Plates and Shells

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Module-04
Lecture-12
Axisymmetrical Bending of Circular Plate under Pure Moment and Uniformly Distributed Load

So, today I am starting the lecture 3 of module 4, that I was continuing this loading condition. Then we can convert the partial differential equation into ordinary differential equation, since the deflection and this load and other stress resultant will not be a function of θ. So, in that case we have seen that equation can be arranged in the desired deflection. So, after the deflection is obtained, then we can calculate the bending moment, shearing force and other bending moment and shearing force. Because in that case the radial moment and tangential or circumferential moment, these two moments are of important in axisymmetric problem, but this $M_{r\theta}$ that you know because of symmetry, and Q_θ is also 0. So, now, let us show further application of this axi-symmetrical formulation by giving more examples in practical cases.

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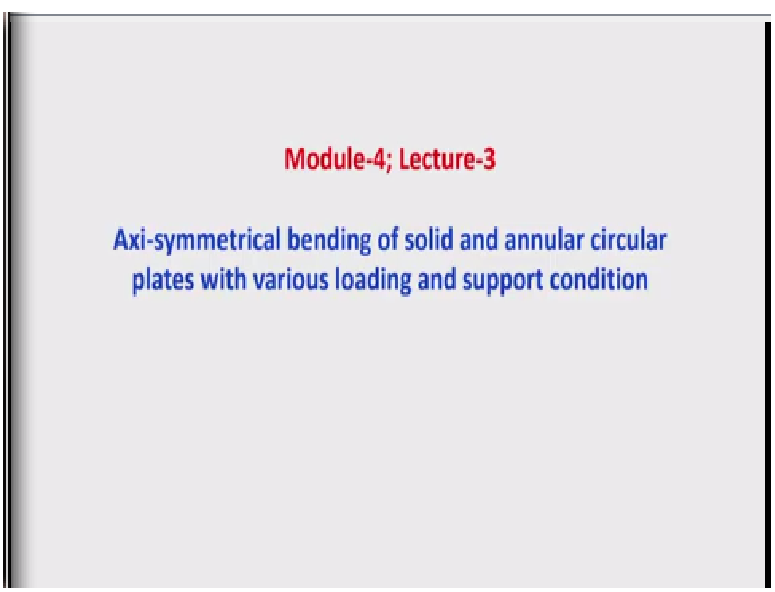

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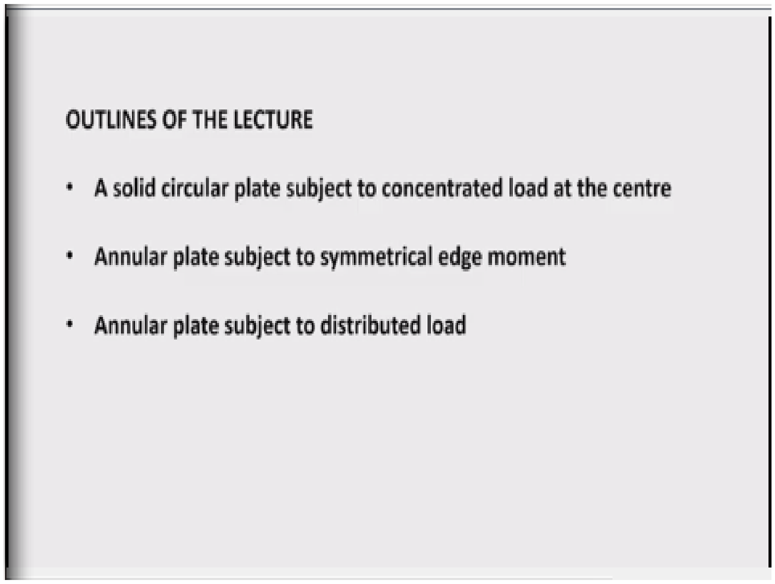

So, we will today discuss the bending of solid plate as well as annular plate supported in different manners. And carrying the load specially in case of this annular plate, or in some case you will find that specially for concentrated load that I have given in the first item. There you will find that discontinuity arises at the application of the point of concentrated load in the expression of slope and this is when the concentrated load is acting just at the below the point of application of the concentrated load, the slope and deflection cannot be found out.

But in the vicinity of the load this can be found out, in fact there is no load which is concentrated. Any load has some distributed area, so practically if such load occurs in the plate, specially the concentrated load at the center, then we can find in the vicinity of the load the bending moment and then we take to the right and left of the load. And we can take the average of these 2 quantities to find the bending moment at the center. So, let us go to the topic.

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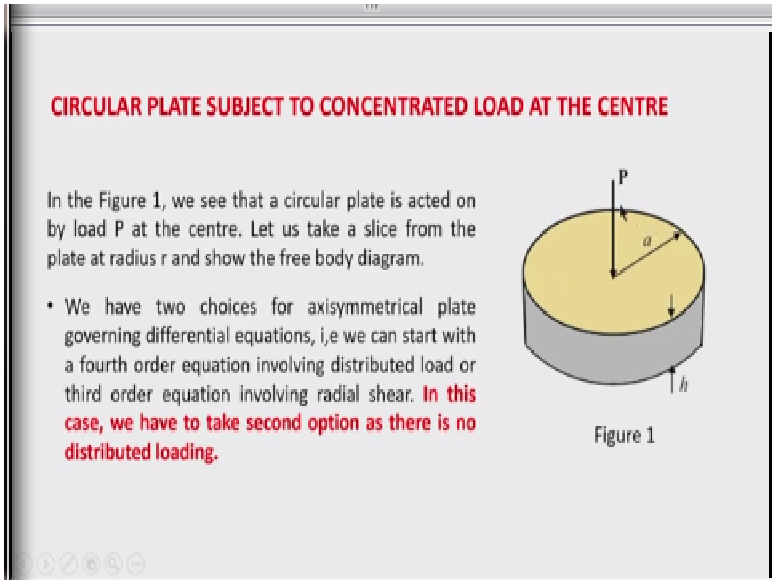

So, here a plate is shown of circle circular shape thickness is h and the radius is a, it is subjected to a concentrated load P. Now this plate may be supported in manners say that it may be clamped along the periphery or it may be simply supported along the periphery. But since it is axi-symmetrical problem, this support condition must be also symmetrical about the rotational axis. Now, here line of application of the load.

Now, solve such type of problem by taking the differential equation of the equilibrium that we develop for axi-symmetrical plate. But we can adopt any of these 2 equations, one equation is that relating the fourth derivative of deflection to the distributed load, and the second one is relating the shear forces to the externally applied load. So, since there is no distributed load, the second option will be better in this case. Because in absence of distributed load the forcing functions of the differential equation cannot be written here.

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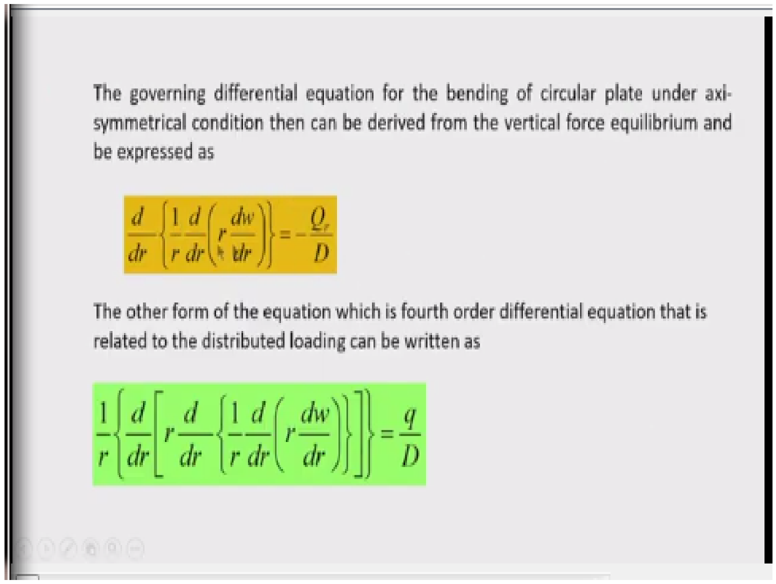

So, let us go to this equation that the second equation which relates deflection with this third derivative of deflection with the shear force that can be taken here with convenience. But another form is also existing; this relating the fourth derivative of the deflection to the distributed load, but this form cannot be used here because of concentrated load action. So, what do we do actually?

