Plates and Shells

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Module-03
Lecture-10
Rectangular Plate with Levy's Boundary Condition Subjected to Edge Moment

Hello everybody, today I am starting the lecture 3 of module 3. So, the topic of our today's lecture will be the application of Levy's method, 2 various types of problem. So, earlier we have seen that Levy's method can be applied to a specific boundary conditions for rectangular plate. That means 2 opposite edges must be simply supported and other 2 opposite edges may have other conditions.

So, based on that we found the solution for different types of problem where the loading was distributed fully, partially and strip loading. And boundary conditions we have taken that it is simply supported on 2 opposite edges and other 2 opposite edges was fixed as well as simply supported. The simply supported on all edges condition was earlier analyzed by Navier's method.

But in the last class, I have shown you that this type of problem can also be solved by Levy's method. Because Levy's method although the derivation is slightly complicated, but computation becomes easier because of single summation. So, that is one advantage of Levy's method and there is more generality in application due to relaxation of boundary condition that all the edges were simply supported which were the essential or the only requirement in case of Navier's method.

Now, we shall use the Levy's method to solve the problems of rectangular plate which is simply supported along all edges, but subjected to edge moment. The edge moment problem is important to obtain the solution for other edge condition. For example, this all edges clamped, so that type of plate can be solved by using the Levy’s method, superimposing the 2, 3 solutions obtained by Levy’s method, so today we will discuss these things.

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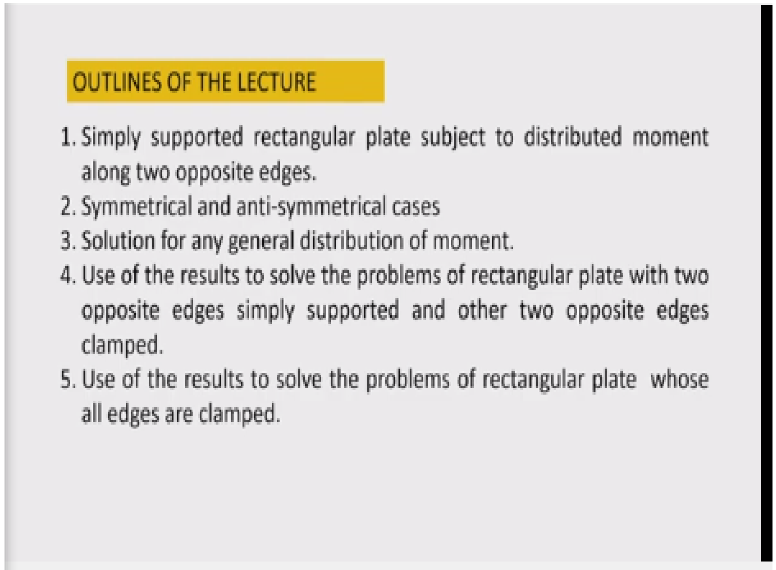

So, today's outlines of the lecture are, simply supported rectangular plates subjected to distributed moment along 2 opposite edges. Then we shall consider symmetrical and anti symmetrical cases for the moments. Then number 3, solution for any general distribution of moment. In this second point, I mentioned that symmetrical and anti symmetrical cases, but if there is any general type of distribution of moment, say at one edge it is say uniformly distributed moment of magnitude say 5 kNm/m.

In the other opposite edges, the moment is say 10 kNm/m. So, in that type of situation it is neither anti symmetrical nor symmetrical. So, any general type of distribution also can be handled by Levy’s method and that we will discuss today. Then use of the results, so far we obtain in case of the uniformly distributed load or partially distributed load and then the edge moment will be applied to find the solution of the plate when 2 opposite edges are simply supported and other 2 opposite edges are clamped.

So, that type of problem actually was solved in my earlier classes using Levy’s method imposing the boundary condition at the other 2 edges because the plate was symmetrical about the x axis.

So, we applied the boundary condition that y = + or - $b/2$ for a rectangular plate of size $a \times b$. Now, that result will be obtained by superimposition of edge moment conditions. So, after delivering this topic that the problem of rectangular plate with 2 opposite edges simply supported and other 2 opposite edges clamped.

