Plates and Shells
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Module-04
Lecture-11
Transformation of Plate Equation from Rectangular Coordinates to Polar Coordinates

Hello everybody, today I will start the module 4. So, first lecture I am starting with a topic that covers the plate equation in polar coordinate system. So far we have introduced and solved some problems of rectangular plates and other plates also some special cases like circular plate or elliptical plate and triangular plate also. Now, it will be advantageous when we use the plate equation for some specific problem where the axial symmetry is there.

For example, circular plate loaded symmetrically and boundary condition is also symmetric with respect to an axis of rotation passing through the centre of the plate. Then it is possible to find the solution of the system equation in the polar coordinate system.

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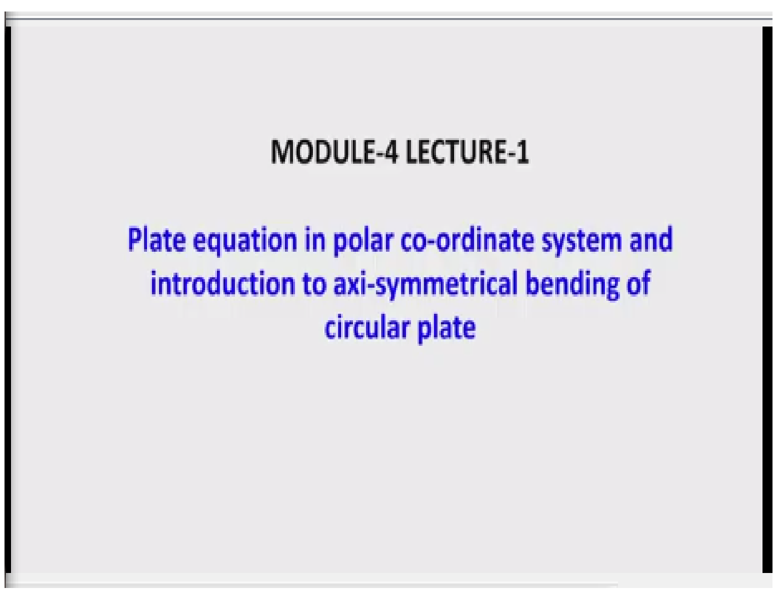

So, today our discussion will be governing differential equation of the plate in polar coordinate system and introduction to axi-symmetrical bending of the circular plate. So, specifically we will use the polar coordinate system to circular plate problem.

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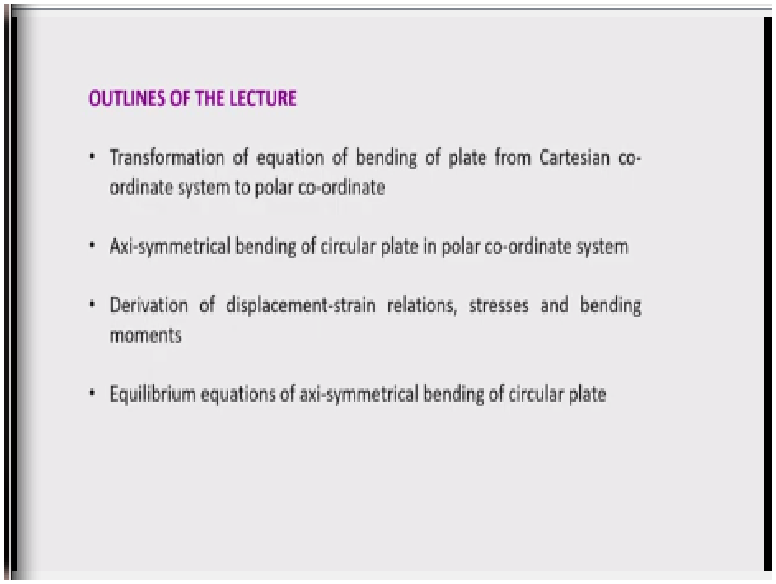

So, today I want to cover the following topics, transformation of equation of bending of plate from Cartesian coordinate system to polar coordinate system. Then axi-symmetrical bending of circular plate in polar coordinate system, then derivation of displacement-strain relations, stresses and bending moments, so this expression how it changes? Because we are familiar with the rectangular coordinate system, the bending moment expression for plate or shear force expression for plate and stresses also we could find.

But now let us see how this relationship will change when we carry out the transformation of the coordinate system. Then we will derive, specifically for axi-symmetrical bending of circular plate, the equations of equilibrium for an infinitesimal element and from that we get the differential equation for the bending of circular plate with axi-symmetrical condition. Now, after getting this equilibrium equation, we can proceed to solve several problems encountered in practice because most of the problem in circular plate comes with the axi-symmetrical loading or axi-symmetrical conditions of boundary and loading. Then we can find the deflection equation and bending moment, shear force, for that type of boundary condition and the loading condition for a circular plate. And make use of these results for designing of the plate element or slab which is modeled as a plate. So, these are the topics that I want to cover in today's lecture.

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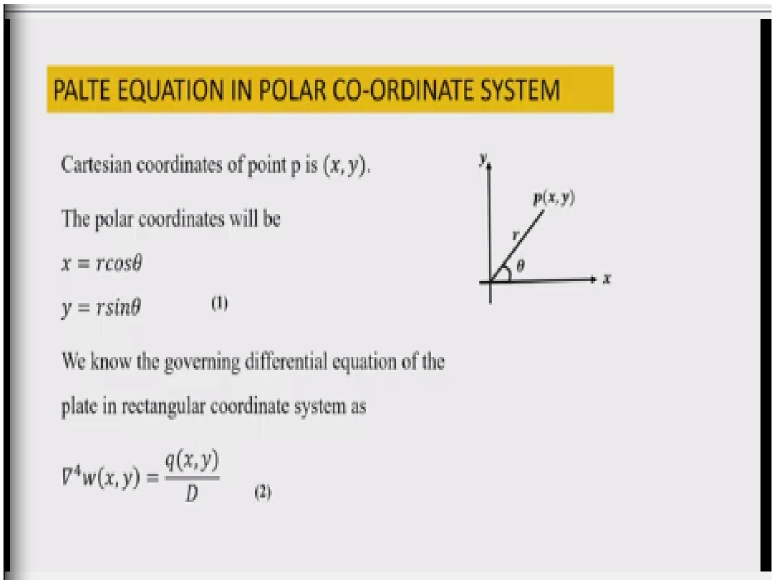

So, let us see first how this system equation or the plate equation can be transformed from rectangular coordinate system to polar coordinate system. So, you know very well that polar coordinate system is expressed in terms of 2 variables. The radial distance of the point from the origin r and the angle that the radial line makes with the reference axis, here we take the x axis. So, the point $p(x,y)$ in Cartesian coordinate system will be transform to a point p with coordinates r and θ, where r easily you can see that x coordinate equal to $rcos\theta$ and y coordinate is equal to $rsin\theta$. And you can also see that $tan\theta$ will be y/x. So, with that relationship $x = rcos\theta$, $y = rsin\theta$, will be able to transform the equations of bending of plate derived in the rectangular coordinate system like $\nabla^4 w(x,y) = q(x,y)/D$, where D is the flexural rigidity of the plate it is nothing but equal to $Eh^3/12(1-v^2)$, where v is the Poisson's ratio of the plate.

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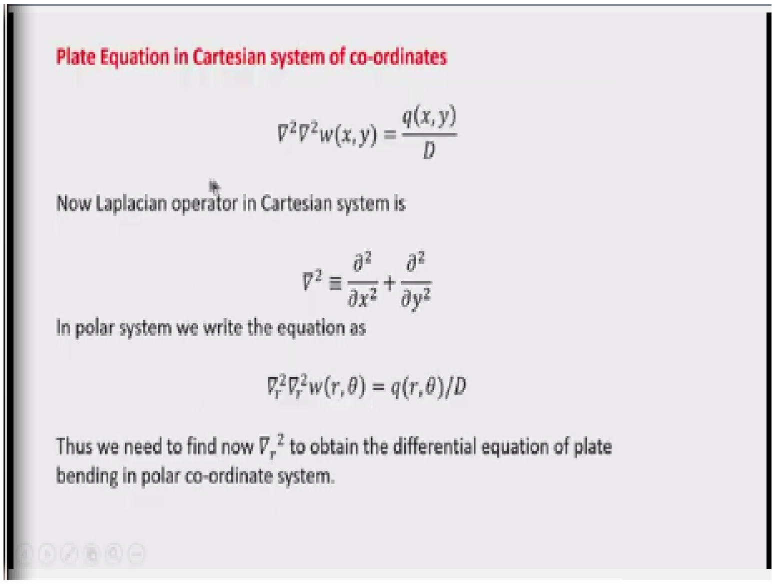

So, plate equation now can be written like this also, say ∇^2 is the Laplacian operator is very well known to the students of mechanics or physics. So, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, this is the Laplacian operator in differential calculus. And $\nabla^2 w(x,y)$, again we operate on *w*, so plate equation can be written in this form $\nabla^2\nabla^2 w(x,y) = q(x,y)/D$. Now, our intention is to change the Laplacian operator in the polar coordinate system.

So, the equation of the plate in transform form can be written as ∇_r^2 operated on $\nabla_r^2 w$, where *w* is a function of *r*, *θ* and *q* is also a function of *r*, *θ*, D remains same whether you change the system of coordinate it will not differ. Now here you can see that *w* and *q* which was previously the functions of *x* and *y*, now it becomes functions of *r* and *θ*. So, you can see that, now our intention is to or aim will be to transform the Laplacian operator which is ∇^2 into the polar form ∇_r^2.

So, you can see that ∇^2 contains the operator $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$, that means second derivative of this quantity. So, first let us transfer this or transform this ∇^2 Laplacian operator in rectangular system to polar coordinate system, then we will be able to write the equation in polar coordinate system.