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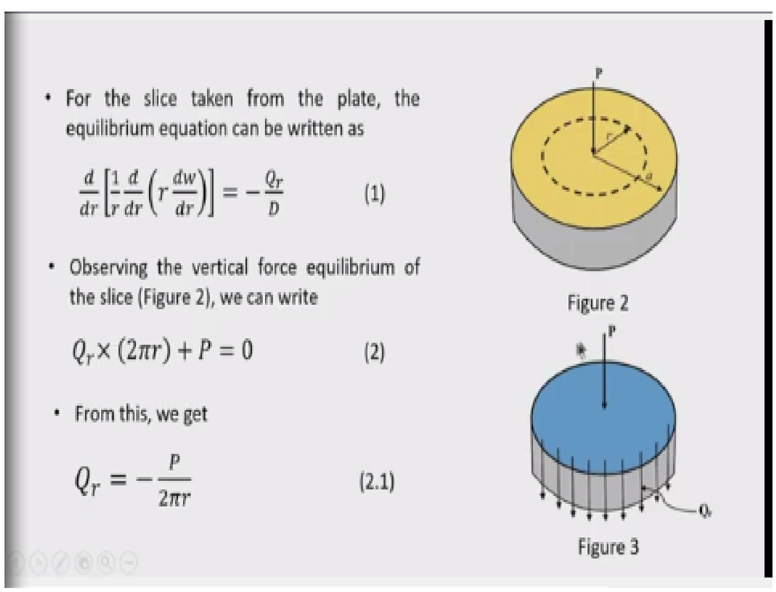

We take a slice of the plate at a radial distance r and consider the equilibrium of this slice. So, this slice is considered here and the equilibrium of vertical forces specially to relate the shear force to the externally applied load will consider the equilibrium of the vertical forces. So, free body diagram showing the vertical forces as shown here in this figure, and now we can write the differential equation, third derivative of the deflection to the shear force per unit length.

So, the equilibrium equation now takes the form $\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] = -\frac{Q_r}{D}$. So, this equation you can see, this quantity is the slope $\frac{dw}{dr}$ and then it is taken to obtain the third derivative and to relate this to the shearing force. Now observing the vertical force equilibrium of this slice in figure 2, you figure 3 whatever you call, then we can write that any radial distance the total vertical shear that is Q_r multiplied by the length of this slice. Now, the slice is also circular in shape, so the perimeter will be $2\pi r$, so $Q_r \times (2\pi r)$ is the total force, this must be balanced, we can write this with a positive sign and equate to 0. From this equation 2, we get $Q_r = -P/2$. So, this is the equation for shearing forces per unit length subjected to a concentrated load at the center at a radial distance r. So, now Q_r can be substituted in the equation 1 and then we can get the differential equation relating the concentrated load with the third derivative.

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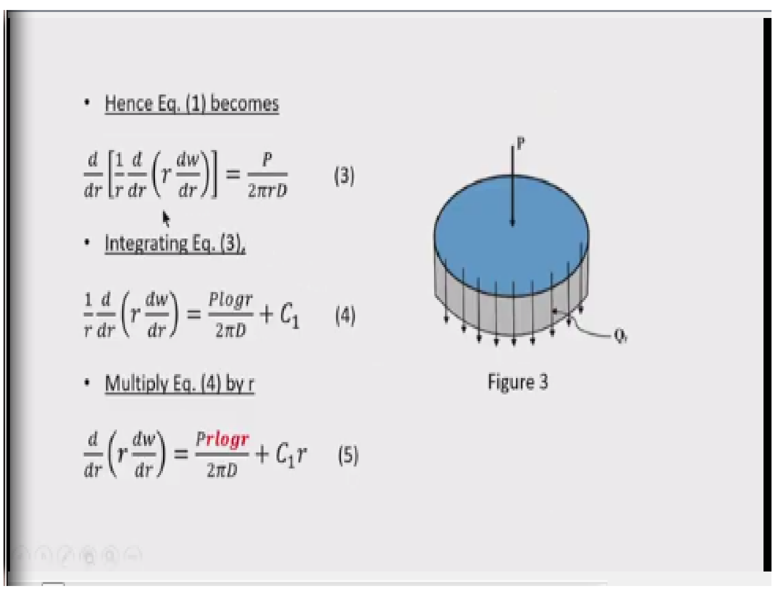

Now, in order of this I write this substitute this Q_r with this $-\frac{P}{2\pi r}$. So, this equation now becomes this $\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] = \frac{P}{2\pi rD}$. So, this equation is actually the shearing force equation and that has to be integrated to extract the deflected surface w. Now, the steps are underlying, so integrating equation 3. So, if I integrate the equation 3, then I get the expression inside the third bracket directly.

So, I get $\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)$, and in the right I will get $\frac{P}{2\pi D}$ is a constant, so $\frac{P}{2}$ constant I can write here. But $\frac{1}{r}$ integration will be $logr$, so $Plogr$ is given in the numerator, so $\frac{Plogr}{2\pi D}$. And then constant of integration appears and we name it as C_1. Now, you can see that $logr$ term is coming because of reciprocal of *r* that is appearing in the equation 3.

And when we integrated *dr*/*r* the $logr$ term appears, but mind that this $logr$ has to be evaluated with respect to the base e. So, this is a natural logarithm that we have to evaluate. Now, let us proceed go ahead, so what we will do here? Again you see we have to integrate, but before integration you are seeing that $\frac{1}{r}$ term is associated in the left hand side. So, if I integrate this equation, then integration by part will give you a complicated expression and you will not be able to get this *w* so easily.

So, what can I do? I eliminate $\frac{1}{r}$ that means I multiply both sides by *r*. So, the right hand side you can see then right hand side there will be no differential coefficient involved. So, whatever the order of *r* or degree of *r* in left hand side it will not give trouble for integration. So, multiplying both side by *r*, we now write $\frac{d}{dr}\left(r\frac{dw}{dr}\right)$ and $\frac{Prlogr}{2\pi D} + C_1 r$.

That is just after multiplying with *r*, we get this expression, equation number 5. Now, you can see I have marked it with the red colour $rlogr$ because this appears as 2 functions of *r*. That means when again I integrate I required to integrate this expression $rlogr$ with respect to *r* using the integration by parts rule.

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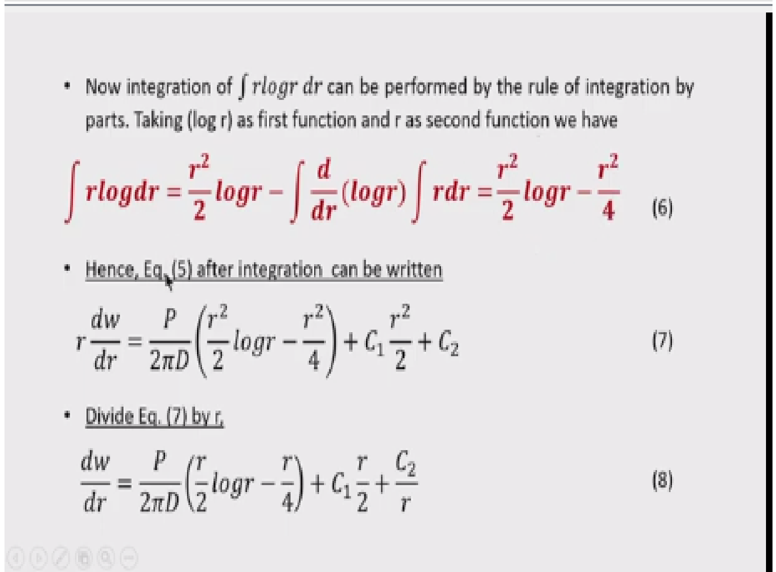

So, now $rlogrdr$ can be performed by the rule of integration by parts. Now, you have to note here very clearly, that when I integrate this $rlogr$, then $logr$ has to be taken as the first function and r has to be taken as the second function. So, after integrating we get in the right hand side, that $\int rlogrdr = \frac{r^2}{2}logr - \frac{r^2}{4}$. So, this is the integration of $rlogrdr$. So, hence, equation 5 after integration becomes this $r\frac{dw}{dr} = \frac{P}{2\pi D}$ and this result is substituted here. So, $\left(\frac{r^2}{2}logr - \frac{r^2}{4}\right) + C_1$, r was there $C_1 r$, so integration will be $C_1\frac{r^2}{2} + C_2$ another constant of integration. Now, here we are integrating a third order equation, so naturally 3 constants of integration will appear. Now we are reaching our target that means we have to obtain now the w, the deflected surface. So, then one more step is there, so divide equation 7 by r, so we have done.

So, that we get sole expression of $\frac{dw}{dr}$ that after this final integration, we will be able to get w. So, divide both sides by r we get in the left hand side $\frac{dw}{dr}$ that is the slope of the plate $= \frac{P}{2\pi D}\left(\frac{r}{2}logr - \frac{r}{4}\right) + C_1\frac{r}{2} + \frac{C_2}{r}$. Now, these expressions represent the slope along any radial distance because it is a axi-symmetrical problem. So, at any angular direction it will be same. Now, you can see the features of this expression.