Then we can further proceed to find the solution of the problems of rectangular plate whose all edges are clamped. Because clamp places are very common in practical application and we required to find the analytical solution if exist. So, it is shown by superimposition principle using the Levy's method, the exact solution of the clamp plate that is clamp plate I mean that all the 4 edges are clamped can be obtained.

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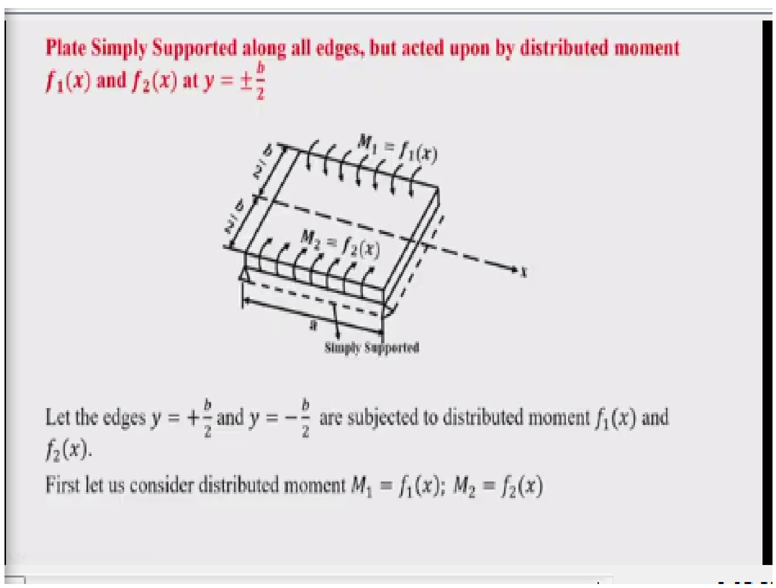

So, now let us consider a plate, rectangular plate, length is a and width is b subjected to distributed moment along the edge $y = 0$, the moment is M_1 which is distributed along the x axis in a functional form that I have written $f_1(x)$, this may be a constant also or may have any other variation. Then on the opposite edge $y = b$, here you will see that the moment M_2 is acting and which is in the functional form is expressed as $f_2(x)$, f_2 is a function of x.

But since the x axis considered running through the centre of the plate, so the edge conditions are defined as $+b/2$ and $-b/2$. If this is the direction of positive x axis the upward direction, then the top edge will be condition at $y = +b/2$, bottom edge will be at the condition $y = -b/2$. So, we consider that moment $M_1 = f_1(x)$ and $M_2 = f_2(x)$, these $f_1(x)$ and $f_2(x)$ may be constant also, so in that case the edge may have the uniform distribution of moment.

And sometimes this we consider a variation of moments say sinusoidal or any other type of moment acting linearly varying moment along the edges. So, that condition can also be handled by this method.

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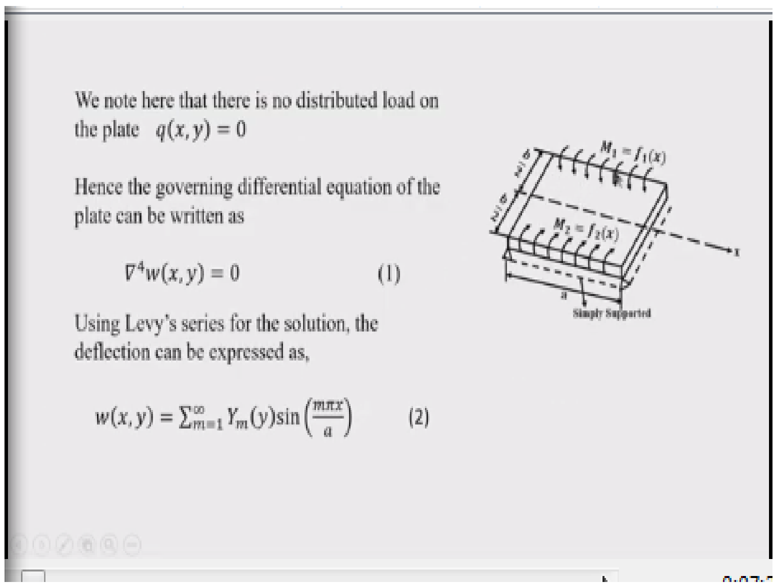