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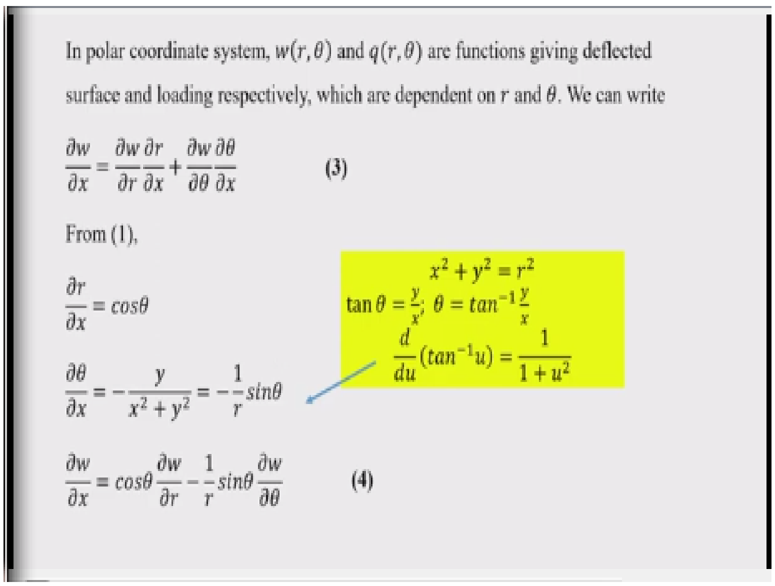

In polar coordinate system $w(r,\theta)$ and $q(r,\theta)$ are functions of r and θ, and giving the deflected surface and loading respectively. So, it depends on both quantity r and θ, but for axi-symmetrical cases where it has the plate posses rotational symmetry then w and q will be only functions of r, so they will not depend on the θ. So, to transform the Laplacian operator in rectangular coordinate to polar coordinate system, let us write the derivative $\partial w/\partial x$ using the chain rule introducing the variable r.

So, $\partial w/\partial x$ can be written as $\partial w/\partial r \times \partial r/\partial x + \partial w/\partial \theta \times \partial \theta/\partial x$. Now, if we look back to equation 1, equation 1 is this $x = rcos\theta$ and $y = rsin\theta$. So, you can find relation between r with x and y, so $r^2 = x^2 + y^2$. So, that relation will be using here, so $x^2 + y^2 = r^2$, that is found because x is expressed as $rcos\theta$ and y is expressed as $rsin\theta$. So, when x is squared and y is squared and both are added, you will get r^2. Now, in order to find out this $\partial r/\partial x$, let us differentiate this quantity with respect to x. So, if you differentiate this with respect to x, for this you will get $2x$ and here because this is not containing any x, so differentiation of y^2 with respect to x will be 0. So, $2x = 2r$ into this dr/dx.

Because first we are differentiating with respect to r, then we are introducing this variable dr/dx. Now dr/dx is now x/r, and x/r is you know that it is $cos\theta$, so you got this quantity**.** Similarly, when we want to find these $d\theta/dx$, now let us see what is θ? $\tan\theta = y/x$, so θ will be $\tan^{-1}(y/x)$. So, differentiation of an inverse function is given by say u, $\tan^{-1}(u)$ is an inverse trigonometrical function. So differentiation with respect to variable u will be $1/(1 + u^2)$. Now, here u is y/x, so substituting this you will get that $\partial\theta/\partial x = -\,y/\left(x^2 + y^2\right)$.

And you know that $x^2 + y^2 = r^2$ and y you know that $y = rsin\theta$. So, naturally it will be $\partial\theta/\partial x = (-1/r)\sin\theta$. So, these 2 quantities we got in this equation, now we substitute this $\partial w/\partial x = \cos\theta(\partial w/\partial r) - (1/r)\sin\theta(\partial w/\partial\theta)$. So, first derivative is obtained, so this task is completed, now let us go to the second derivative. Because to find out the Laplacian operator must get the second derivative, then we can isolate the operator.

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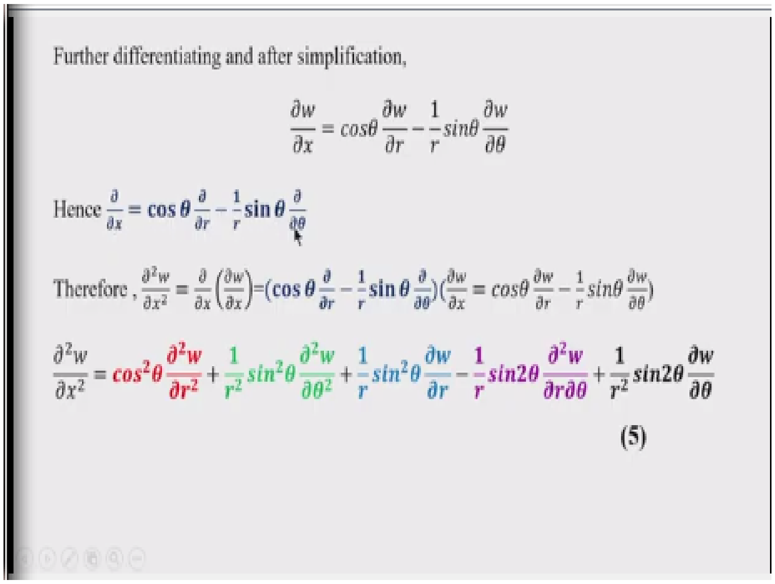

In the second derivative of this w, we got this already, so this operator $\partial/\partial x = \cos\theta(\partial/\partial r) - (1/r)\sin\theta(\partial/\partial\theta)$. So, you can see this, this operator I have written, I have isolated this operator, differential operator and other variables, so w is isolated, separated. Now to take the second derivative of that $\partial w/\partial x$, we write $\partial^2 w/\partial x^2$ = $\partial/\partial x(\partial w/\partial x)$, that is differentiation of $\partial w/\partial x$ again with respect to x to find out the second derivative.

Now here you can see that $\partial/\partial x$ that is the operator we have already found out, so we write here this operator $\cos\theta(\partial/\partial r) - (1/r)\sin\theta(\partial/\partial\theta)$. Then $\partial w/\partial x$, we already found it here, so we write here again $\partial w/\partial x$ that quantity will be $\cos\theta(\partial w/\partial r) - (1/r)\sin\theta(\partial w/\partial\theta)$. So, this quantity is $\partial w/\partial x$, so this quantity is this.

So, after operating with this operator this function, then you will get $\partial^2 w/\partial x^2 = \cos^2\theta(\partial^2 w/\partial r^2) + (1/r^2)\sin^2\theta(\partial^2 w/\partial\theta^2) + (1/r)\sin^2\theta(\partial w/\partial r) - (1/r)\sin 2\theta(\partial^2 w/\partial r\partial\theta) + (1/r^2)\sin 2\theta(\partial w/\partial\theta)$

.Now you can note here I have used different colours for different terms, so I will explain what is the significance of that. So, you can see the second derivative of w is now expressed in terms of variable r and θ**.**

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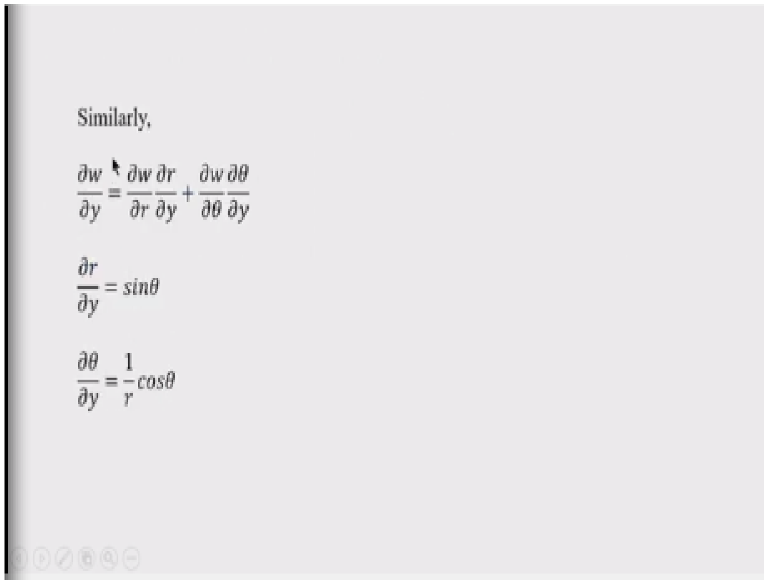

So, similarly one can obtain these $\partial w/\partial y$ that using the chain rule of differentiation $\partial w/\partial y$ can be written $(\partial w/\partial r)(\partial r/\partial y)+(\partial w/\partial \theta)(\partial \theta/\partial y)$, so this is the expression of $\partial w/\partial y$ using the chain rule. Now, already we know that $\partial r/\partial y = \sin\theta$, that we can prove it. Because if we go to the first expression, this expression then differentiating again with respect to y you will get this $2y$ here and here you will get $2r$ $\partial r/\partial y$.