At the point of application of the load that is load is applied at the center because it is a concentrated load. So, at $r = 0$, you will get that $\frac{dw}{dr}$ is undefined quantity. So, that means, in the vicinity of center only this slope can be evaluated but not exactly the center. This is because of concentrated load acting at a single point. In fact, there is no such load acting at a single point as some distributed area, whatever small maybe. So, finally integrate equation 8, so after final integration again you see inside bracket $rlogr$ term appears. So, this expression can be used again here for integration.

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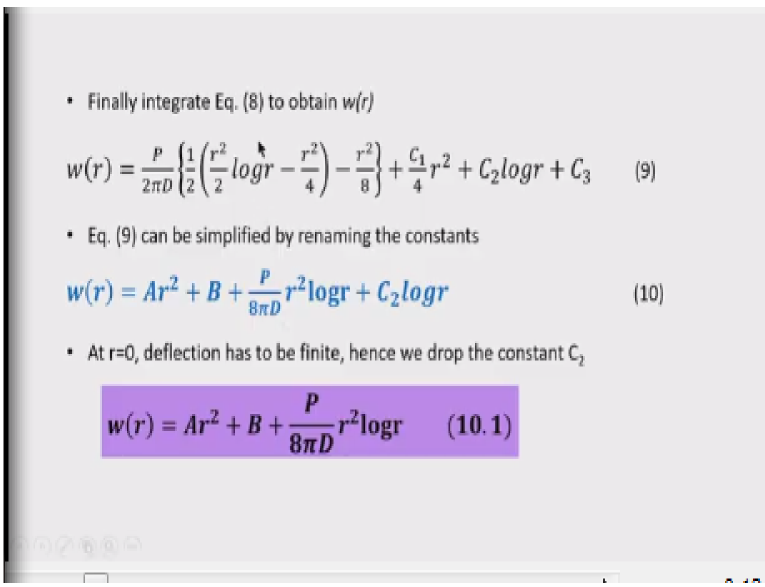

So, after final integration, we get $w(r)$, that is w is a function of r deflected surface is the function of r because it is axi-symmetrical problem. So, there will be no variable θ in the deflection expression because at any orientation the deflection value will be same along the radial direction. So, $\frac{P}{2\pi D}\left\{\frac{1}{2}\left(\frac{r^2}{2}logr - \frac{r^2}{4}\right) - \frac{r^2}{8}\right\} + \frac{C_1}{4}r^2 + C_2 logr$. Because $\frac{C_2}{r}$ term is there, so after integration it becomes $C_2 logr$ and the final constant is C_3. After this step, no more integration is required because we obtained $w(r)$ that is the deflected surface. But still some terms you can see that there are terms which contains P, that is the load, there are terms which contains $r^2 logr$, there are terms which contains only r^2 and there is also a term which contains only $logr$ and a constant.

So, the constant arranged or clubbed together and we can write that $w(r) = Ar^2$ that is r^2 term, we have isolated all r^2 term that is here term is there. That this constant is named as *A*, so Ar^2. Then only this sole constant the C_3 we have taken here and named as *B*, fresh we have named fresh as *B*. Then the term with *P*, so bending moment whatever we get is due to load *P* only, *P* does not exist in the expression with fresh then there is no value of this solution.

So, here we get this term with *P* as $\frac{P}{8\pi D}$, you can see the product of this $r^2 logr$. And other terms we have just clubbed together with a constant, *B*. Suppose $\frac{P}{2\pi D}$ $\frac{r^2}{4}$ their constant term is there with r^2, so it is plot together with the constant *A*. So, final expression for deflection; now can be written $Ar^2 + B + \frac{P}{8\pi D}r^2 logr + C_2 logr$. So, constant of integration are *A*, *B* and C_2 that have to be evaluated applying the boundary condition at the edges.

Now you can examine the nature of the expression and you will be interested because at the point of application of the load will get the maximum deflection as well as you will get the maximum bending moment, it is obvious. But in this expression if you put *r* = 0, then deflection is not finite here. Because this term is becoming unbounded and this is r^2, so 0 multiplied by any high number will be 0, so this term will give trouble in the expression.

So, therefore drop constant C_2 and therefore final expression for the deflection can be written as $w(r) = Ar^2 + B + \frac{P}{8\pi D}r^2 logr$. So, this is the final expression for the deflection. Now, with this expression that is concentrated load acting at the center, we can now proceed to evaluate the value of deflection, bending moment etcetera, based on the boundary condition.

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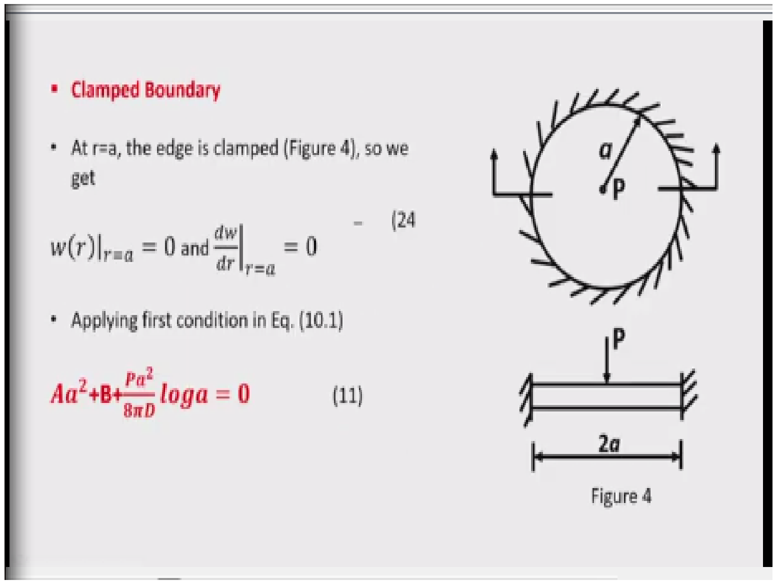

Now, first let us consider a clamped boundary, that is plate is clamped at the periphery, a circular plate clamped or welded a plate is welded at the periphery. Or say for example, a circular slab in assembly hall or somewhere in a temple; it is supported on a ring beam. So, it is subjected to a concentrated load, there may be concentrated load or a load which is distributed over a very small area. So, in that case let us find out the maximum deflection bending moment etcetera. So, at $r = a$, the edge is clamped, so the limit of the plate is from 0 to a, a is the radius of the plate. See this figure and here therefore we get the deflection at the boundary is 0 and slope at the boundary is 0 along the radius of the clamped condition. Now applying the first condition in equation 10.1, what is 10.1? Let us see this 10.1. If I put $r = a$, then equate to 0, so Aa^2+B+ $\frac{Pa^2}{8\pi D}loga = 0$. So, this we get here Aa^2+B+$\frac{Pa^2}{8\pi D}loga$. So, this is the first equation, finding the constant of integration A and B.

The constant of integration has to be obtained by applying the condition of slope. At the clamped end the slope along the radial direction is 0. So, the first derivative of w is 0, we have 2 derivative of w, since the expression for w is now known in terms of constant. So, we can obtain the first derivative of w, you can obtain the first term derivative with respect to r will be $2Ar$ derivative of B will be 0 because it is a constant, derivative of third term that $\frac{P}{8\pi D}$ will be constant term. And here products of 2 terms are there, so therefore you have to differentiate r^2

first and coefficient will be remaining as $logr$ plus you differentiate $logr$ and with that multiplication of r^2 will be there.

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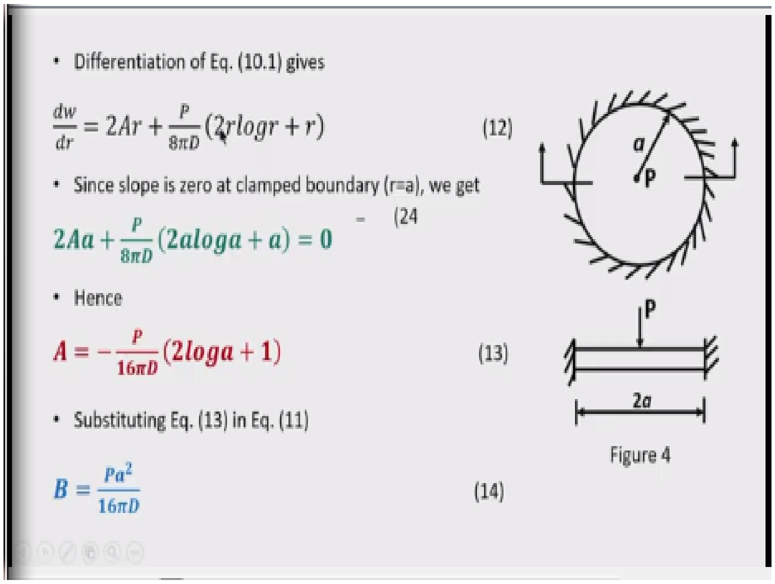

So, using this we obtain the first derivative of this expression as $2Ar + \frac{P}{8\pi D}(2rlogr + r)$. So, you can see here this expression is the result of the differentiation of $r^2 logr$. So, after differentiation of $r^2 logr$, it is decomposed into 2 terms. Now, since the slope is 0 at the clamped boundary $r = a$, now we apply the clamped boundary condition. Clamped boundary condition is $2Aa + \frac{P}{8\pi D}(2aloga + a) = 0$, this is the condition.