So, let us see since there is no distributed load, so, $q(x,y) = 0$, that parameter we know. So, plate equation is reduced to the homogeneous solution of this 4th order differential equation $\nabla^4 w(x,y) = 0$, because there is no load acting on the plate, the edge moments are applied but that is to be fulfilled by boundary conditions. So, the plate surface free from loading, therefore we have written the differential equation of the plate as $\nabla^4 w(x,y) = 0$, w is of course a function of x and y.

Now since we are adopting the Levy's condition, so we will be using a single sine series as expressed by $Y_m \times \sin(m\pi x/a)$, where Y is a function of y that remains unknown. And $\sin(m\pi x/a)$ is the condition that is taken to satisfy the boundary condition at $x = 0$ and $x = a$. Because $x = 0$, $x = a$ boundary condition for simply supported, so it satisfies the condition at $x = 0$, $x = a$.

But other edges the boundary condition is here we are taking simply supported, but it contains edge moment. So, sin($m\pi x/a$) will not directly satisfy the boundary condition here, because the edge moment condition have to be applied to find out the solution of the boundary value problem. After substitution of the equation 2 in equation 1, it can be readily verified that the 4th derivative of Y with respect to y will be the first term. Then second term will be this $-2m^2\pi^2/a^2$ into second derivative of Y with respect to y. And the third term will be $m^4\pi^4/a^4 \times y = 0$. Because sin($m\pi x/a$) will be appearing in the summation term.

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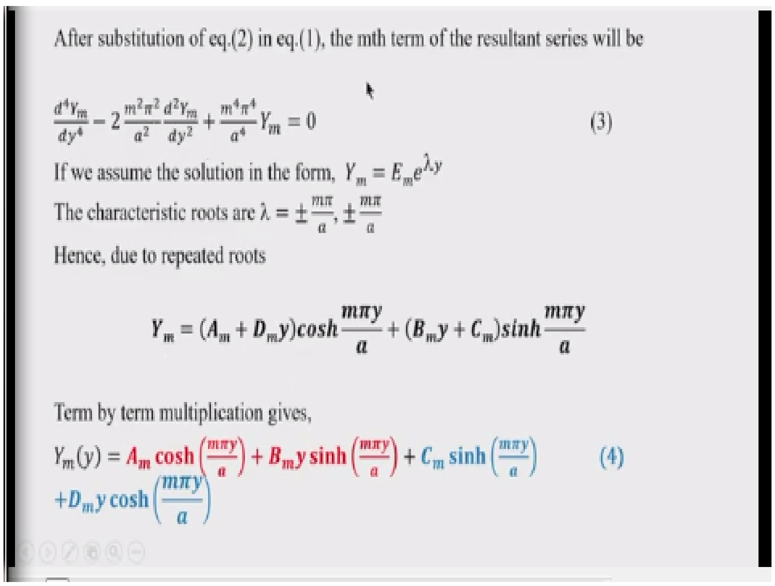

And taking a general term of this series, say mth term, we can write the differential equation as equation 3. So, it is reduced to an ordinary differential equation of 4th order. So, we have seen the solution of this differential equation. That is if I assume the solution in the form $E_m e^{\lambda y}$, where E_m is a constant. And after substituting this we get a characteristic equation, which will be λ^4 $-2m^2\pi^2/a^2$ into $e^{\lambda y}$, $e^{\lambda y}$ of course will be common to all the terms.

So, we need not consider it, because it will be cancelled. So, the characteristic equation will be λ^4 $-2m^2\pi^2/a^2 + m^4\pi^4/a^4 = 0$. By solving this characteristic equation, we get 4 roots because this is a 4th order equation from 4th order polynomial, we get the 4 roots. You can see the roots are repeated $+m\pi/a$, $+m\pi/a$ is repeated and similarly $-m\pi/a$, $-m\pi/a$ will be repeated.