So, you can find $\partial r/\partial y = y/r$, and y/r we know that in polar coordinate system y is r sinθ, r will get cancel**.** So, then $\partial \theta/\partial y$ you will get again in this similar fashion $(1/r)$ cosθ**.** So, now this expression can be written as $\partial w/\partial y = (\partial w/\partial r)\sin\theta + (\partial w/\partial \theta)(1/r)\cos\theta$ **.**

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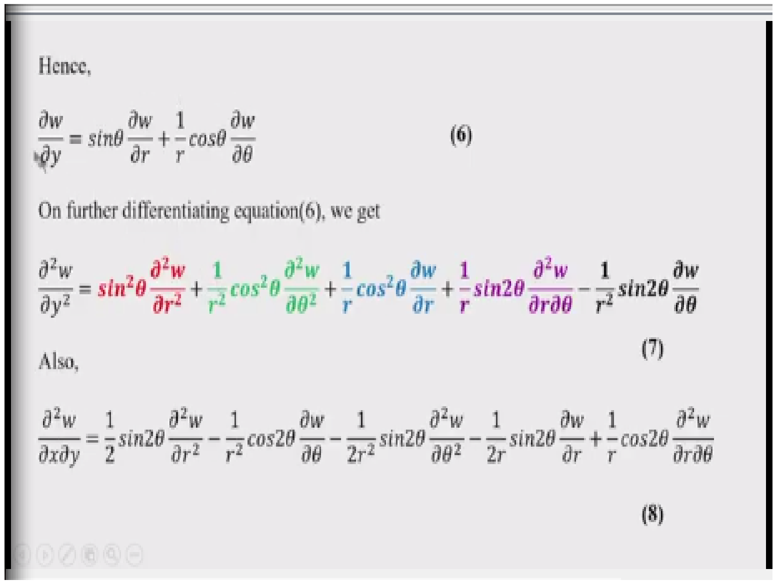

So, this expression is written like that, now on further differentiation as we have done earlier, that means we can now separate the operator $\partial w/\partial y$. And we can write this operator $\partial/\partial y$ this is the differential operator = $(\partial/\partial r)\sin\theta + (\partial/\partial\theta)(1/r)\cos\theta$. So, then differentiation of $\partial w/\partial y$ again with respect to y will yield the equation like that. So, here you can see I have also used the different colours for different terms.

Previously also in the equation 5, I have used different colours for different terms. Now, here you can see if I see the red colour term, $\cos^2\theta\left(\partial^2 w/\partial r^2\right)$. Say let us take the first term of this Laplacian equation that is $\partial^2 w/\partial x^2$, very popular equation. So, $\partial^2 w/\partial x^2$ square in the polar form there are say 1, 2, 3, 4, 5 terms, so let us take the first term which is marked with the red colour. So, $\cos^2\theta\left(\partial^2 w/\partial r^2\right)$ in the equation 5, now we go to the equation 7, equation 7 we have got $\sin^2\theta\left(\partial^2 w/\partial r^2\right)$, so this is also red colour term. So, if I add this because our intention is to find the Laplacian operator in polar coordinate that is transforming the Laplacian operator from rectangular coordinate to polar coordinate, to find out the equation of the plate. So, if I add equation 5 and 7, you see the addition of the red term here and red term here will yield you this $\partial^2 w/\partial r^2\left(\cos^2\theta+\sin^2\theta\right)$, so, this is one.

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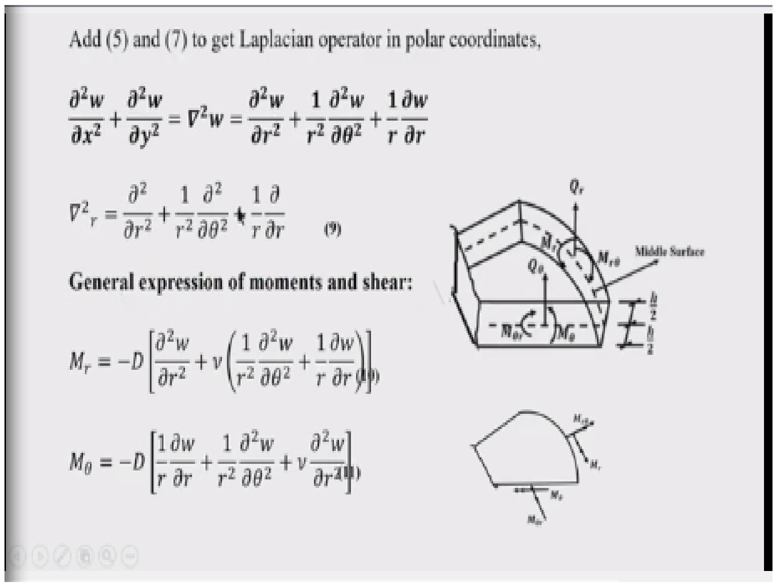

So, after adding you will get first term as $\partial^2 w/\partial r^2$, that is sure. Now, let us come to the second term, this second term of the equation 5 which is $\left(1/r^2\right)\sin^2\theta\left(\partial^2 w/\partial\theta^2\right)$, you can see $\partial^2 w/\partial\theta^2$ and with that coefficient is $\left(1/r^2\right)\sin^2\theta$. So, this is written with the green colour in this equation 5, similar term we will find in the equation number 7 with the green colour. Now, if I add again you are finding that common term is there in both the equation $\left(\partial^2 w/\partial\theta^2\right)\left(1/r^2\right)$. So, naturally if I take common $\left(1/r^2\right)\left(\partial^2 w/\partial\theta^2\right)$. And then these other terms in the parentheses will be $\cos^2\theta+\sin^2\theta$. So, naturally the another term in the polar system will be $\left(1/r^2\right)\left(\partial^2 w/\partial\theta^2\right)$.

So, originally you have seen there are 5 terms, but here you can see that only 3 terms are there, that means other terms will get cancelled**.** So, if I now see this term $\left(1/r\right)\sin^2\theta\left(\partial w/\partial r\right)$ here it is blue coloured, equation number 5, equation number 7 let us see, equation number 7 also it is blue colour and the coefficients is with $\cos^2\theta$. So, naturally when it is added then it will be $(1/r)(\partial w/\partial r)$ r. So, the other terms the $(1/r)\sin 2\theta\left(\partial^2 w/\partial r\partial\theta\right)$, - $(1/r^2)\sin 2\theta(\partial w/\partial\theta)$ have alternate sign, equal but opposite sign, so they will get cancelled. So, after transformation, we will get the Laplacian operator in polar coordinate system to be consisting of 3 terms $\partial^2 w/\partial r^2 + (1/r^2)(\partial^2 w/\partial\theta^2) + (1/r)(\partial w/\partial r)$.

So, our one of the important task is over now, because to find the plate equation 4th order plate equation we need the Laplacian operator, then we will operate again with the equation $\nabla^2 w$. So, this term, this operator or this differentiation or the addition of these 2 curvature have been already carried out and we have converted to the polar system. Now, we can conveniently write $\nabla^2 r = \partial^2/\partial r^2 + (1/r^2)(\partial^2/\partial\theta^2) + (1/r)(\partial/\partial r)$.

So, this is the Laplacian operator in the polar coordinate system. Now, we know the general expression of moments and shear, that can be written now in this form $M_r = -D\left[\partial^2 w/\partial r^2 + \nu\left(1/r^2\,\partial^2 w/\partial\theta^2 + 1/r\,\partial w/\partial r\right)\right]$, so this is equation 10. Then the M_θ the bending moment in the circumferential direction, M_r is the bending moment in the radial direction.

So, M_θ is the bending moment in the circumferential direction is $M_\theta = -D\left[1/r\,\partial w/\partial r + 1/r^2\,\partial^2 w/\partial\theta^2 + \nu\,\partial^2 w/\partial r^2\right]$. This can also be found because the bending moment expression contains the second derivative of the quantities. So, second derivative will convert now, we have converted into polar coordinate system and using this we can find this expression, that is equivalent expression in the polar coordinate system.

So, the stress resultant that is we consider in the circular plate or in the polar coordinate system or M_r the radial moment, M_θ is the circumferential moment and $M_{r\theta}$ is the twisting moment, Q_r is the shearing force and Q is in the direction along r. And Q_θ is the shearing force along the or in another perpendicular direction. So, that means Q_r and Q_θ these are the shear forces that is present in case of plates of circular shape, when the polar coordinate system is used. Interestingly you will find that for axi-symmetrical problem the quantity will be only dependent on r.

So, anything which is involving θ that is dependent on θ should be dropped, that is w in case of axi-symmetrical problem will be only a function of r. So, this term does not come into picture when we consider the axi-symmetrical condition, so that will come later. But now you have understood that transformation of Laplacian operator from rectangular coordinate system to the polar coordinate system**.**

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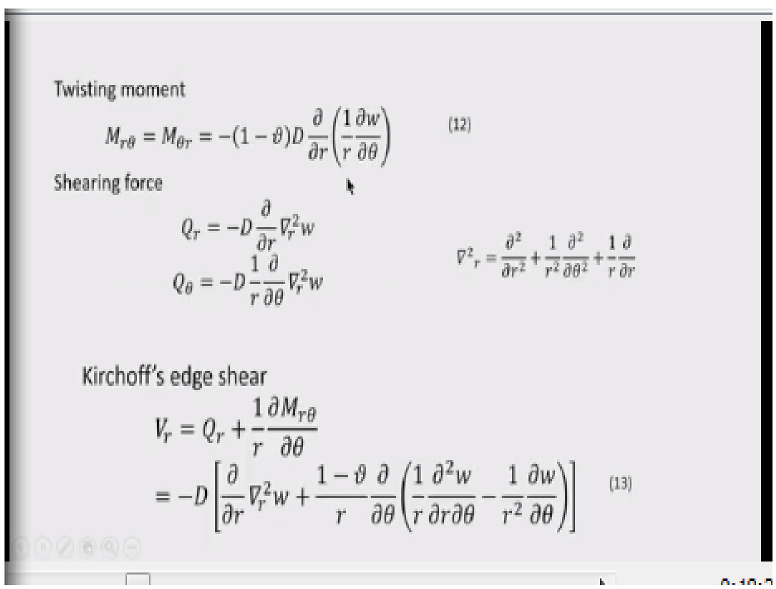

So, the twisting moment $M_{r\theta}$ or $M_{\theta r}$ is also written as $-(1-\nu)D$, D is the flexural rigidity of the plate which is the same quantity as we have found in the rectangular plate, (D has no relation with the change of coordinate system) into the partial derivative with respect to r of the function $(1/r)(\partial w/\partial\theta)$. Then sharing force $Q_r = -D\frac{\partial}{\partial r}\nabla_r^2 w$ and this is the Laplacian operator in polar coordinate system consisting of 3 terms, Laplacian operator in rectangular coordinate system consists of only 2 terms.

