1+v)D(a^2-b^2)}$$

Then

$$C_2 = \frac{a^2b^2\,(M_2-M_1)}{(1-v)D(a^2-b^2)}$$

So, these 2 constants are obtained where a is the outer radius and b is the inner radius. As you have seen in this figure and M one is the inner moment and M_2 is the outer moment.

So, that has been shown in the figure. So, other constant C_3 is found from the fact that in the simplest supported condition at the edges say at outer edge the plate is simply supported. So, at r = a deflection must be 0. So, based on that we get another equation. So, substituting r = a here in this exhibition, we get $C_1/\ 4\ r^2 + C_2$ log a + C_3 = 0. So, this equation we get. Now combining this term that is

$$C_3 =- \frac{C_1}{4}r^2 + C_2 log\frac{r}{a}$$

So, these 2 terms we have got now, 2 terms C_1 and C_2 already calculated. So, now substituting C_1 and C_2, you can now get C_3. So, C_3 is calculated as this

$$C_3 = -\frac{a^2 (a^2 M_2 - b^2 M_1)}{2(1+\nu)D(a^2 - b^2)}$$

So, 3 constants of integration is now completely known, because we require only 3 constants C_1, C_2, C_3. Three constants are appearing because we have integrated a third order equation. Instead of fourth order equation we have integrated the third order equation, so, 3 constants are appearing. Now, 3 constants here are completely known, so, we can write the expression for deflection as this. Now, when $M_2 = 0$. For example, here there is no moment at the outer edge only the inner edge moment is acting.

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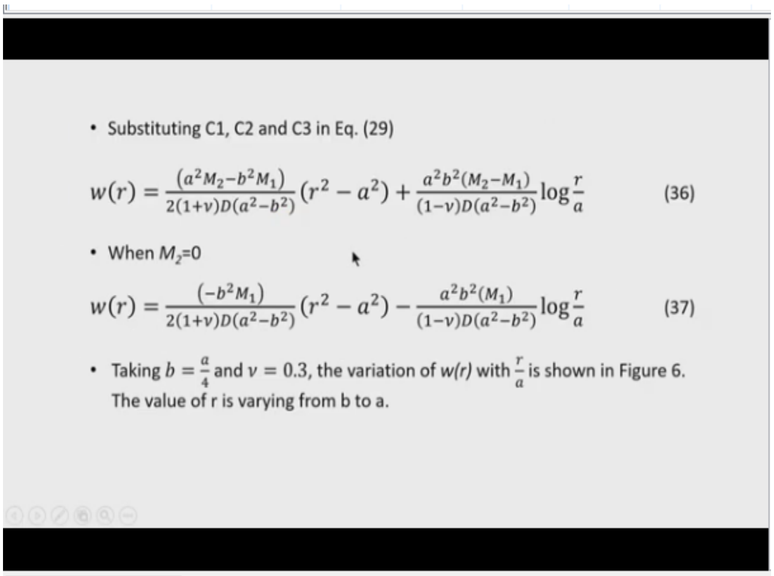

So, put $M_2 = 0$ in this expression and then we can get the expression for w(r) as this. Now, let us see how the deflected surface varies with r by a ratio? So, the range of r is from b to a, b is the inner radius and a is the outer radius. So, we have taken the inner radius as one fourth of the for example to illustrate the solution we have taken the inner radius as the one fourth of the outer radius and we have taken it is a steel plate whose poisson ratio is 0.3.