So, according to the theory of linear differential equation, when the repeated root occurs, the solution can be written as $Y_m = (A_m + D_m y) \cosh(m\pi y/a) + (B_m y + C_m) \sinh(m\pi y/a)$. So, here you see I have taken the constant not in sequence just arbitrary constant you can take anything. But interestingly you can see because of this cosine function, cos hyperbolic function and sine hyperbolic function. And when it is multiplied by a function y, then different characters of the function will be reflected. Now let us see first, say if I multiply A_m with $\cosh(m\pi y/a)$, $\cosh(m\pi y/a)$ is a symmetric function, so therefore this term will be symmetric. But when I multiply $D_m y \cosh(m\pi y/a)$, you can note it that y is your odd function whereas $\cosh(m\pi y/a)$ is even function.

So, product of odd and even function again will be even function. So, product of this anti symmetric term and symmetric term will be your anti symmetric term. Again you come here the second term $B_m y \sinh(m\pi y/a)$, here you can note that y is your this is anti symmetric term, whereas sine hyperbolic is a symmetric term. So, product of anti symmetric and with another anti symmetric term will be symmetric term, and then $C_m \sinh(m\pi y/a)$ is a anti symmetric term.

So, I have grouped the symmetric and anti symmetric term here in this final solution after term by term multiplication. Now here you can see the red colour terms are all symmetric terms and blue colour terms are anti symmetric terms. So, it can be easily verified that when the symmetric moment occurs or symmetric moment acts at the opposite 2 edges, then the deflection function should not contain any anti symmetric term. So, therefore we can drop this the blue colour term which is here the anti symmetric term.

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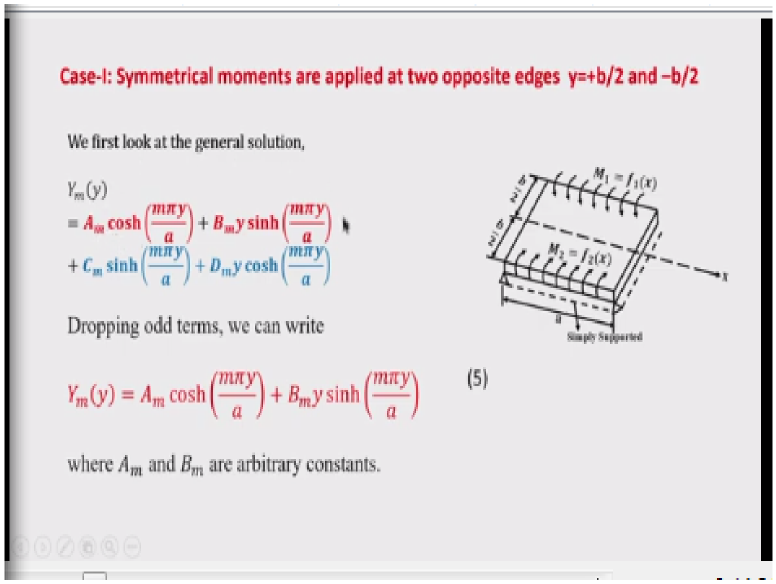

So, now in our problem, we first consider the symmetrical moments are applied on the 2 opposite edges that is $M_1 = f_1(x)$ and $M_2 = f_2(x)$. But $f_1(x) = f_2(x)$, so we get the solution only with the red terms of the expression that is the symmetric functions. So, dropping the odd terms, odd terms are blue colour terms here. So, accordingly I have written this solution as $Y_m(y) = (A_m) \cosh(m\pi y/a) + (B_m y) \sinh(m\pi y/a)$.

Now note here that A_m and B_m are arbitrary constants of integration and it can be only found by imposing the boundary conditions. Now, let us go to the boundary condition here, at the 2 opposite edges the x = 0 and x = a, the plate was simply supported as seen here. So, the plate equation of deflected surfaces written in this form, that along the x direction the sine function $\sin(m\pi x/a)$ is taken.