But Laplacian operator in polar coordinate system for general condition when these quantities are dependent on r and θ contains 3 terms. So, $Q_\theta = -D\frac{1}{r}\frac{\partial}{\partial\theta}\nabla_r^2 w$, ∇_r^2 is introduced to you, I have given the detailed derivation of ∇_r^2. Kirchoff's edge shear, that is at the free edge or at the other edge where the twisting moment, shear force as well as bending moment 3 quantities are there. But it is seen that the 3 forces are not necessary actually, 3 quantities are not necessary for expressing the boundary condition, because the twisting moment has contribution to the shearing force. So, that have been pointed out by these previous authors and Kirchoff modified the equation of edge shear, that is the shear force along the edge, he has expressed as $V_r = Q_r + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta}$. So, this is the quantity where the $M_{r\theta}$ or $M_{\theta r}$ has to be substituted from this quantity and Q_r has to be substituted from here.

So, ultimately you will get the edge shear force or radial edge shear force $= -D$ and this $(\partial/\partial r)\nabla_r^2 w$, ∇_r^2 is the Laplacian operator in polar system + $[(1-\nu)/r]\partial/\partial\theta\left(1/r\,\partial^2 w/\partial r\partial\theta - 1/r^2\,\partial w/\partial\theta\right)$. You can note here that this term $\partial r\partial\theta$ is due to contribution of $M_{r\theta}$ in the edge shear, so that you have noted and this V_r is written like that.

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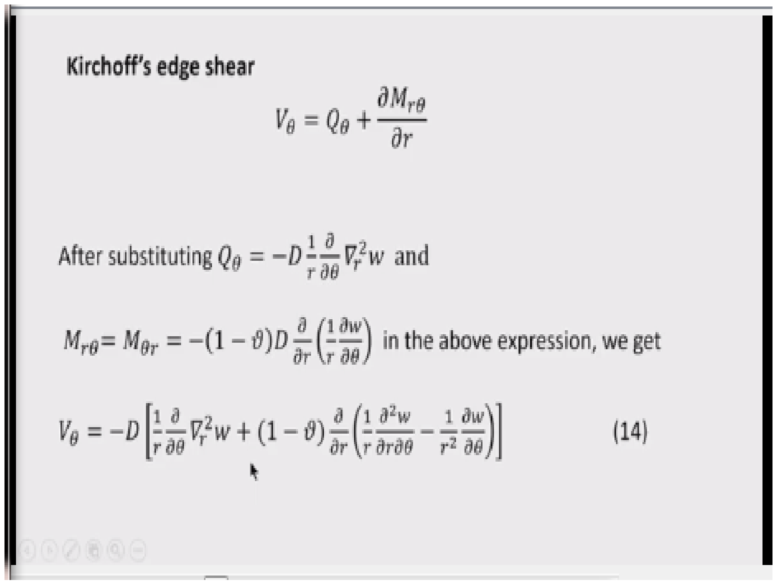

Similarly, the other edge shear that is V_θ can be written as $Q_\theta + \partial M_{r\theta}/\partial r$. After substituting Q_θ as $-D\frac{1}{r}\frac{\partial}{\partial\theta}\nabla_r^2 w$ and $M_{r\theta} = M_{\theta r} = -(1-\nu)D\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)$ in the above expression, we get this quantity after simplification. So, this is a very large equation and sometimes if the boundary condition is such that we have to impose V_θ. Then we have to use this equation

$$-D\left[\frac{1}{r}\frac{\partial}{\partial\theta}\nabla_r^2 w + (1-\nu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial w}{\partial\theta}\right)\right]$$

. So, the expression for the radial moment, circumferential moment, then shear forces Q_r, Q_θ and the Kirchoff's edge shear V_r, V_θ are given to you.

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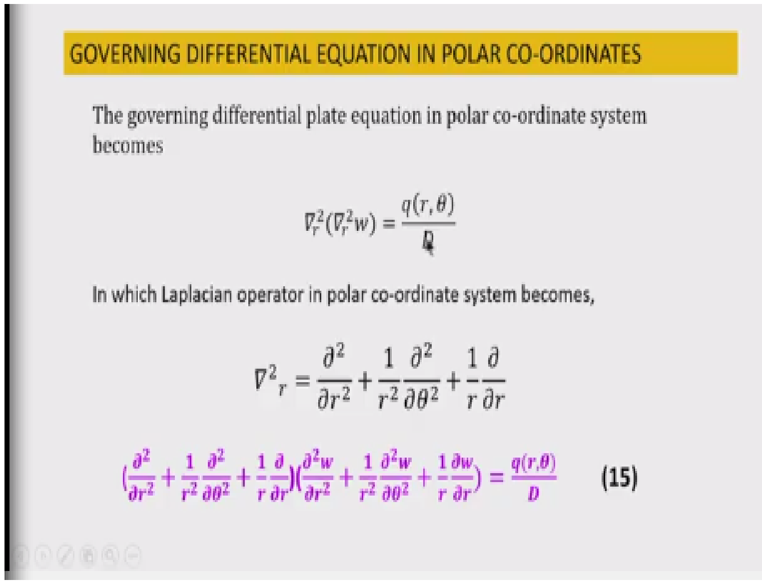

Now, let us see what will be the change in differential equation? So, differential equation originally was ∇^2 operated on $\nabla^2 w$ = q/D, where ∇^2 was the Laplacian operator in rectangular coordinate system. But we have now known this transformation of Laplacian operator in rectangular system to polar coordinate system. So, we can now write this plate equation in polar coordinate system as $\nabla_r^2\left(\nabla_r^2 w\right)$ that is ∇_r^2 is the operator which is consisting of these term $\partial^2/\partial r^2 + \left(1/r^2\right)\left(\partial^2/\partial\theta^2\right) + \left(1/r\right)\left(\partial/\partial r\right)$.

So, when it is operated with this function $\nabla^2 w$ then we get the full plate equation. So, you can write this thing in this form after getting, and then after operating you will get the 4th order differential equation of the plate.

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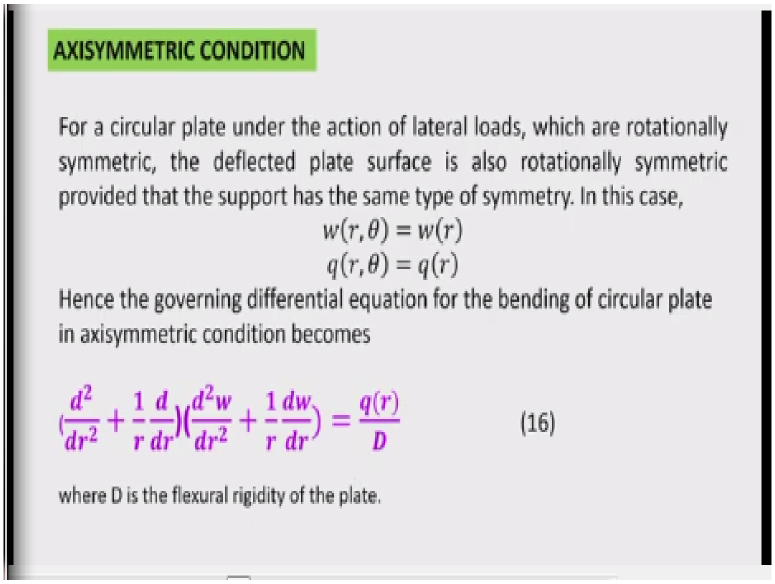

Now, let us derive the condition for axi-symmetrical loading and boundary conditions. Because the plate equation that originally contains w as a functions of r and θ and q is also a function of r and θ, involves the 4th order partial differential equation. Now, for axial symmetry for a circular plate under the action of lateral load, which are rotationally symmetric, rotationally symmetric means the stress, deflection or loading anything you take all will be dependent on r at any direction or at any orientation θ there will be no change. So that condition is known as axi-symmetrical condition. For example, a circular plate which may be simply supported at the edges or maybe clamped or maybe similar kind of in the same edge condition and it is continuous along the boundary.

So, in that case we can call that the plate is having the axisymmetric condition and then there is deflection and this q that is load that is applied on the plate will only be a function of r. That is for example, you take a circular plate and it is uniformly loaded. So, at any radial distance you will get the load is q that is the uniformly distributed load, intensity. Now, if you take say another condition say a circular plate and it is loaded with a linearly varying load, gradually increasing from 0 to maximum at the edges linearly. In that case also at any radial distance you will get this same load intensity. So, that means, it also falls under the category of axi-symmetrical condition. So, in the axi-symmetrical condition w will be $w(r,\theta)$ which is actually the function dependent variable of the functions, then it will be converted to only $w(r)$ and $q(r,\theta)$, r and θ are the dependent variables of the function, so it will be converted to $q(r)$.

So, with this simplification, now we can very easily see that this operator has no meaning, Because $\partial^2/\partial\theta^2$ will go to 0 because this derivative does not exist because it will be a function of r only. So, this will remain, and here this quantity will remain, this will have no significance because $\partial^2 w/\partial\theta^2$ again will be 0 because w will be only a function of r, so this quantity will remain, and q will be only a function of r.

So, instead of partial derivative, now we can write it as an ordinary derivative differential equation. So, partial differential equation is not necessary here, we will get the ordinary differential equation. So, therefore the Laplacian operator reduces to only 2 terms here $\partial^2/\partial r^2 + (1/r)(\partial/\partial r)$ and then this is the quantity that we get and = $q(r)/D$. So, this is very simplified situation and most of the cases this situation exists in practical field. So, we take this differential equation of the plate to find out the deflection and other stress resultant.