So, now we get 2 equations, one is this equation relating A and B here and here also another equation but here B term does not appear because B is a constant which appearing as a sole term and when we differentiate it, it vanishes. So, in this expression we get $2Aa + \frac{P}{8\pi D}(2aloga + a) = 0$. From this expression one can see that the constant A obtained as minus I take this term in the right hand side, so it becomes $-\frac{P}{16\pi D}$, you can see 2 is brought here, so 8×2, $16\pi D$ bracket $2loga$ because it is divided by a, so $2loga$ and it is a/a is 1.

So, we get the quantity or constant A as $-\frac{P}{16\pi D}(2loga + 1)$ this equation 13. In equation 11, equation 11 is this because it gives the value of A in terms of known parameters. Because D is known the flexural rigidity depends on the material constant E then ν and also the thickness of the plate is known, radius of the plate is known, so the constant A is fully evaluated.

So, after knowing the constant A, we can find the constant B from the expression 11. So, A is substituted from the previous equation here and we can now evaluate B. After evaluating B, this B expression comes as $B = \frac{Pa^2}{16\pi D}$. So, we get the desired constants of integration that is required to fully know, completely know the deflected surface. Now, substituting A and B in the deflected surface that we have found from the solution of the differential equation that is 10.1.

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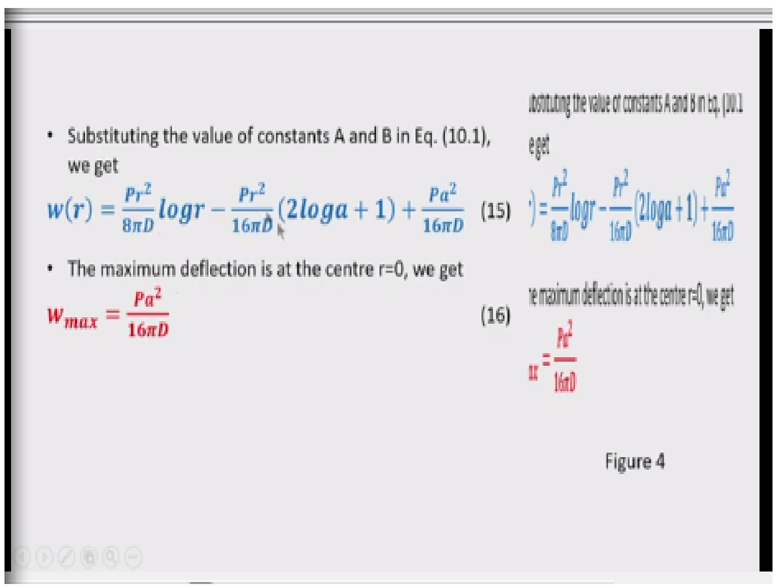

We will find that deflection equation now becomes $w(r) = \frac{Pr^2}{8\pi D} logr - \frac{Pr^2}{16\pi D}(2loga + 1) +$ the constant term.

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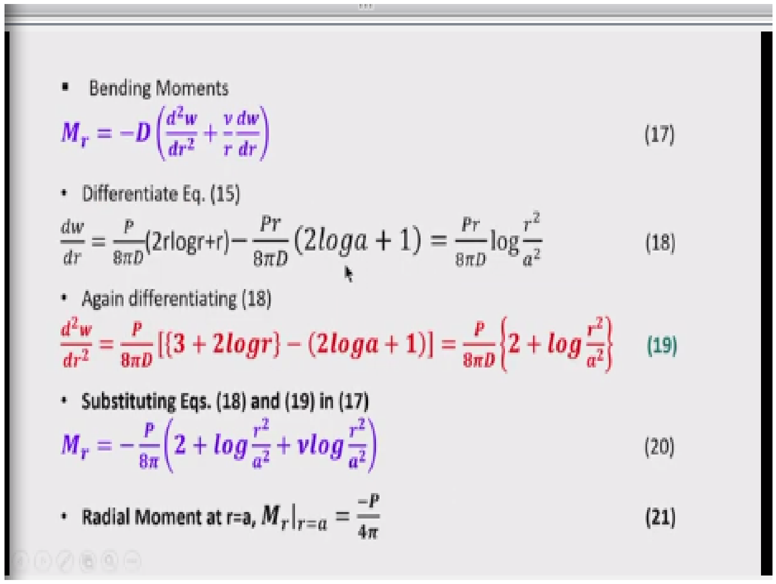

So, bending moment expression becomes the $-\ D\ \frac{d^2w}{dr^2} + \frac{\nu}{r}$, ν is the Poisson ratio $\frac{dw}{dr}$. Now after differentiating the w, 2 times and then we can get equation 19. So, bending moment expression is now written as $-\frac{P}{8\pi}\left(2\ +\ log\frac{r^2}{a^2}\ +\ \nu log\frac{r^2}{a^2}\right)$. You can see the expression for radial moment, examine the expression for radial moment.

The radial moment at the center does not exist because the slope is discontinuous, the first derivative is discontinuous, so the second derivative also. So, therefore you have to find this the radial moment in the vicinity of the application of the load. But at the edges, the clamped bending moment that is fixed end bending moment by putting a in the expression we can get the maximum bending moment.

So, when you put this $\frac{r^2}{a^2}$, r is substituted as a. So, this becomes 1, so $log 1$ is 0, similarly this is also becoming 1. So, therefore the maximum radial moment that is occurring at the clamped edge is equal to $\frac{-P}{4\pi}$. So, this is the value of maximum radial bending moment and you can see that this quantity does not depend on the Poisson ratio. Now, let us come to the moment.

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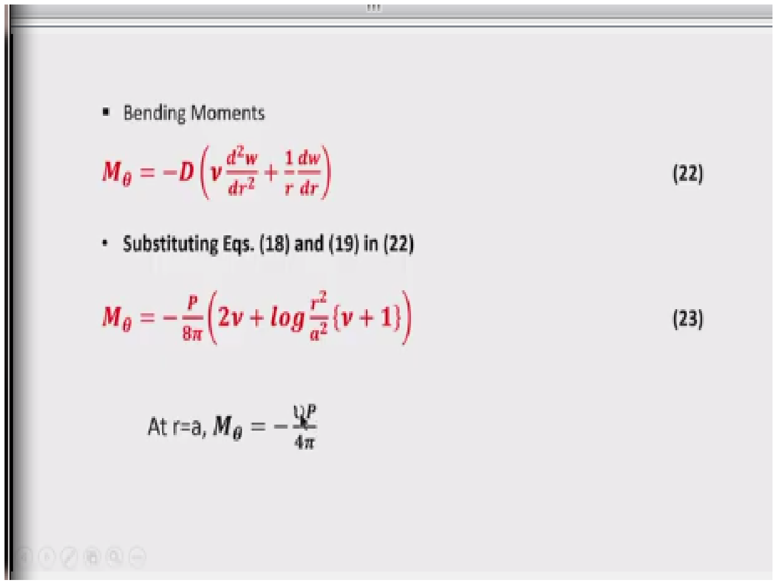

The circumferential bending moment can be obtained by this expression $-D\left(\nu \frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right)$. Now again substituting the second derivative and first derivative of the deflected surface, we get that $M_\theta = -\frac{P}{8\pi}\left(2\nu + log\frac{r^2}{a^2}\{\nu + 1\}\right)$. Now, here you can see this expression M_θ is not only dependent on the value of the load and the ratio of the r/a but also it depends on the material property ν specially.

So, at $r = a$ again this term becomes 0 because $log1$ is 0. So, we get at the edges is $M_\theta = -\frac{P}{4\pi}$. Now compared to the earlier value of the moment $\frac{-P}{4\pi}$, if you calculate these value then you will find this moment is reduced because of this Poisson ratio effect.