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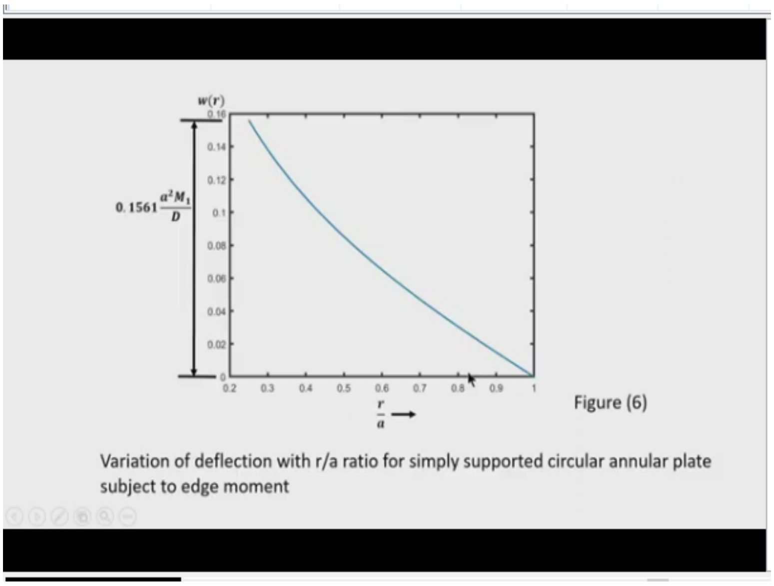

The variation of w(r) with r/a is shown in this figure 6 and 0 is the center, this is the center and this is the outer radius. So, this variation of deflection with r by a ratio for simply supported circular annular plate subjected to as moment is shown only for this edge moment M_1. Now, you can see here that at the simply supported edge the deflection is 0 obviously as expected and it is not going at the center because center there is no material.

So, at 0.25 because r is equal to only 0.25 a because we have taken r = a /4. So, you can see that at r= a/4 that is at the inner edge because this is free the maximum deflection is occurring here.

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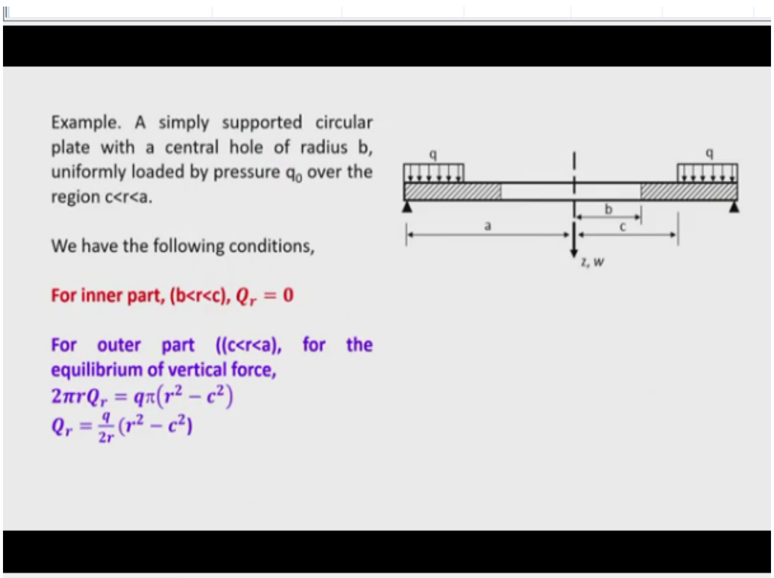

So, in such a manner, we can find the expression for the annular plate. Now here another variety of problem I am discussing this problem will be slightly complicated, because the loading is not continuous, because there is a break in the load although it is axisymmetrical, but the load is not containing the full plate, this is also a plate with hole, the radius of the hole is b, but the loading portion is from c to a, that is c is the radial distance from where the uniform load starts and it ends at the outer edge a.

The hole with the radius b in the plate edges. Now, here we have to obtain the solution in 2 cases, that means first we have to get the solution for inner region and then we have to get the solution for outer region. Inner region I am calling that region where there is no load acting, in the outer region I am calling this portion where the load is acting. So, inner part is say b to c where no load is acting and outer part is from c to a.

Now, if I see the vertical force equilibrium in the outer part, we can see that total shearing force at any slice will be $2\pi r Q_r$ equal to total particle load acting on the slice. So, total vertical load acting on the slides will be the area will be $\pi(r^2 - c^2)$ and q is the load acting on the slice. So, this is the total external load that is acting on this slice should be equal to the total shearing force. So, from that condition we can get