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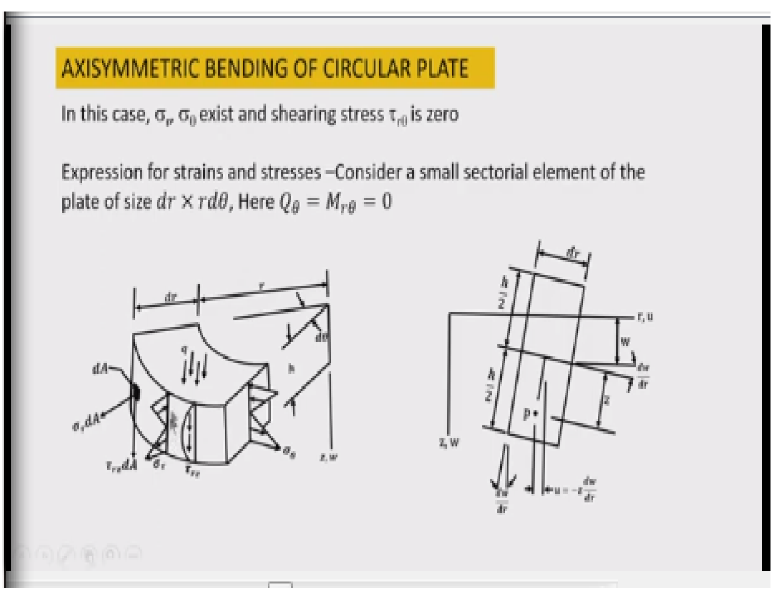

Now let us see, what are the quantity that we have to take into account? We have to take into account this bending moment in the radial direction and bending moment in the circumferential direction. And due to axi-symmetrical condition the Q_θ and $M_{r\theta}$ will be 0, only Q_r will be existing. Now say this is a portion of a plate, of very small length the dr at a radial distance r and which subtends an angle dθ and q is the transverse load acting on the plate.

So, you can see at any element of the plate the stresses in the radial direction is σ_r, so force on the element will be σ_r dA, where dA is area of the plate. And the shear stress that is acting along the edges will be 0, because in axi-symmetrical condition there will be no shear stress. But vertical shear stress will exist and vertical shear stress is on this τ_{rz}, and it will have a parabolic distribution and it is neglected in our thin plate theory.

So, in axi-symmetrical condition $\tau_{r\theta}$ is not to be considered only the stress that is σ_r and σ_θ are of importance, h is the thickness of the plate. So, this is demonstrated here, this stress distribution will be linear but we will prove it later on with the expression. Now, let us see an element of the plate of width dr that is shown in the bent configuration. So you can see that this slope of the plate is dw/dr.

So, due to deflection of the plate, it will undergo the displacement along the radial direction and that displacement is u, it is shown here. Now since it is the negative direction of the r because here the axis is the important direction is r, here in the radial direction and w is the vertical displacement, and u is the radial displacement**.**

So, you can see that u that is the this or radial displacement we can call, can be expressed in terms of if this point is located at a height z and the slope is same as this slope dw/dr, then it can be expressed as z dw/dr. So, this height into tanθ and because of small deflection, we take the tanθ, that is dw/dr = θ. The negative sign is taken because the u is measured in the negative direction of this axis r**.** So, the strain u is now - z dw/dr.

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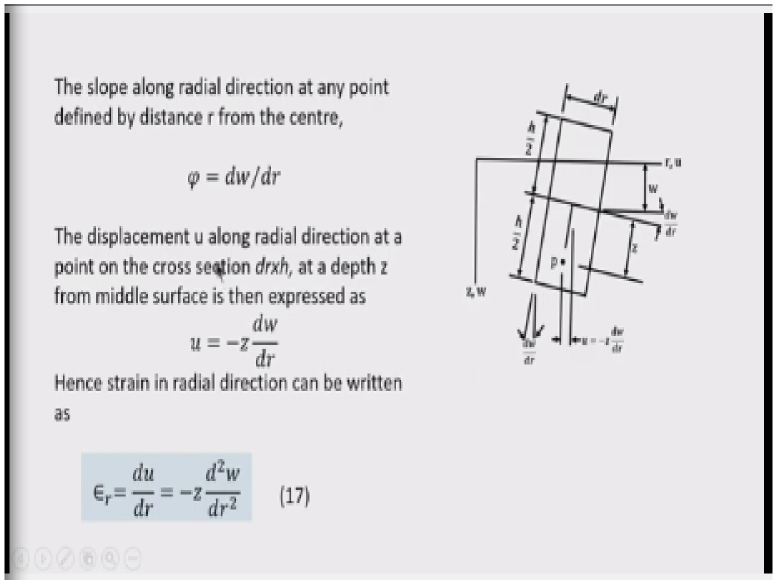

Now, let us take this slope $\varphi = dw/dr$ or you can call it θ also, any angular symbol you can give here. So, u is - z dw/dr that is the displacement in the radial direction. So, naturally the strain in the radial direction will be ϵ_r = the derivative of u with respect to r. So, if this is known, u is known then it will have this strain in the radial direction is - z d^2w/dr^2. So, staying in the radial direction is known.

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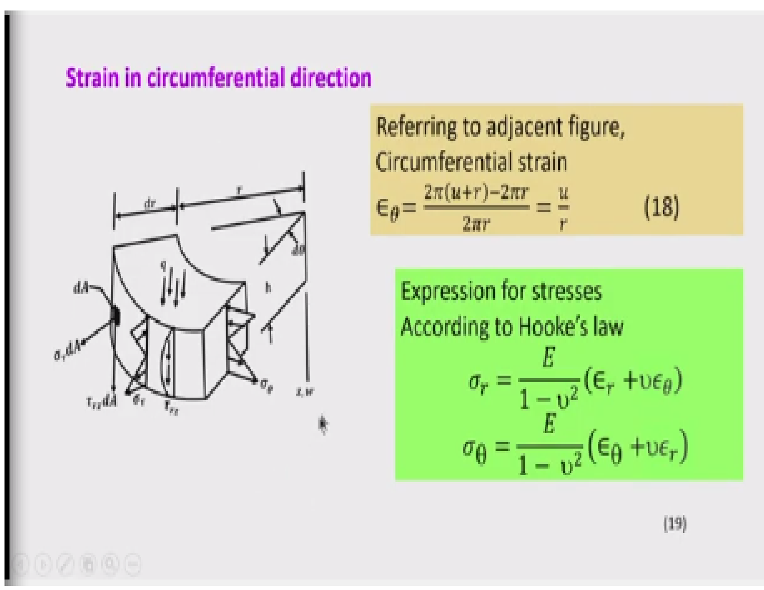

Staying in the circumferential direction, now can be obtained like that, because of displacement u in the radial direction the radius changes from originally it was r, now it will be $u + r$. So, the change of the circumferential length will be $\{2\pi(u + r) - 2\pi r\}/2\pi r$, change in length divided by the original length will give you the strain in the circumferential direction. So, you can see from this quantity, the strain in the circumferential direction is nothing but u/r.

So, after obtaining this strain in the radial and circumferential direction, we can now express the stresses using the Hooke's law. So, because we are dealing with the linear elasticity, so we will use the Hooke's law. So, using the Hooke's law this is $\sigma_r = \frac{E}{1-\nu^2}(\in_r + \nu \in_\theta)$, $\in_r$, $\in_\theta$ are the strains in the radial and circumferential direction.

Similarly, σ_θ that is the strain in the circumferential direction is written as $\sigma_\theta = \frac{E}{1-\nu^2}(\in_\theta + \nu \in_r)$, $\in$ is used to denote the strains, which are actually small quantity. So, knowing the strains and we already know the quantity $\in_r$, now we can express σ_r, σ_θ in terms of the distance from the middle surface. Because middle surface is our reference plate from the distance measured from the neutral surface, positive upward and negative downward.

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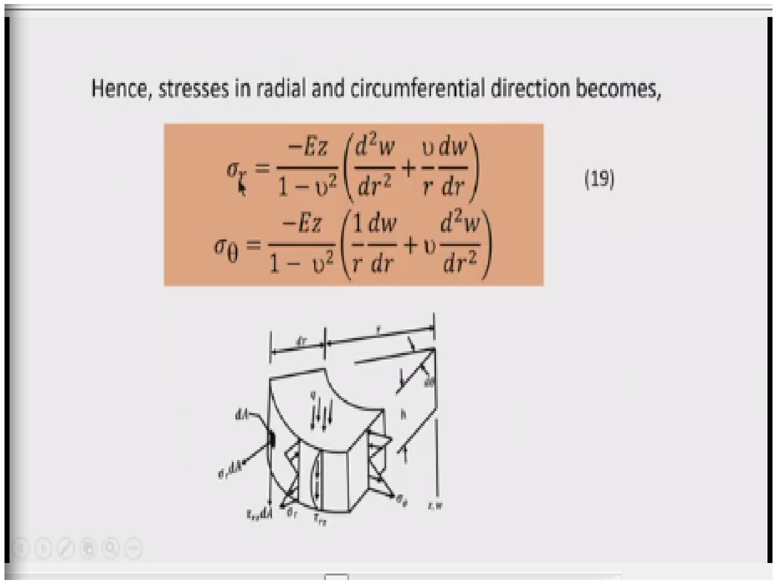

Then, we can express σ_r = -Ez, E is the modulus of elasticity of the material of the plate divided by $\left(1-\nu^2\right)$ × $\left(d^2w/dr^2+\nu/r\, dw/dr\right)$. Because here only the total derivative is of importance, because the partial derivative has no meaning here, because this w is a function of only one variable. So, similarly σ_θ, there is the stress in the circumferential direction can be written as $-Ez/\left(1-\nu^2\right)\left(1/r\, dw/dr+\nu\, d^2w/dr^2\right)$.

You can note here that in this stress expression of in the radial direction or circumferential direction whatever you call it both quantities say slope as well as curvature, both are involved. So, slope has contribution in the stress and also the curvature has also contribution towards the stress. So, these are the expression for the stresses in the radial and circumferential direction. From this expression 19, one can note that σ_r and σ_θ varies linearly with z.