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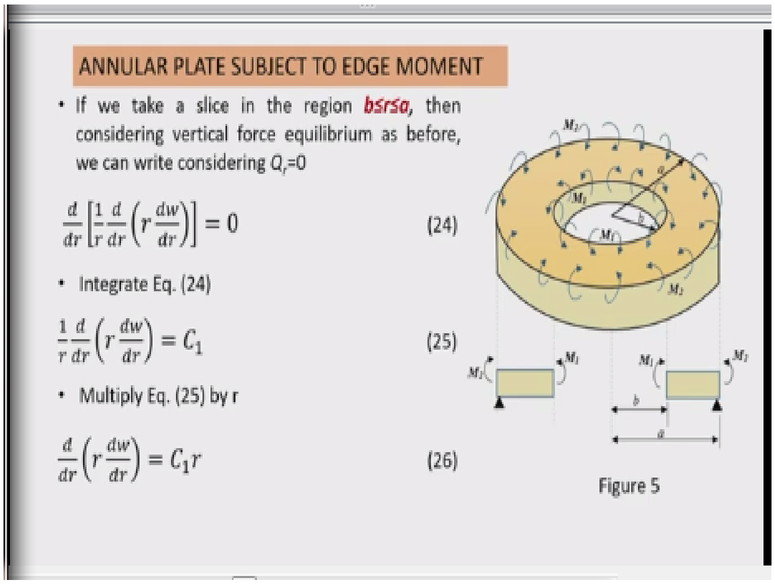

So, then one can find also the shearing force, but shearing force can be found from the equilibrium in the vertical direction or from the calculation finding the third derivative and equating to the shearing force because we know this deflected surface. So, we can now obtain the derivative up to third order easily. Let us come to a case where the plate has a hole at the center.

So, as I told you this type of plate is encountered in practical application in several occasions. For example, a circular shaped elevated water tank is to be constructed and then we arranged the column in this circumference of this circle. So, for this column which are arranged in the circumference of the circle, it will be convenient to adopt an annular rough foundation accommodating all the column.

So, this type of problem is found application in practice and plate theory for finding these stress resultant can be used. Now, here we are examining a case where the annular plate is subjected to a moment at the edges. So, at the outer edge, the moment is M_2; at the inner edge, moment is M_1. And you can see here that the inner hole, inner edge has a radius of b, that means, the radius of the hole is b and the outer radius is a.

So, total radius of the complete plate is a, but there is a hole at the center concentric hole the radius of the hole is b. So, let us solve this problem. Now you can see that in the region, r ranging from b to a, this region, there is no distributed load. So, that means shearing force will be

0. So, based on that we can write this equation $\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] = 0$. So, this is the equation or expression for shear force equal to 0 because no shearing action is taken place here because no load is there.

So, solve homogeneous equation to find the deflected surface. Now integrate this equation, after integration we get the quantity inside this third bracket and the quantity is $\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = C_1$. So, because this is 0, so after integration at constant term will appear. Now multiply equation 25 by r, if I multiply equation 25 by r then it becomes $\frac{d}{dr}\left(r\frac{dw}{dr}\right) = C_1 r$, this expression can be again integrated.

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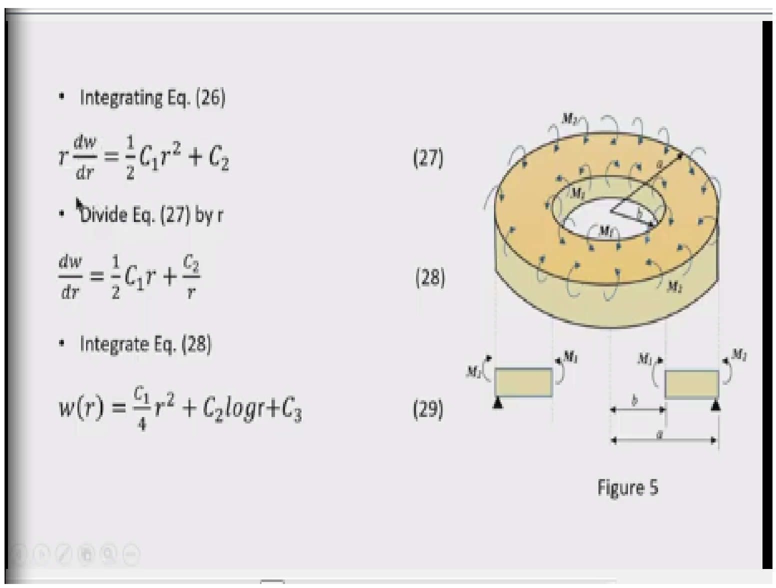

So, after integrating this expression, we get $r\frac{dw}{dr} = \frac{1}{2}C_1 r^2 + C_2$. Then to obtain w again the tricks have to be applied that we now divide these both sides of the equation with r. So, dividing both sides of the equation by r we get $\frac{dw}{dr} = \frac{1}{2}C_1 r + \frac{C_2}{r}$. Now integrate equation 28, after integrating equation 28 we now finally arrived at the desired expression, so w is nothing but $\frac{C_1}{4}r^2$ because integration of r with respect to dr will be $r^2/2$. So, naturally 1/4 term, a factor is coming here and this term will be $C_2 log r + C_3$. Now, here we are getting 3 constants of integration as expected after final integration. And now 3 constants of integration have to be found out applying the boundary condition at the edges. Let us consider the edges are simply supported. So, that means at this outer edge a moment M_2 is applied and the inner edge M_1 is applied.

And in the region 0 to b, there is no material that is a hollow portion of the plate. So, for at $r = 0$, there is no question of any stress resultant or deflection because no material is there. So, this expression $w(r) = \frac{C_1}{4}r^2 + C_2 log r + C_3$ is valid only for the region r in between b to a. So, only in this region it is valid, so we can keep now all the terms. So, $\frac{C_1}{4}$, C_2, C_3 all the terms now are important to find out the deflection of the plate.

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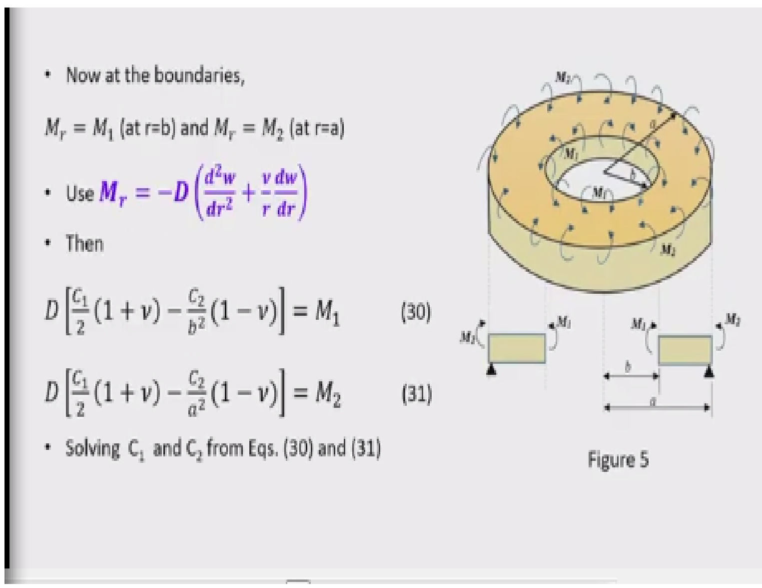

Now at the boundaries, we examine that the radial moment M_1 at $r = b$. So, at the inner edge, the moment that is applied is M_1 and it is a symmetrically applied moment. Edge moment uniform along the periphery, inner periphery and outer periphery, so that you have this bending under this symmetrical moment because if the moments are anti-symmetrical condition, the differential equation cannot be used.

So, M_1 is existing at the inner edge. So, therefore at $r = b$, we now equate the bending moment M_r to M_1. Now expression for M_r is $-D$ bracket the curvature that is the second derivative of w with respect to r and then $+ \frac{\nu}{r}$, ν is the Poisson ratio and first derivative of w with respect to r. First derivative and second derivative both have to be found out. Now here you can see after differentiation and applying the condition at $r = b$ and $r = a$ that is the inner edge and outer edge, we get 2 equations.

The 1st equation is after application of the condition of radial moment at the inner edge at $r = b$ we get, $D\left[\frac{C_1}{2}(1 + \nu) - \frac{C_2}{b^2}(1 - \nu)\right] = M_1$. And then after applying the boundary condition at the outer edge that is $r = a$ we get, $D\left[\frac{C_1}{2}(1 + \nu) - \frac{C_2}{a^2}(1 - \nu)\right] = M_2$. So, 2 equations now we get and 2 equations can be solved for C_1 and C_2.

And after solving C_1 and C_2 we go for finding these another constant C_3. But application of 2 boundary conditions gives only the 2 equations. You see, there are 3 constants of integration but we get only 2 equations. So, let us see how the third constant can be evaluated. First let us obtain the 2 constants of integration C_1 and C_2. Solving C_1 and C_2 from equation 30 and 31 that can be solved, because this is a linear equation with C_1 and C_2. So, it can be solved by Cramer's rule or by simply any method you apply from algebra you can get the value of C_1 and C_2.