$$Q_r = \frac{q}{2r}(r^2 - c^2)$$

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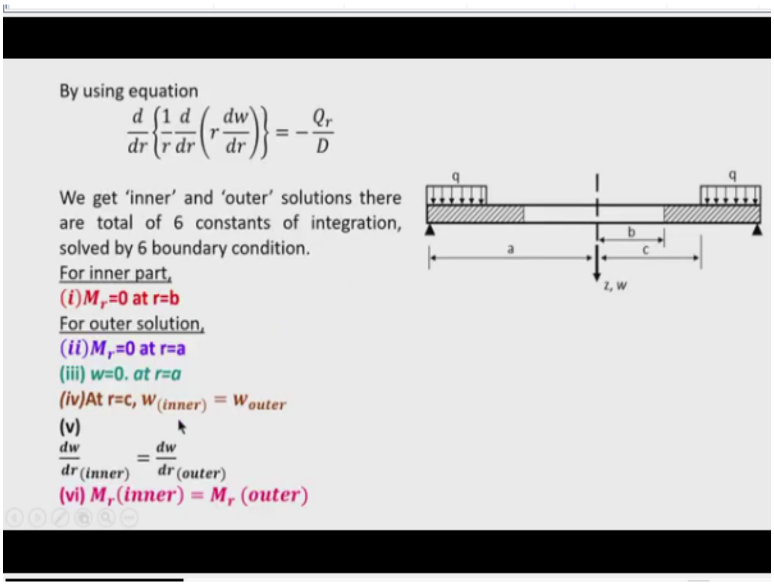

Then we should find the deflected surface by integration procedure and we adopt the third order equation where shear force is related to the third derivative of the deflection. Now, here if you solve this by substituting Q_r that we have obtained here, we can get a solution w. Now, this solution contains 3 constants for the outer region. Then the inner region also we will get a solution inner region actually know these forces acting.

So, naturally this right hand side will be 0, but we get 3 constants of integration. So, 3 + 3 = 6 constants of integration have to be known by applying the boundary condition. So, boundary condition will require 6 in numbers to be applied for to know all the constants of integration and then we can finally know the deflected surface. For the inner part, if you see that r is equal to b, no moment is acting, because it is a free end and no externally applied as moment is there.

So, therefore, we take $M_r = 0$ at r = b. So, this is one condition. Then for outer region or outer solution if you call that we have obtained where the load is considered q. Second condition is at simply supported end that is the radial moment is 0 i.e. M_r is 0 that is second condition then deflection is 0 at the simply supported end that is another condition. So, we get 3 conditions to be imposed only 2 differential equations involving 6 constants of integration.

Then another 3 constants are found at r = c that is this point deflection found from the inner part should match with the deflection found from the outer part for satisfying the compatibility of deflection. So, compatibility of deflection has to be satisfied at the common point. So, at r = c, $w_{inner} = w_{outer}$. Then fifth condition is the slope compatibility has to be satisfied at the common point.

So, at the r = c again $\left(\frac{dw}{dr}\right)_{inner} = \left(\frac{dw}{dr}\right)_{outer}$. That means outer and inner solution differentiation has to be taken that has to be equated. Last boundary condition is the bending moment M_r, there is the radial bending moment at this common point on the inner part should be equal to the radial bending moment at this point from the outer solution. So, in this way we get the 6 solutions, 6 boundary conditions and applying 6 boundary conditions, 6 constants of integration can be evaluated.

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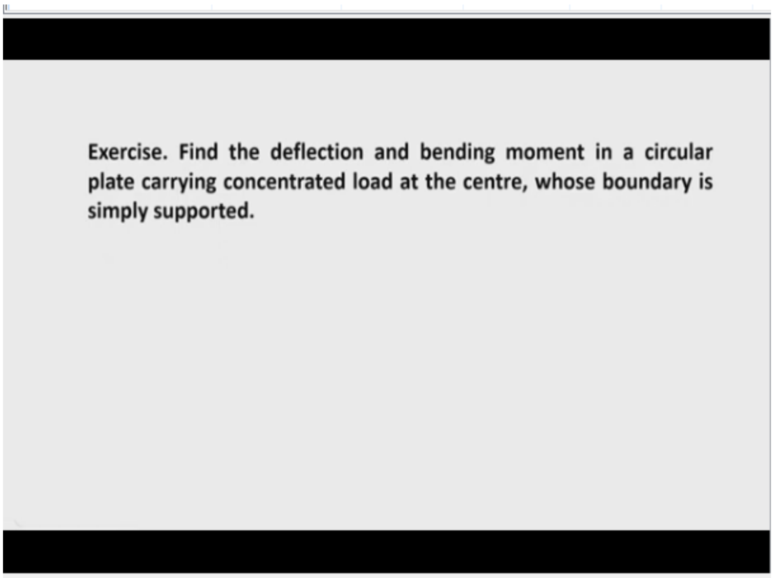