So, the variation here is shown above the neutral axis, this is the middle surface it will be compressive and below the middle surface it will be tensile. Similarly, this σ_θ variation on the other edges it is shown like linear variation and $\tau_{r\theta}$ is 0 but τ_{rz} will exist, it will have a parabolic variation, but it is value is negligible and we will not consider in deriving the plate equation.

So, we have already obtained the differential equation of the bending of plates, circular plate under axi-symmetrical condition. So, this equation is valid, you can directly now use this equation to solve the axi-symmetrical problem of circular plate. But here I will demonstrate also the derivation of the differential equation by using the force balance that is using the concept of mechanics.

Previously, I just used the calculus to transform the differential equation in this rectangular system to polar coordinate system. Now, I intend to do with the help of the principle of mechanics.

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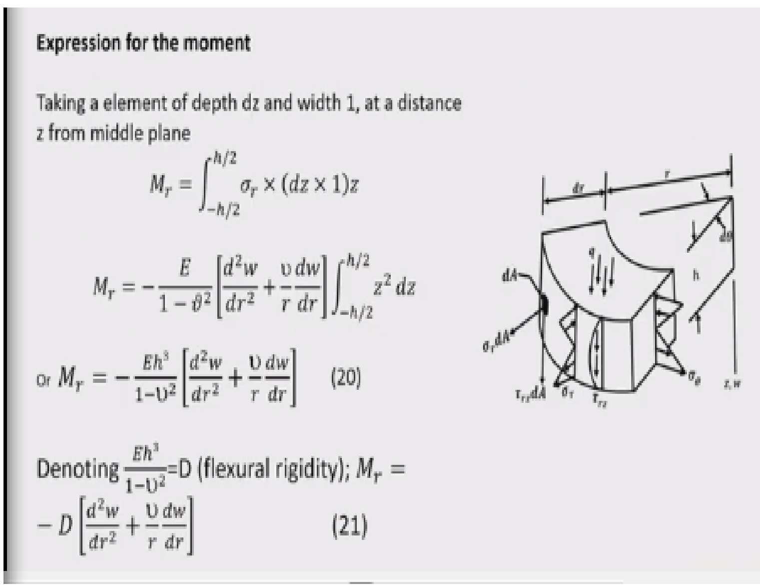

So, before that let us find the expression for radial moment and circumferential moment**.** So, radial stresses we have obtained σ_r and on the small element d*A* the force is σ_r d*A*. What is dA? The element is taken of width 1 and the depth is d*z*, so d*z* × 1 is the area of the element. So, force into distance above the neutral axis will give the moment, and it is moment on the element force acting on the element.

After integrating along the depth of the plate -*h*/2 to + *h*/2, we get the moment in radial direction which varies. This complete radial moment is obtained by taking the contribution of all the differential element. So, M_r after integration is converted to like that

$E/(1-\nu^2)(d^2w/dr^2+\nu/r\,dw/dr)$. These are only the ordinary differential coefficient is taken, so there is no necessity to use partial derivative sign.

Then z^2 dz it is integrated in the limit -h/2 to + h/2. After integration and after cancelling some term or arranging some term it will be converted in this form $-Eh^3/(1-\nu^2)$ and inside the bracket the curvature $d^2w/dr^2+\nu/r\,dw/dr$. So, this quantity you can see the constant $Eh^3/(1-\nu^2)$ is same as the parameter that we have found in case of rectangular plate, this is called the flexural rigidity of the plate.

So, there is no difference in the flexural rigidity of the plate because it depends on this thickness of the plate as well as the material properties. Similarly, that you can write it after denoting this $Eh^3/(1-\nu^2)$ with the symbol D.

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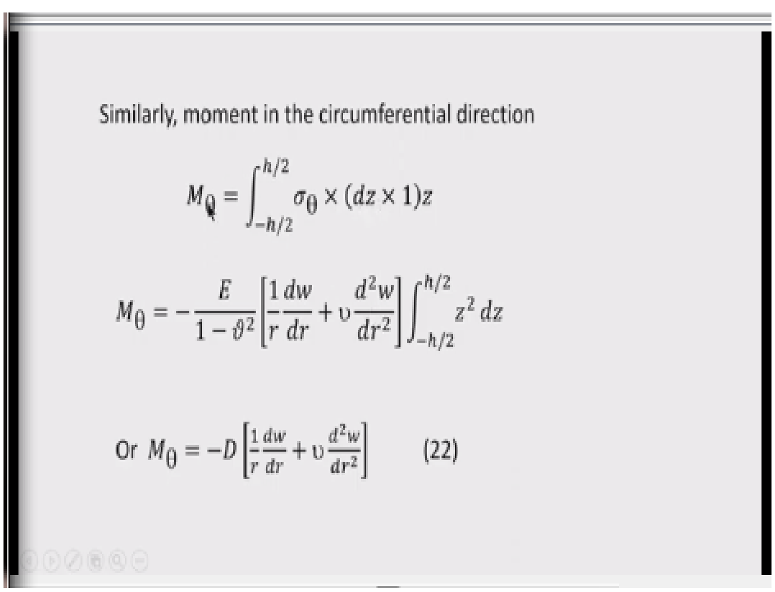

Similarly, the circumferential moment M_θ can be found, in the circumferential direction if I take an element then with the procedure I can calculate this or find an expression for M_θ, after substituting the value of the expression for σ_θ. So, then M_θ becomes $-D(1/r\, dw/dr + \nu\, d^2w/dr^2)$. So, these are the expressions for bending moment in the radial direction as well as circumferential direction.

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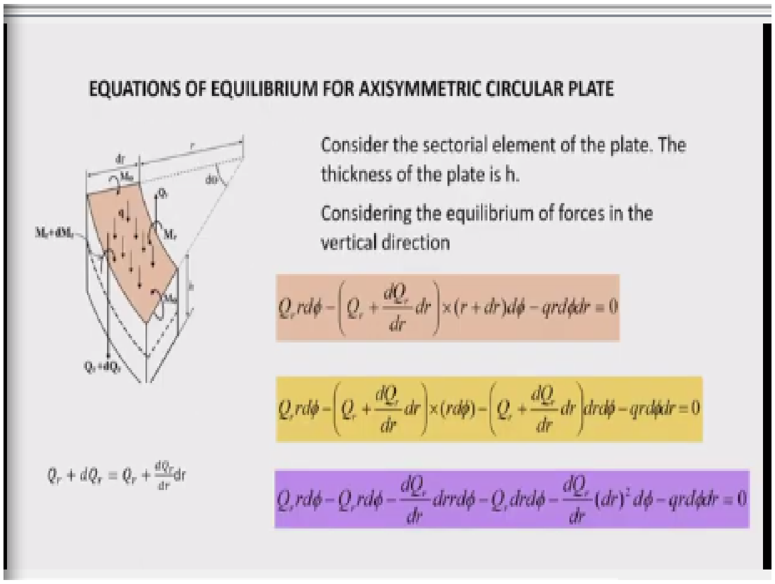


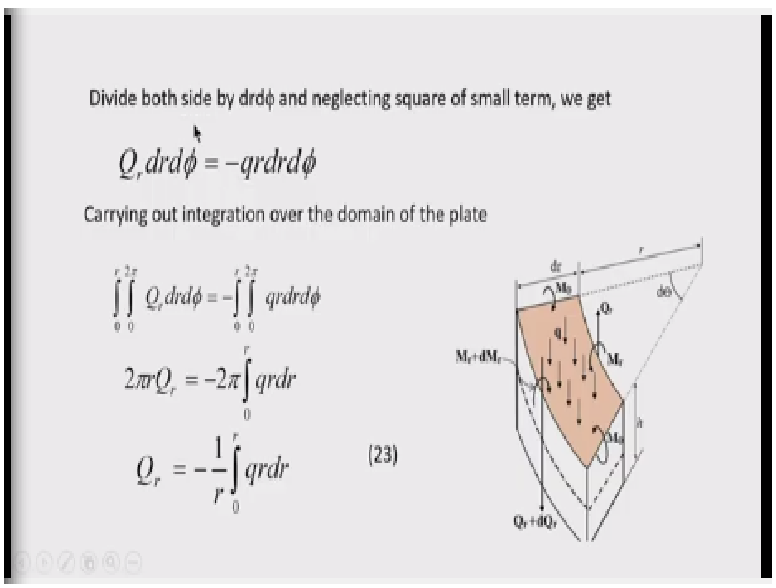
Now equations of equilibrium of the plate under axi-symmetrical condition, so let us consider a portion of the circular plate and the forces acting on the elements or the moments acting on the elements are shown. This is Q_r, on the opposite edges it is $Q_r + dQ_r$, dQ_r has to be understood as this differential incremental term $dQ_r/dr \times dr$. Then M_r, an incremental term is dM_r which has to be understood as the $dM_r/dr \times dr$. So, this I am using short form, I have written here.

So, in the circumferential direction M_θ, and along the other direction for keeping the equilibrium it is M_θ. So, M_θ will remain same, there will be no increment along the circumferential direction. So, consider this sectorial element of the plate, the thickness of the plate is h, the equilibrium condition is considered, first let us consider the force equilibrium. So, force equilibrium in this z direction that is the vertical direction if I call it.

Then the length of this element is, if this length is r then $rd\theta$ is the length of the element. So, $rd\theta$ or $d\phi$ whatever you call this is the differential angle. So $Q_r\ rd\theta$ is the force along this edge, vertical force. On the other edge, this is (Q_r + dQ_r/dr × dr) × (r + dr) dϕ. So, this as length is increased, because this radius, this radial distance r + dr, so the length is (r + dr) dϕ, so this you can note it.