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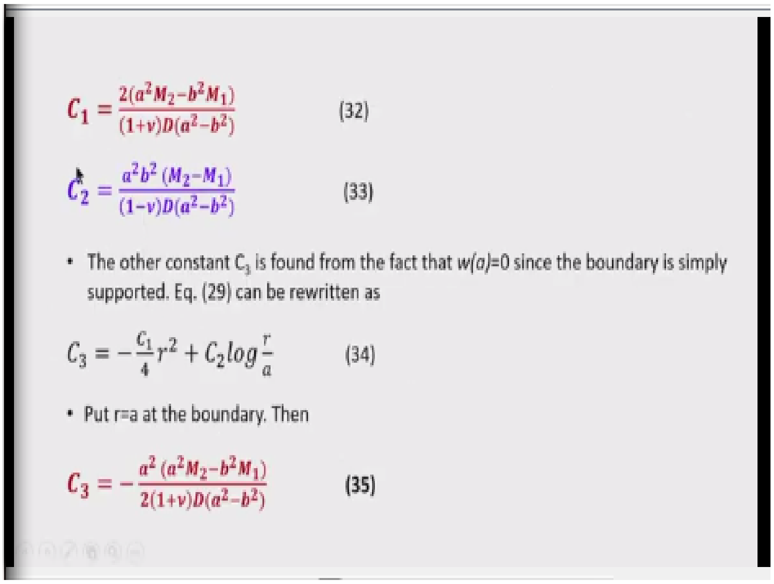

So, obtaining the value of C_1 and C_2 we can write now $C_1 = \frac{2(a^2M_2 - b^2M_1)}{(1+\nu)D(a^2-b^2)}$. Then $C_2 = \frac{a^2b^2(M_2 - M_1)}{(1-\nu)D(a^2-b^2)}$. So, these 2 constants are obtained where a is the outer radius, b is the inner radius, M_1 is the inner moment and M_2 is the outer moment. So, that has been shown in the figure.

So, other constant C_3 is found from the fact that $w(a)$=0 since simply supported condition at the edges, say at outer edge the plate is simply supported, so at $r = a$ deflection must be 0. So, based on that we get another equation. So, substituting $r = a$ here in this expression, we get $\frac{C_1}{4}a^2 + C_2 log a + C_3 = 0$, so this equation we get. Now combining this term that is $C_3 = -\frac{C_1}{4}r^2 + C_2 log\frac{r}{a}$.

So, these 2 terms we have got now, 2 terms C_1 and C_2 already calculated, so now substituting C_1 and C_2 you can now get C_3. So, C_3 is calculated as this, $C_3 = -\frac{a^2(a^2M_2 - b^2M_1)}{2(1+\nu)D(a^2-b^2)}$, M_2 is the moment at the outer edge. So, 3 constants of integration are now completely known. Because we require only 3 constants C_1, C_2, C_3, 3 constants are appearing because we have integrated a third order equation. Instead of fourth order equation we have integrated the third order equation, so 3 constants are appearing. Now 3 constants here are completely known.

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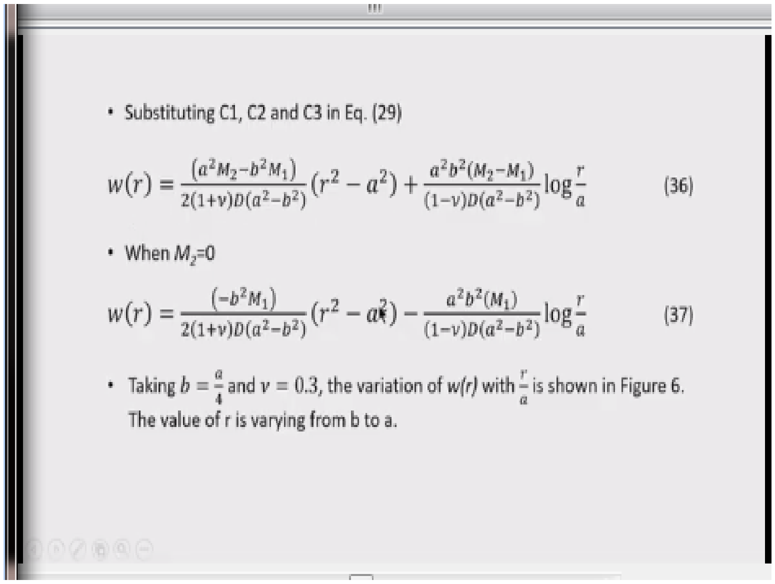

So, we can write the expression for deflection. $M_2 = 0$, for example here there is no moment at the outer edge, only the inner edge moment is acting. So, put $M_2 = 0$ in this expression and then we can get the expression for $w(r)$ as this. Now, let us see how the deflected surface varies with r/a ratio. So, the range of r is from b to a, b is the inner radius and a is the outer radius. So, for example to illustrate the solution we have taken the inner radius as the one fourth of the outer radius. And we have taken it is a steel plate whose Poisson ratio is 0.3. The variation of $w(r)$ with the r/a is shown in this figure 6.

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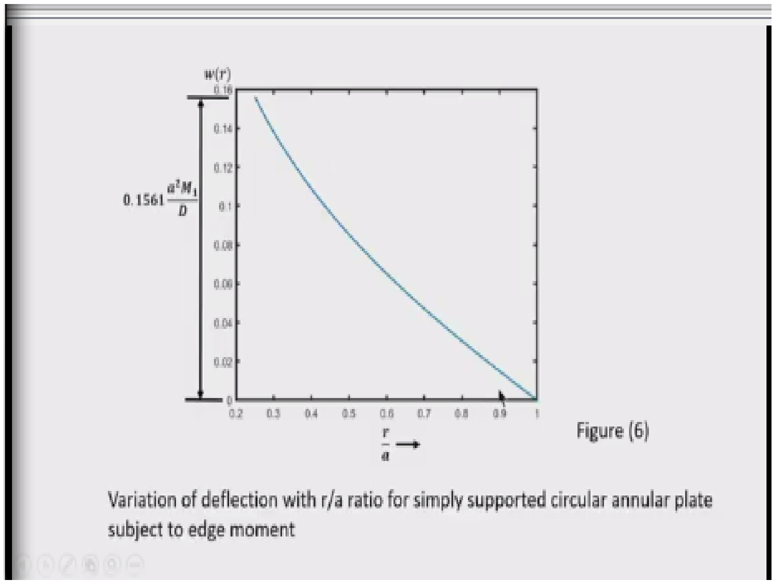

Here you can see this is the end, 0 is the center, this is the center and this is the outer radius. So, variation of deflection with *r*/*a* ratio for simply supported circular annular plate subjected to edge moment is shown only for this edge moment M_1. Now you can see here that at the simply supported edge the deflection is 0 obviously as expected and, it is not going at the center because at center there is no material. So, at 0.25 because *r* is equal to only 0.25 *a* because we have taken *r* = *a*/4. So, you can see that at *r* = *a*/4, that is at the inner edge because this is free, the maximum deflection is occurring here.

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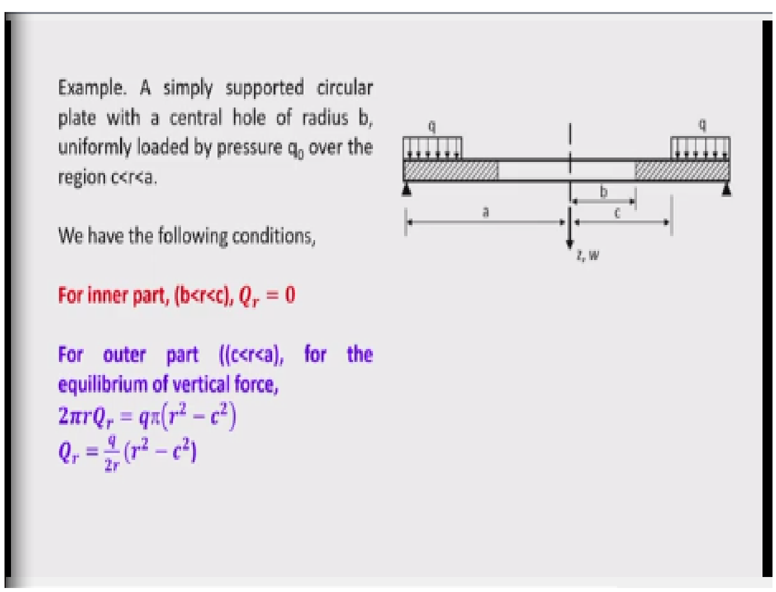

So, in such a manner we can find the expression for the annular plate. Now here another variety of problem I am discussing. This problem will be slightly complicated because the loading is not continuous. Because there is a break in the load although it is axi-symmetrical, but the load is not containing the full plate. This is also a plate with hole, the radius of the hole is b but the loading portion is from c to a. That is c is the radial distance from where the uniform load starts and it ends at the outer edge a, the hole with the radius b in the plate exist. Now here we have to obtain the solution in 2 cases that means first we have to get the solution for inner region and then we have to get the solution for outer region. Inner region I am calling that region to where there is no load acting and the outer region I am calling this portion where the load is acting.