But along the y direction that function that $Y_m(y)$ requires to be known completely, once we can find the A_m and B_m.

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At the edges $y = \pm \frac{b}{2}, w = 0 \Rightarrow Y_m = 0$

$$\Rightarrow A_m \cosh\left(\frac{m\pi b}{2a}\right) + B_m \frac{b}{2} \sinh\left(\frac{m\pi b}{2a}\right) = 0$$

$$\Rightarrow A_m = -B_m \frac{b}{2} \frac{\sinh\left(\frac{m\pi b}{2a}\right)}{\cosh\left(\frac{m\pi b}{2a}\right)}$$

$$\alpha_m = \frac{m\pi b}{2a}$$

$$\Rightarrow A_m = -B_m \frac{b}{2} \tanh\left(\frac{m\pi b}{2a}\right) \qquad (6)$$

Or $A_m = -B_m \frac{a\alpha_m}{m\pi} \tanh \alpha_m$

So, A_m and B_m are the arbitrary constants which are found by applying the edge condition. So, in the plate you can see the x axis, this axis of symmetry is passing along the centre of the plate in the x direction, so it is symmetrical about x axis. So, at y = b/2 or at y = -b/2 same moment exists. So, we take here y = + b/2, w is 0 because this is simply supported; now since *w* is 0, the Y_m function will be 0.

Now, Y_m is composed of symmetrical terms of the solution. So, I can write a here instead of y I have written b/2 and instead of y I have again here written $b/2 \times \sinh(m\pi b/2a)$. So, this is the equation that obtained by applying the boundary condition on deflection on the edges y = + b/2, -b/2 also you can obtain the similar equation.

Now from that equation, it is readily seen that A_m can be expressed in terms of B_m. So, A_m is expressed in terms of B_m and A_m is now equal to - B_m $b/2 \times \sinh(m\pi b/2a)/\cosh(m\pi b/2a)$. So, this ratio is tan hyperbolic, so I have written this function as - B_m $b/2 \times \tanh(m\pi b/2a)$. Now, note here that this quantity inside the parentheses is written as a number α_m which is nothing but $m\pi b/2a$ and it is a constant, depends on the harmonic numbers.

So, $m\pi b/2a$ is substituted as α_m and A_m is written in the concise form that is - B_m $a\alpha_m /m\pi \times \tanh(\alpha_m)$. Because from this substitution we can see $b/2$ is nothing but $a\alpha_m /m\pi$. So, this is the equation relating 2 constants of integration A_m and B_m. Now we want to impose another boundary condition, so that the constants A_m and B_m are completely known.

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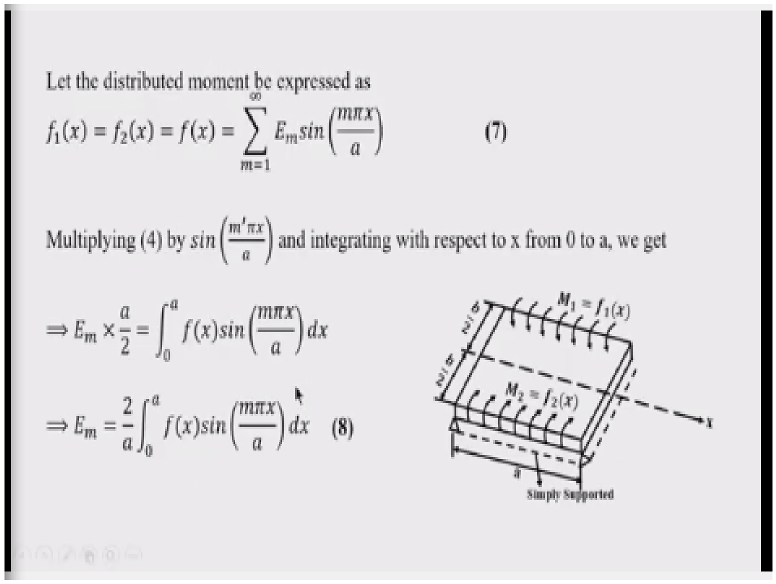

Now let us see the second boundary condition. Second boundary conditions have to be applied on the moment. So, moment that is acting along the edge are same $M_1 = M_2$ here, because this symmetric functions are taken as a deflection series, it means that moment should be also symmetry. So, $f_1(x) = f_2(x)$ we have taken f(x) which is a distribution of the moment along the x, y = + or - b/2.