So, these 2 forces we have written and then the transverse load, the vertical load that is acting on the surface of the plate is q per unit area. So, total area of this element that is coloured here is q × dr × rdϕ, so q × dr × rdϕ, so this is the total force on the element. So, equating to 0, and then getting simplifying this, that is first I carried out term by term multiplication.

And then you can get these many terms are common, that can be in many terms, some terms are having a product of square of small quantity dr. So, taking this advantage, that is canceling common terms and ignoring the term with the square of small quantity.



Then we can write the equation in this form. After dividing both sides by dr dϕ, so both sides are divided by dr dϕ and then we get this equation in this form. So, Q_r dr dϕ = - qr dr dϕ, neglecting the square of this small term. Now, carrying out integration over the domain of the plate, now let us integrate it. So, domain of the plate is for radius at any radial distance r, the radial distance varies from 0 to r, so 0 to r is the limit for radius. And angle after full rotation because it is axi-symmetrical condition, so full rotation has to be there, so 0 to 2π. So, we carry out integration of the both sides, we get here $2\pi r\ Q_r = -2\pi \int_0^r qrdr$. And then from that quantity we can write $Q_r = \frac{-1}{r}\int_0^r qrdr$, so this is one equation.

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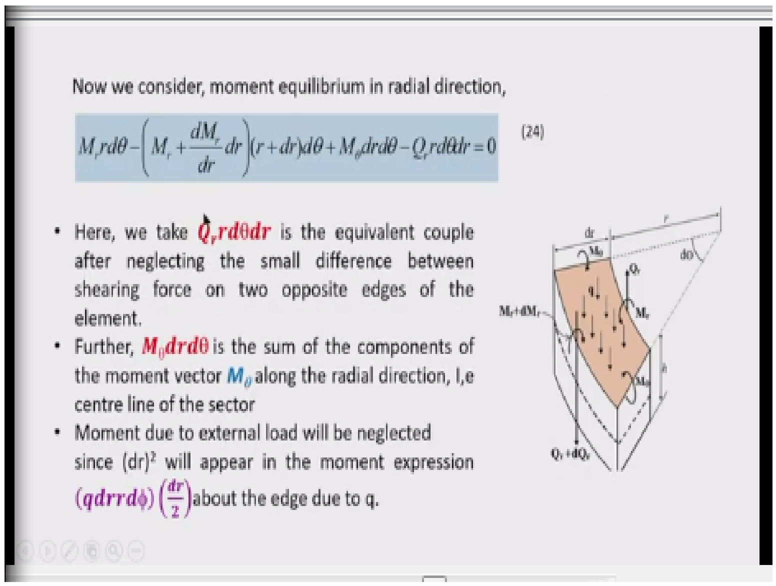

Then second equation is considered after taking the moment equilibrium, we consider moment equilibrium in radial direction. So, if I take the moment equilibrium in radial direction in this side this is M_r and total moment along the side is $M_r\ r d\phi$, because all the quantities are expressed in terms of quantity per unit length. So, M_r, M_θ etcetera Q_r or Q_θ whatever maybe these will be in terms of the parameter or quantity per unit length.

So, therefore to get the total force on moment, we have to multiply it by corresponding length. So, $M_r \times rd\theta$ is total moment along this edge, and along this edge, you will get $M_r + dM_r/dr \times dr$ and this length, the arc length $(r + dr)d\theta$. So, the moment of all the forces are taken, this is first we take the moment. And then you can note here in this side the force is upward Q_r on the other side $Q_r + dQ_r$.

So, we take $Q_r\ r\mathrm{d}\theta\mathrm{d}r$, because this term is small, so we can take these 2 forces, that Q_r and Q_r which are equal and opposite in nature, so it produces a couple. So, we take the advantage of this small distance or small value of $\mathrm{d}Q_r$. Therefore, we have stated that $Q_r\ r\mathrm{d}\theta\mathrm{d}r$, this quantity is the equivalent couple, after neglecting this small difference between the shearing force on 2 opposite edges of the element.

Further, you can see that this term is coming, how this term is coming? If you see M_θ and if you use the right hand rule, then it will be a vector, which is pointing towards the center. Then in this side if you consider M_θ and use the right hand rule, then you will see that is a vector which is away from the center. And if you resolve it in the radial direction at the center, then $\mathrm{d}\theta/2$, $\mathrm{d}\theta/2$ will be there and it will be this $M_\theta \cos \mathrm{d}\theta/2$ and here also $M_\theta \cos \mathrm{d}\theta/2$.

Since $\mathrm{d}\theta$ is small quantity, so you can see we can take it that $\cos \mathrm{d}\theta = 1$. Now, since both the vectors though it is pointing if you use the vector notation for the moment, and it is pointing towards the center, and another is pointing away from the center. But the effect is same, both the moments are symmetrical and it is causing a sag in the plate. So, therefore this is added, the components are added and we have written $M_\theta\ \mathrm{d}r\ \mathrm{d}\theta$, as the another term in the equilibrium equation.

Further $M_\theta\ \mathrm{d}r\ \mathrm{d}\theta$ is a sum of the component of the moment vector M_θ along the radial direction, so that quantity we have taken. And another thing is that when we take the moment of the external load about any edge, so this length is $\mathrm{d}r$. So, $\mathrm{d}r/2$, because the total load again $\mathrm{d}r$ is coming when you compute the area, and distance also $\mathrm{d}r/2$ will come. So, this $\mathrm{d}r \times \mathrm{d}r$, $\mathrm{d}r^2$ will appear in the moment expression, that we neglect it. Because it will be small quantity and it is not necessary to increase with the terms which has no significance.

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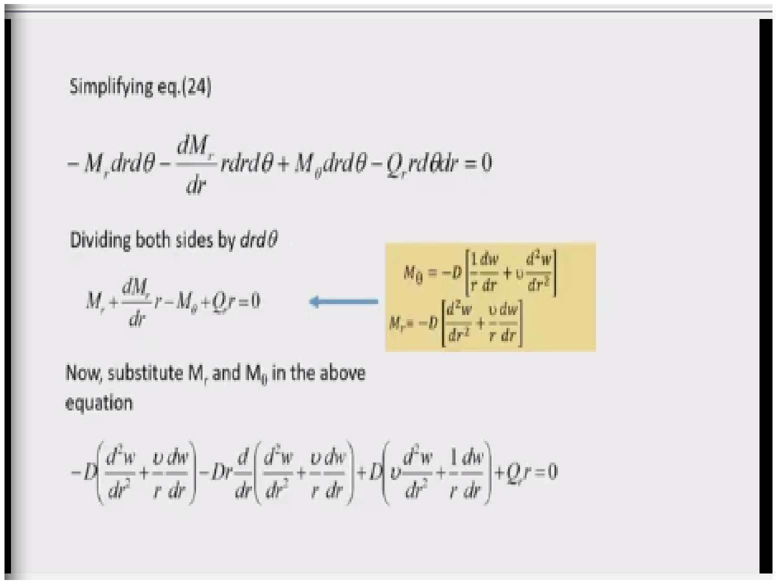

So, simplifying this we write this $-M_r\, dr\, d\theta$ - $dM_r/dr \times r\, dr\, d\theta + M_\theta\, dr\, d\theta$ - $Q_r\, r d\theta dr = 0$. Dividing both sides by $dr\, d\theta$, we ultimately get this quantity $M_r + dM_r/dr \times r$ - $M_\theta + Q_r\, r$, so Q subscript r indicates this radial shear along the edges equal to 0. Now, the expression for M_θ is this, and expression for M_r is this.

So this is the expression for M_θ, and this is the expression for M_r. So after substituting this expression and then we can write this $M_r = -D\left(d^2w/dr^2 + v/r\, dw/dr\right)$. Again this M_r is differentiated, so this differential coefficient is written, d/dr. And D is there, D is there, flexural rigidity, r is there, M_θ we have expressed, this is the expression for M_θ and then + $Q_r\, r$, so this is written.

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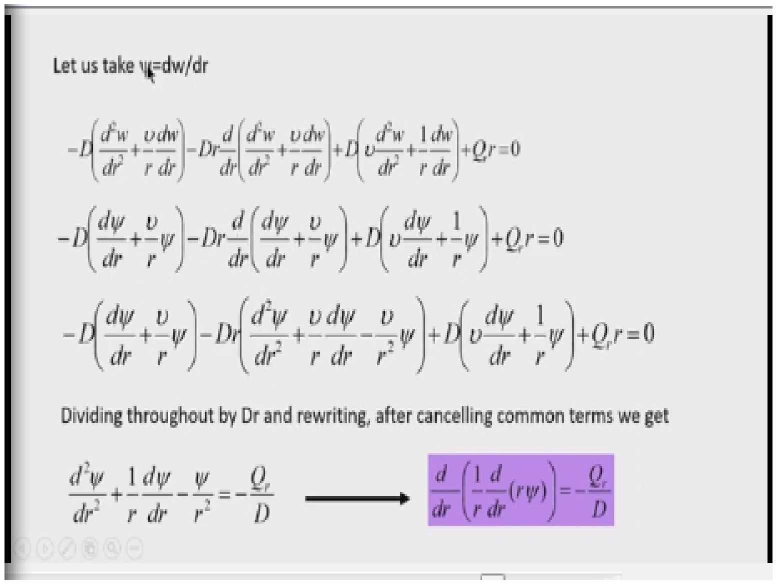

And now to simplify the equation, let us take this symbol, the slope as ψ, angle ψ. So, substituting the slope here dw/dr as ψ. We can now express the quantity dψ/dr + $v/r \times \psi$, here D r d/dr. And then the curvature is derivative of the slope like that we have written + $v/r \times \psi$ + D ($v \times$dψ/dr + 1/r ψ) + Q_r r = 0. So, here what we do actually?

We now differentiate this, and after term by term differentiation we get this quantity inside the bracket. Because this is a product of 2 functions 1/r ψ, which are both are functions of r, so it is differentiated and we get one more term here, after differentiation. Previously, it was 2 terms, now we get 3 terms, because after differentiation with respect to r, it will be $d^2\psi/dr^2$.