So, inner part is say b to c where no load is acting and outer part is from c to a. Now if I see the vertical force equilibrium in the outer part, we can see that total shearing force at any slice will be $2\pi r Q_r$ is equal to total vertical load acting on the slice. So, total vertical load acting on the slice will be the total vertical load acting on the area, which will be $\left(r^2 - c^2\right)$ and q is the load acting on the slice. So, this is the total external load that is acting on this slice should be equal to the total shearing force. So, from that condition we can get $Q_r = \frac{q}{2r}\left(r^2 - c^2\right)$.

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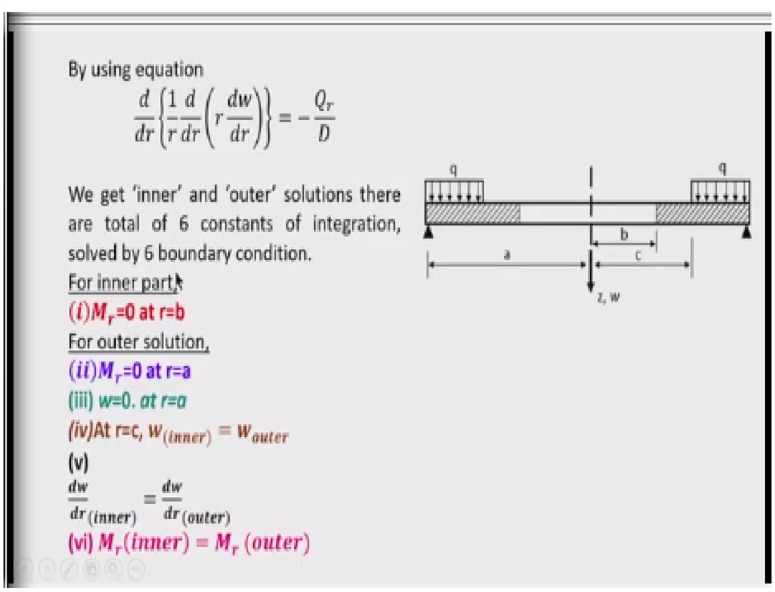

Then we should find the deflected surface by integration procedure and we adopt the third order equation where shear force is related to the third derivative of the deflection. Now here if you solve this by substituting Q_r that we have obtained here, we can get a solution *w*. Now this solution contains 3 constants for the outer region. Then the inner region also will get a solution, inner region actually no force is acting, so naturally this *q* will be 0, but we get 3 constants of integration.

So, 3 + 3 = 6 constants of integration have to be known by applying the boundary condition. So, boundary condition we require 6 in numbers to be applied to know all the constants of integration and then we can finally know the deflected surface. For the inner part if you see that *r* = *b* no moment is acting because it is a free end and no externally applied edge moment is there. So, therefore we take M_r=0 at *r* = *b*, so this is one condition.

Then for outer region or outer solution, if you call that we have obtained, where the load is considered *q*. Second condition is at simply supported end that is the radial moment is 0, M_r is 0, that is second condition. Then deflection is 0 at this simply supported end that is another condition. So, we get 3 conditions to be imposed on the 6 in the 2 differential equations involving 6 constants of integration.

Then another 3 constants are found at *r* = *c*, that is this point, deflection found from the inner part should match with the deflection found from the outer part for satisfying the compatibility of deflection. So, compatibility of deflection has to be satisfied at the common point. So, at *r* = *c*, $w_{(inner)} = w_{outer}$, then 5th condition is the slope compatibility has to be satisfied at the common point. So, at *r* = *c* again $\frac{dw}{dr}_{(inner)} = \frac{dw}{dr}_{(outer)}$.

That means, outer and inner solution differentiation has to be taken and has to be equated. Last boundary condition is the bending moment M_r, that is the radial bending moment at this common point on the inner part should be equal to the radial bending moment at this point from the outer solution. So, in this way we get the 6 solution, 6 boundary conditions and applying 6 boundary condition, 6 constants of integration can be evaluated.

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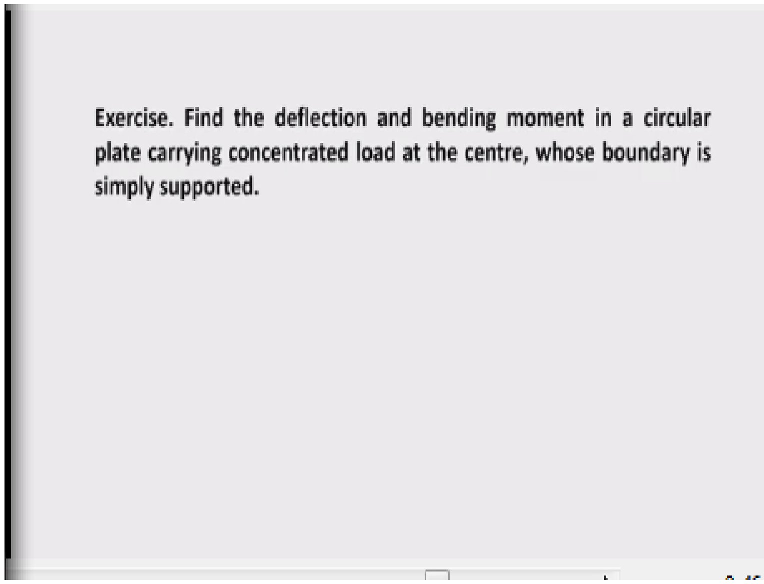

Now lastly I want to discuss a problem of finding the deflection and bending moment in a circular plate carrying concentrated load at the center, whose boundary condition is simply supported. So, that type of problem is also occurring in practice, that is the simply supported edges and let us see how to solve such problem.

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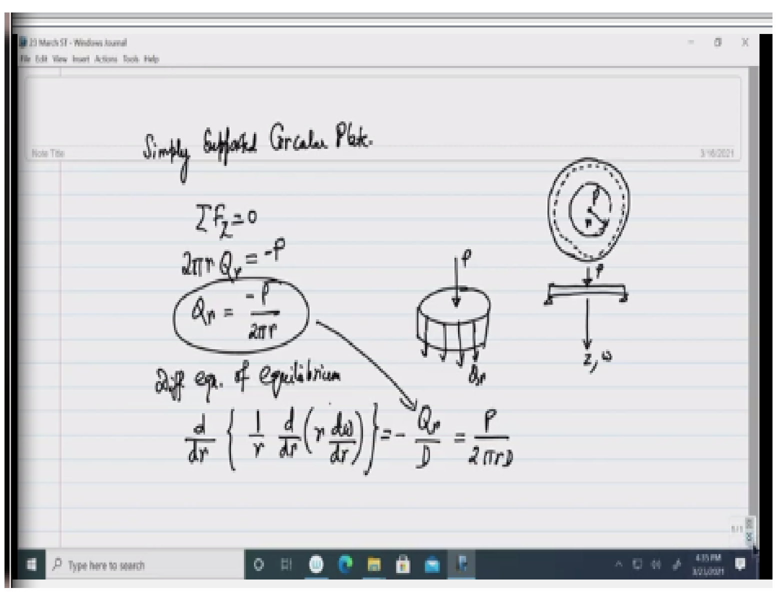

So, we have here this problem of simply supported plate, circular plate. That means we have a plate which is simply supported at the edges and carrying a concentrated load at the center, that means if you see a section in the center there is a load. So, let us solve the problem of finding the deflection. The equation of the plate deflection is known, we can take the third order equation, we can take the fourth order equation.

Now here we shall take a slice around the load and consider the equilibrium of this slice. So, load is acting here P and we have the shearing force that is acting around the slice of magnitude Q_r per unit length. Now from vertical force equilibrium that is summation of forces in z direction is 0, z direction is the vertical, this is the z direction and direction of w is also same. So, taking the equilibrium of forces, summation of forces in the z direction to be 0, we now get $2\pi r Q_r$ because we take this slice at a distance of r equal to this P. Or we can say because both are taken this downward, so it will be minus. So, $Q_r = -\frac{P}{2\pi r}$ with the minus sign, so this is the Q_r. Now let us write the equations of equilibrium, differential equations. So, differential equation, let us write like this $\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] = -\frac{Q_r}{D}$. Instead of Q_r now I put this term. So, this term is brought here and now differential equation can be written is $\frac{P}{2\pi r D}$. So, instead of Q_r, we have now written $-\frac{P}{2\pi r}$, so this equation now becomes $\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] = \frac{P}{2\pi r D}$.