So, it is expressed as a Fourier series m = 1 to ∞ that summation goes up to ∞ and the coefficient of Fourier term that is E_m a general coefficient is taken into $\sin(m\pi x/a)$. Because any function you can express in terms of Fourier series. Now, let us see what is E_m? Now, in that case if I multiply, this equation by $\sin(m'\pi x/a)$ another equation and integrate with respect to x from 0 to a.

We get because of integration with respect to dx, this integral will be $\sin^2(m\pi x/a)\ dx$ and due to arithmetic orthogonality condition of this sine function, you know that integration of $\sin^2(m\pi x/a)$ dx with the limit 0 to a will be $a/2$. So, therefore this right hand term becomes $E_m \times a/2$, and the left hand function now it is an integral of f(x) multiplied by sin($m\pi x/a$) with respect to dx and in the limit 0 to a.

Now we can find the E_m, coefficient of the Fourier series that is assumed for expressing the moment applied in the edges. So, $E_m = \frac{2}{a}\int_0^a f(x)\sin(m\pi x/a)dx$, so this equation represents the coefficient of Fourier series which is used to represent the edge moment. Now, suppose the edge moment is a constant quantity that is uniformly distributed moment.

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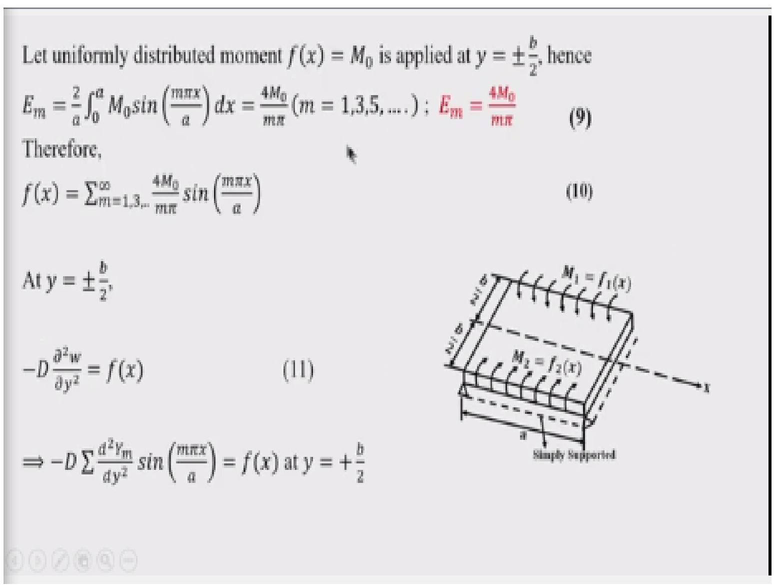

Then in that case, we assume that f(x) is M_0 where M_0 is a constant value is applied to y = + or - b/2, that is here at this edge your moment is also M_0, at this age also moment is M_0. So, substituting this f(x) as M_0 here, we required to integrate this function to find the coefficient of the Fourier series. So, after integration you can find that sin($m\pi x/a$) dx integration will be cos($m\pi x/a$) and with the limit 0 to a.

So, when you put the limit and consider only odd terms, for even term the value of the integral will be 0. So, only for odd term we get the coefficient of Fourier series which is used to express the edge moment is nothing but $4M_0/m\pi$, where m is the number that takes the odd integers only.

So, the first term of the Fourier series will be $4M_0/\pi$, second coefficient of the Fourier series will be $2M_0/\pi$ and of course that have to be multiplied by the sine function.

That means the moment that can be expressed as $4M_0/\pi \times \sin(\pi x/a) + 4M_0/2\pi \times \sin(2\pi x/a) + 4M_0/3\pi \times \sin(3\pi x/a)$ and so on, so summation will go up to ∞. But you can note that for even integers the function value will be 0, so therefore E_m will be only the $4M_0/m\pi$. Now, after substituting the value of E_m we can now express the distributed moment in a definite form.