And then after differentiation with respect to r this variable ψ, then it will be dψ/dr, then 1/r has to be differentiated again, so this is differentiated and it is written, this quantity is written + Q_r r = 0. Dividing throughout by dr and rewriting after canceling some common terms, you will get a very simple equation. The $d^2\psi/dr^2 + 1/r\, d\psi/dr - \psi/r^2 = -Q_r/D$.

This can be in compact form, it can be arranged in this way d/dr{1/r d/dr (r × ψ)} = - Q_r/D. So, this is the compact form of this equation that we have obtained, where you can note that ψ is nothing but dw/dr.

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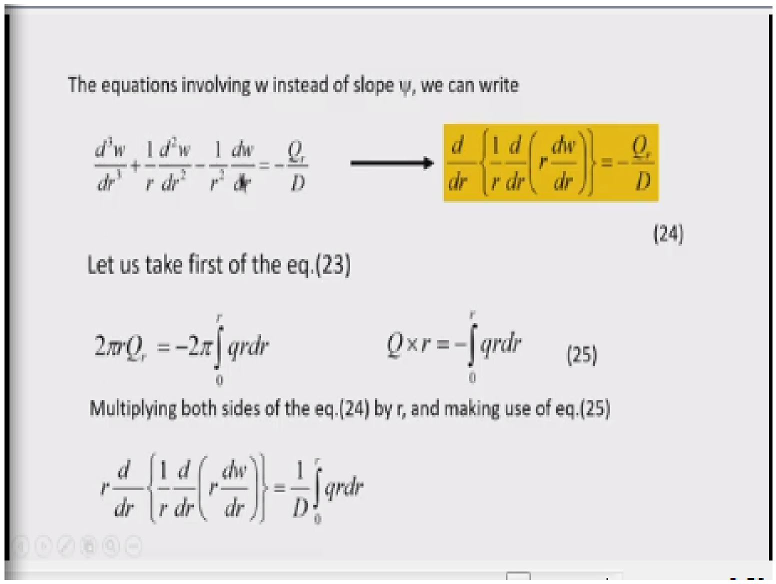

The equation involving w instead of slope ψ can also be written. So, you can see that equation was 2nd order equation with slope, now it will be 3rd order equation. So, shear is involved 3rd order derivative so it is coming, so physically we are getting that, we are approaching in the right direction. So, $d^3w/dr^3 + 1/r\, d^2w/dr^2 - 1/r^2\, dw/dr = -Q_r/D$.

So, this can be written in this form that, again involving the deflection it can be written in the compact form with this only consist of this 3 terms, that is dw/dr and d/dr and d/dr. So, you can see here, the w is here inside this bracket, that it is jacketed by so many variables. So, this is a very convenient form when the Q_r is known in terms of r, then you can integrate successively and find out extract the value of w.

Now let us take the first of the equation 23, so equation let us see, 23. So, this equation let us take, first of the equation this equation we take. So, if we take the first of the equation of 23, then we know this equation is the first equation and $Q \times r$ actually it will be a total force, total shear force that is acting after integration with respect to 0 to r. So, $Q \times r$, that means you can see that Q_r is the radial shear per unit length. So, multiplied by the circumference of this circular ring $2\pi r$ will get this shear force along this circumference, total shear force. So, this is equal to

$-2\pi\int_0^r qrdr$, So multiplying both sides of the equation 24, 24 is this equation by r. And making use of equation 25, you can write this now in this form. Now actually my intention is to remove the term Q and write it in terms of load, small q.

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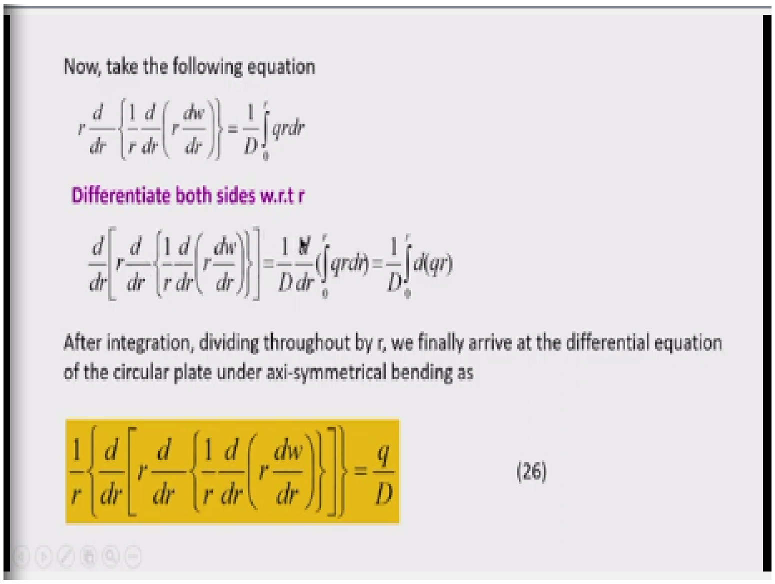

So, we have written like that now, so differentiate both sides with respect to r. So, I have differentiated both sides with respect to r that means d/dr additional term is here. And here again d/dr this term is there. Now interchanging the integration and differentiation, we now write in this form, 1/D, d(qr). So, after integration and dividing throughout by r, we finally arrive at the differential equation of the circular plate under axi-symmetrical bending as this.

Now you can note the difference between these 2 equations, 24 and 25, 24 also involves w but it is a 3rd order differential equation. And the order of differential equation is less that means constant of integration will be also less. So, this Q_r is the shear force and here you are getting that this is a 4th order differential equation, and q is the loading that is distributed loading on the plate. So, both the equation can be used depending on the situation.

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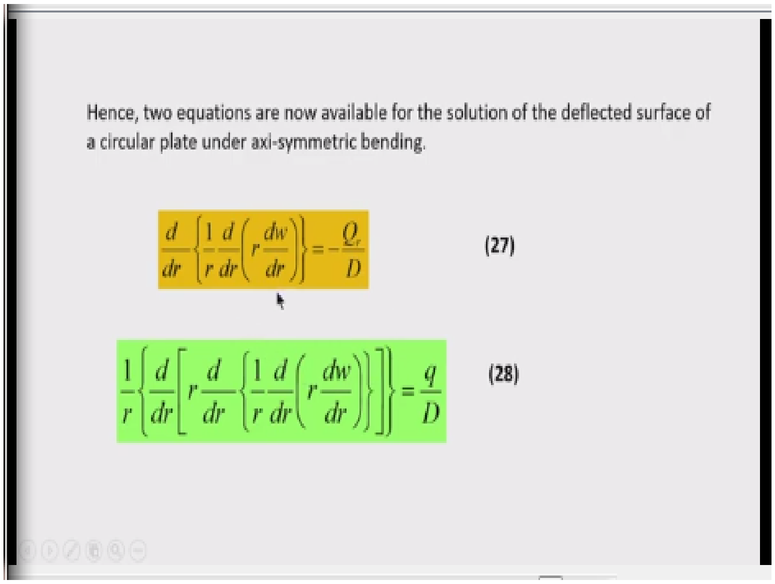

So, it will be demonstrated through different examples in subsequent classes.

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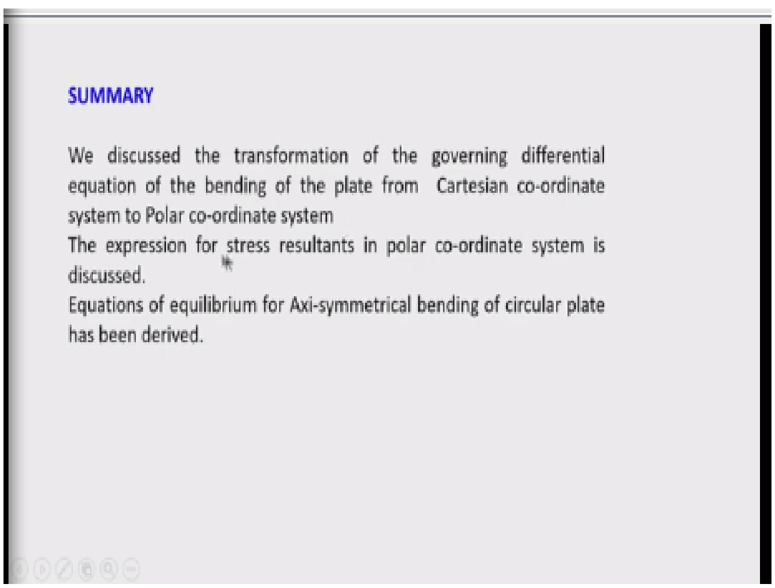

So, today whatever I discussed the transformation of governing differential equation of the bending of plate from Cartesian coordinate system to polar coordinate system. The expression for stress resultants in polar coordinate system is discussed here. So, that is first using the calculus, only I transform the Cartesian plate equation in Cartesian system to a plate equation in polar coordinate system.

So, it is transformed actually from Cartesian system to polar coordinate system. Now, the use of polar coordinate system is shown in case of circular plate using an axi-symmetrical condition. That is having a rotational symmetry in the plate in respect of loading, in respect of edge condition. So, with that condition we have seen that the plate equation is now decomposed into an ordinary differential equation.

And we have found 2 types of differential equation to be used for solving the plate problem, one is the 4^{th} order equation in which the equation the 4^{th} order derivative is related to the distributed loading. In another case, we have seen that it is a 3^{rd} order equation. And 3^{rd} order differential equation is ultimately related to the shearing force on the plate. So, 2 equations are there, now there we derive the equation in the 2 forms.

And each form has it is own advantage and disadvantage. So, depending on the situation and the problem that will be encountered we can use any of these 2 equations. So, today I finish up to this the different applications of the circular plate under axi-symmetrical bending will be considered in the subsequent classes. Thank you very much.