Now, lastly I want to discuss a problem of finding the deflection and bending moment in a circular plate carrying concentrated load at the center, whose boundary condition is simply supported. So, that type of problem; is also occurring in practice that is the simply supported edges and let us see how to solve such problems?

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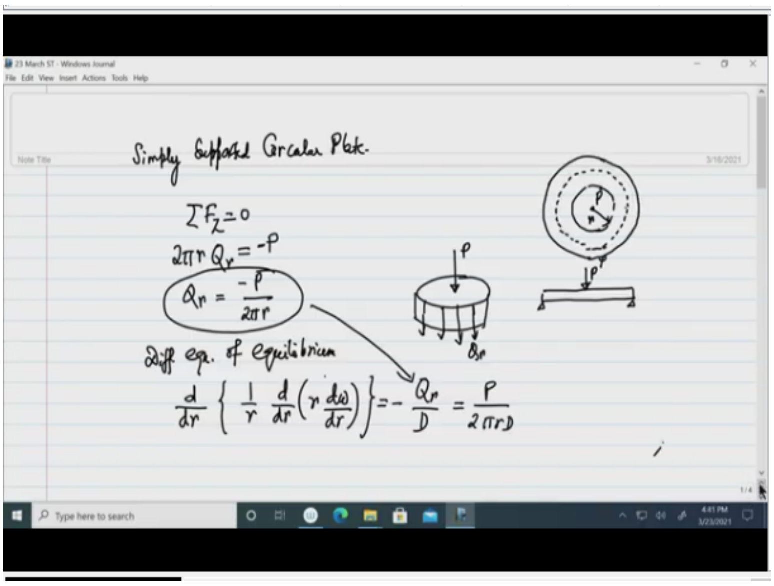

So, we have here this problem of simply supported circular plate that means, we have a plate which is simply supported at the edges and carrying a concentrated load at the center. That means, if you see a section as a center there is a load. So, let us solve the problem of finding the deflection. The equation of the plate deflection is known we can take the third order recursion we can take the fourth order equation.

Now, here we shall take a slice around the load and consider the equilibrium of this slice. So, load acting here P and we have the shearing force that is acting around the slice of magnitude Q_r per unit length. Now, from vertical force equilibrium that is summation of forces in z direction is 0, z direction is the vertical, this is the z direction and direction of w is also. So, taking the equilibrium of forces summation of forces in the z direction to be 0 we now get $2\pi r Q_r$ because we take this slice at a distance of r equal to P.

Both are taken this downward. So, it will be minus. So, Q_r is now equal to $\frac{P}{2\pi r}$ with a minus sign. So, this is the Q r. Now let us write the equations of equilibrium, differential equations. So, differential equation let us write like this

$$\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\} = -\frac{Q_r}{D}$$

Instead of Q_r now, I put this term. So, this term is brought here and now differential equation can be written is $\frac{P}{2\pi r D}$. So, instead of Q r we have now written $-\frac{P}{2\pi r}$. So, this equation now becomes $\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\} = -\frac{Q_r}{D} = \frac{P}{2\pi r D}$