That is $4M_0/m\pi \times \sin(m\pi x/a)$, this coefficient is very important because when the uniformly distributed moment acts on the edges. Then this coefficient represents the amplitude of each of the sine terms which is used to compose the series. Now, we apply the second boundary condition. So, second boundary condition when applied will give you the complete value of the A and B, the coefficient A_m and B_m.

Second boundary condition as you know that it is simply supported along other 2 edges also. So, naturally the bending moment is 0 in the y direction. Now bending moment in y direction is given by $-D\left(\partial^2 w/\partial y^2 + \nu\, \partial^2 w/\partial x^2\right)$, that should be there. But because the edges are supported, so there will be no curvature along the x axis, so therefore the second term has no meaning here at the edges. So, edge moment is $-D\partial^2 w/\partial y^2 = f(x)$, $f(x)$ means here this function, so we have to substitute this $f(x)$ here with this series.

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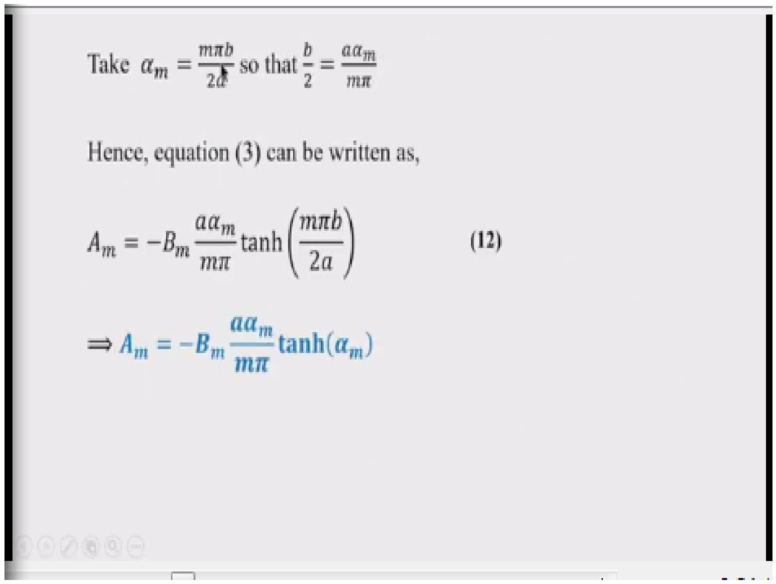

Now, let us see we have substituted $\alpha_m = m\pi b/2a$ and $b/2 = a\alpha_m/m\pi$ for some convenience in calculation or to write the expression in compact form, there is no other intention. So, hence equation 3 can be written as, equation 3 is on A_m that is this equation that you can write it the equation that we have found by applying the condition of displacement on the boundary. Then you can write this function as with $A_m = - B_m\, a\alpha_m/m\pi \tanh(m\pi b/2a)$, and this is substituted as α_m.

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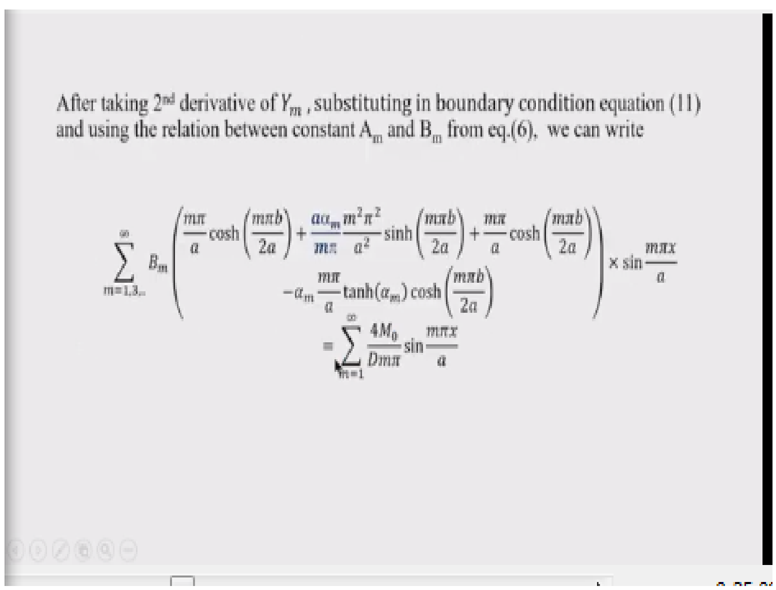