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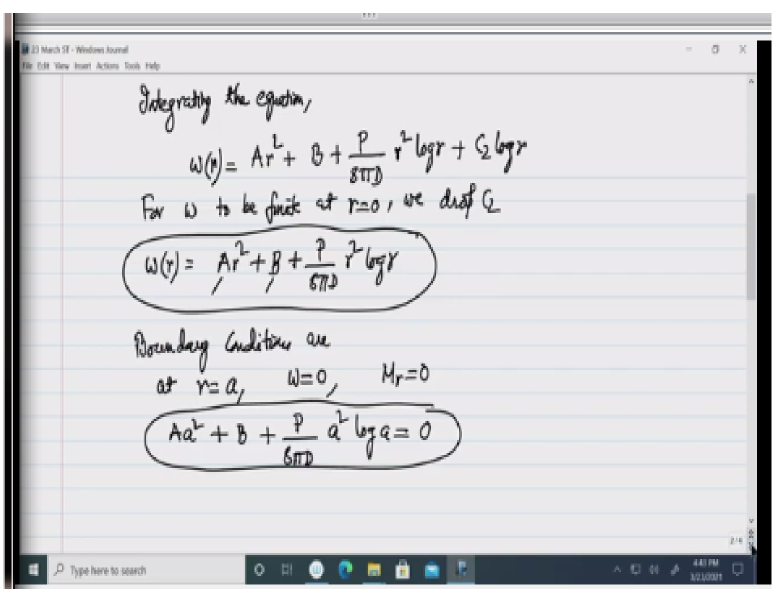

Now integrating the equation, what do we get actually? We will get this $w(r)$ because integration have to be carried out it is result is known to us $Ar^2 + B + \frac{P}{8\pi D}r^2 logr + C_2 logr$. Now for deflection to remain finite for $w(r)$ to be finite at r = 0, that is at the center we drop C_2. So, therefore final expression for deflection becomes $Ar^2 + B + \frac{P}{8\pi D}r^2 logr$. So, this is the final expression for deflection. Now, if you see this deflected surface, then you can find the 2 constants of integration A and B by applying the boundary condition.

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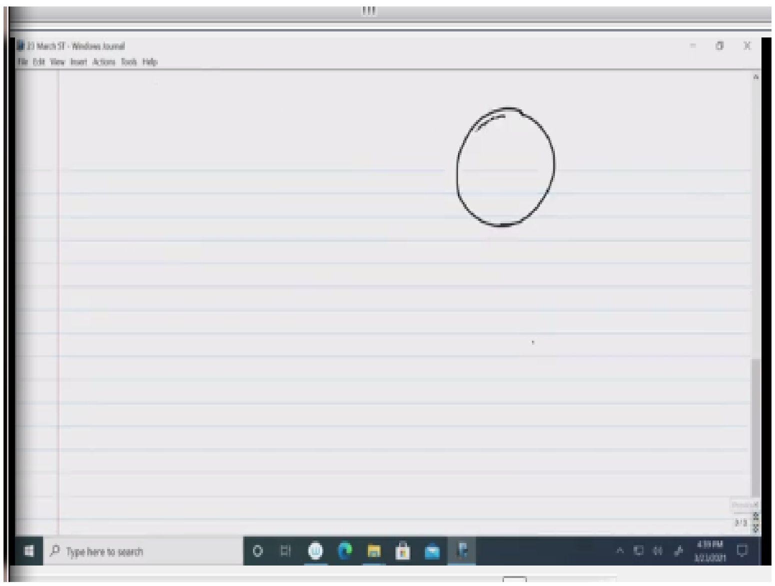

So, if I look towards the boundary condition in the previous slide. Boundary conditions are that is a circular plate was there and these edges are simply supported.

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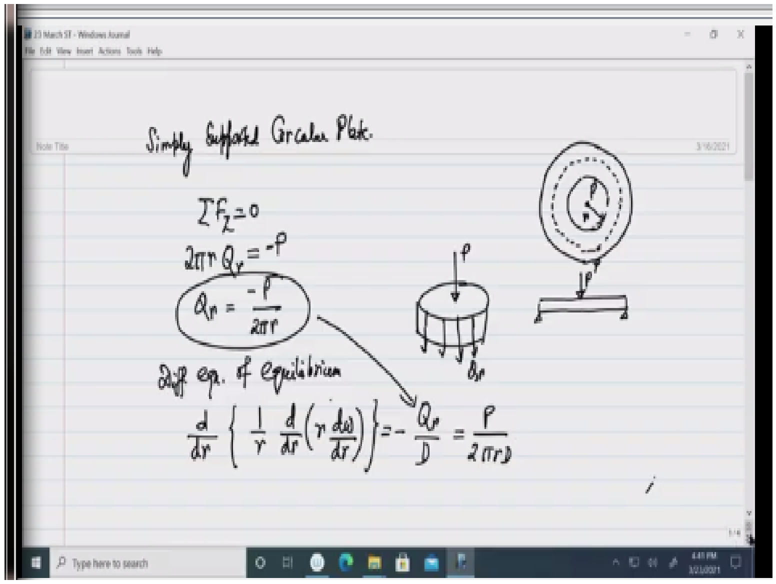

So, the equilibrium of vertical forces is considered for this slice. And we have seen that the Q_r can be related with the externally applied load as $-\frac{P}{2\pi r}$. So, differential equation of equilibrium is now $\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] = \frac{P}{2\pi r D}$. So, obtaining this we now go to finding the constants of integration. That means, the equation can be rearranged in this final form and then we apply the boundary condition.

So, boundary conditions are at say radius say $r = a$, at the boundary the deflection is 0 because it is simply supported as well as $M_r = 0$, that is the radial moment equal to 0. So, first condition gives this $Aa^2 + B + \frac{P}{8\pi D}a^2 log a = 0$, so this is one equation after applying the boundary condition. Second equation is obtained applying the radial moment condition to be 0 at the edges.

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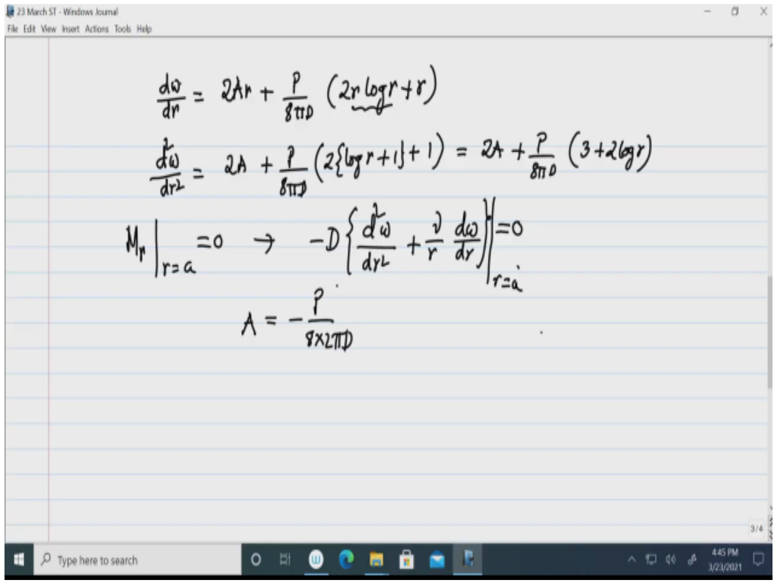

Now if I go finding the radial moment, then we should know the first derivative and second derivative because the radial moment contains the curvature as well as the slope also in axi-symmetrical equation. So, $\frac{dw}{dr} = 2Ar + \frac{P}{8\pi D}(2rlogr + r)$. And second derivative of this $\frac{d^2w}{dr^2} = 2A + \frac{P}{8\pi D}(2 + 2logr + 1)$. You see, this is the product of 2 functions, so we are differentiating. And after differentiating, we are getting say here 2 into say we are getting first say *logr*, then we are getting 1, +1 another term is there.

So, final expression of second derivative is $2A + \frac{P}{8\pi D}(3 + 2logr)$. Now apply the condition of bending moments, so M_r at $r = a$ to be 0. What is M_r? $M_r = -D\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right)$ should be equal to 0 at $r = a$. So, after substituting these values, these $\frac{d^2w}{dr^2}$ and $\frac{dw}{dr}$ with r substituted as a, we now finally get an expression of A, so from that condition, we get directly the expression for A as $-\frac{P}{8\times2\pi D(1+\nu)}\left\{(3 + \nu) + loga^2(1 + \nu)\right\}$, because 2 is there in the left hand side of the equation.