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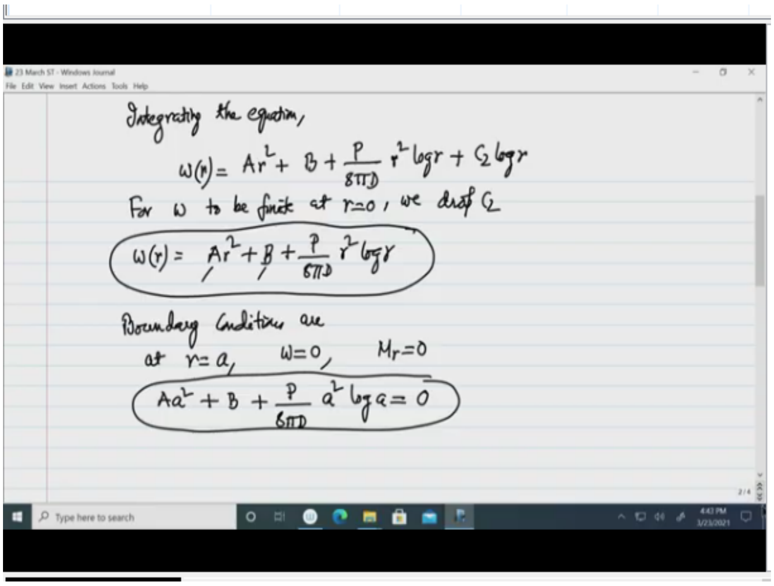

So, after obtaining this we now go to finding the constants of integration that means, the equation can be rearranged in this form final form and then we apply the boundary condition. So, boundary conditions are at r is equal to say a radius a, r = a at the boundary, the deflection is 0 because it is simply supported as well as $M_r = 0$. There is the radial moment equal to 0. So, first condition gives this $Aa^2 + B + \frac{P}{8\pi D}a^2 \log a = 0$. So, this is one equation after applying the boundary condition. Second equation is obtained applying the radial moment condition to be 0 at the edges.

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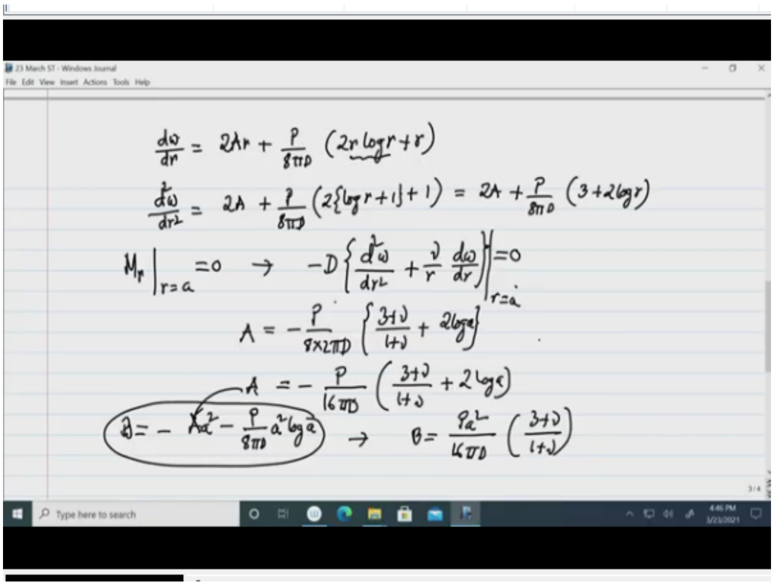

Now, if I go in finding the radial moment then we should know the first derivative and second derivative because the radial moment contains the curvature as well as the slope also in axisymmetrical in polar coordinate system. So, $\dfrac{dw}{dr}$ is now equal to

$$\frac{dw}{dr} = 2Ar + \frac{P}{8\pi D}(2r\log r + r)$$

and second derivative of this

$$\frac{d^2w}{dr^2} = 2A + \frac{P}{8\pi D}\left\{2\left(\log p + 1\right) + 1\right\}$$

Now, apply the condition of bending moments. So, $M_{r|r=a} = 0$, what is the M r?

$$M_{r|} = -D\left\{\frac{d^2w}{dr^2} + \frac{\upsilon}{r}\frac{dw}{dr}\right\}$$ should be equal to 0 at r = a.

So, after substituting these value this d square w by dr square and dw by dr with r substituted as a, we now finally get an expression of a. So, from that condition, we get directly the expression for A as

$$A = -\frac{P}{8 \times 2\pi D}\left\{\frac{3+\upsilon}{1+\upsilon} + 2\log a\right\}$$

So, this expression results after simplification. And ultimately the value of A is now equal to

$$A = -\frac{P}{16\pi D}\left\{\frac{3+\upsilon}{1+\upsilon} + 2\log a\right\}$$

. Substituting a in the expression for deflection that we have previously found at boundary which is equated to 0, we now get the quantity B. So, B is from our deflected surface equation that

$$B = -Aa^2 - \frac{P}{8\pi D}a^2 \log a$$

That means deflection is equated to 0 at the boundary and we get this expression.

So, this expression is evaluated finally after substituting A here, so after substituting a we get after simplification B

$$B = \frac{Pa^2}{16\pi D}\left(\frac{3+\vartheta}{1+\upsilon}\right).$$

So, A and B, two constants are known for this simply supported plate. Let us find this deflected surface.

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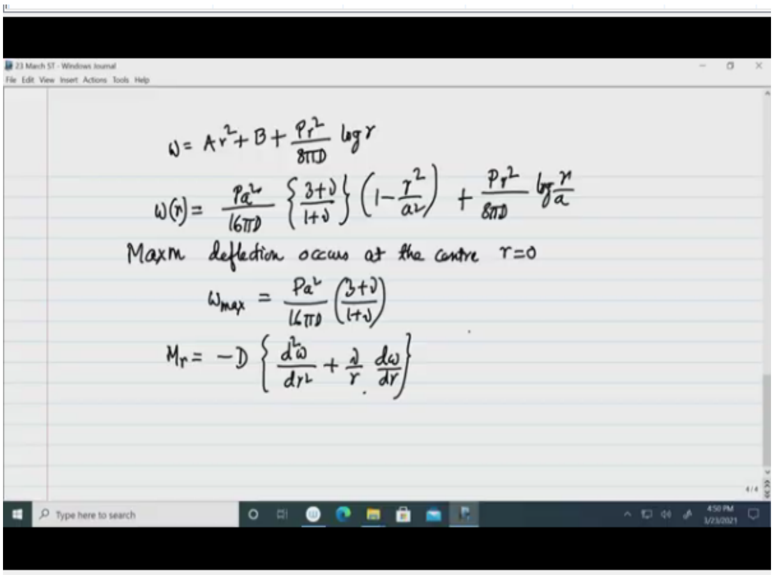

So, once the A and B is known, then we can write

$$w = Ar^2 + B + \frac{Pr^2}{8\pi D}\log r$$

After substituting the value of this A and B, we now finally obtain the deflection in this form that is deflection will be now written or simplified as

$$w(r)=\frac{Pa^2}{16\pi D}\left\{\frac{3+\upsilon}{1+\upsilon}\right\}\left(1-\frac{r^2}{a^2}\right)+\frac{\mathrm{P}r^2}{8\pi D}\log\frac{r}{a}$$

So, this is the expression for deflection. Now, you can see the maximum deflection occurs at the center and it is natural because at the center the load is applied concentrated load at r = 0. So, putting r = 0 here, you will get the maximum deflection. So,

$$w_{\max}=\frac{Pa^2}{16\pi D}\left(\frac{3+\upsilon}{1+\upsilon}\right)$$

This is the maximum deflection. Now, if I want to find the second derivative of this expression and first derivative of expression, then wanted to calculate the bending moment, we can now find the bending moment also. So, the bending moment M_r can be found

$$M_r=-D\left\{\frac{d^2w}{dr^2}+\frac{\upsilon}{r}\frac{dw}{dr}\right\}$$

So, you can see the maximum deflection occurs at the center, but when you find the second derivative and substitute in this expression you will be noticing that at r = 0 the bending moment does not exist because of the discontinuity of the slope and curvature expression. So, because of the concentrated load at the center the slope and derivative exactly at the center r = 0 does not exist.

But we have to find the slope and curvature in the vicinity of the center that 0 + ε or 0 - ε and then we can take ε is a small quantity and then we can take the average of these 2 values to get the bending moment. So, axisymmetrical problem I have fully discussed now and I hope that you can now proceed to tackle the problem of a circular plate subjected to axisymmetrical condition that is on the boundary and also with respect to load.

Two formulations are necessary; one is the shear equation and another is deflection equation that is relating the fourth order derivative to the load; any of these 2 equations you can apply depending on the condition of the problem, thank you.