So, after taking the second derivative of Y_m, because this function the bending moment boundary condition requires second derivative of Y_m. So, when the second derivative of Y_m is taken and then substituting the boundary conditions that is equal to $f(x)$ the moment is $f(x)$. So, $-D\partial^2 w/\partial y^2 = f(x)$ minus of course, then using the relation between the constant A_m and B_m, constant A_m and B_m already found in equation 6.

So, that we have utilized and we have written the series in terms of only one constant. Now previously originally there was 2 constant as you note here A_m and B_m. But we have seen that one constant is related to another constant by use of deflection condition. Then another equation for determining the single constant B_m can be found after applying the moment condition at the edges.

So, applying the moment condition at the edges we have bought this series, complete series say summation B_m and inside the bracket you will get the term which are coming from this second derivative of Y_m. Then it is multiplied by $\sin(m\pi x/a)$ that is there because $\sin(m\pi x/a)$ will be always there when we take the even derivatives. With even derivatives sine term will again appear.

Then right hand term is $f(x)$, and $f(x)$ is nothing but this series $4M_0/D\pi \times \sin(m\pi x/a)$, this is nothing but your E_m that is a constant of the Fourier series. Now, our intention is to find the value of B_m, that is our intention, when you find the value of B_m then you can easily find the other constant A_m, so, how it is found? You can see, this is a series, so when $m = 1$ you will get B_1 multiplied by something $\times \sin(\pi x/a)$.

When $m = 3$, then you will get B_3 and this term substituted with $m = 3$ and this is $\sin(3\pi x/a)$ equal to the right hand side with series of the term with $m = 3$. So, like that you will get the infinite number of terms in the right hand side as well as infinite number of terms in the left hand side. So, in each term you will get the coefficient B_m, that is B_1, B_3, B_5, B_7 like that odd number of coefficients will normally appear and because odd number of integers m is required for the solution.

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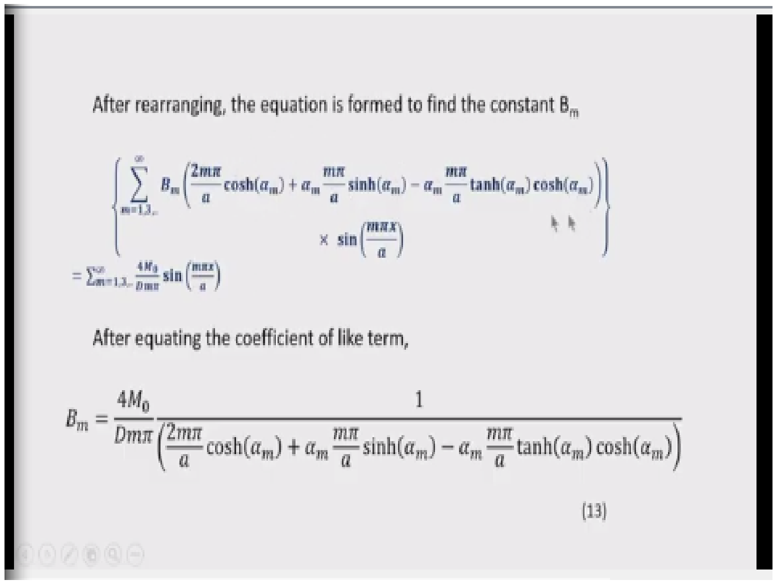

So, comparing the coefficients of the like term that is the technique. So, after rearrangement I have written like that. Because this you can see that $\cosh(m\pi b/2a)$ and $\cosh(m\pi b/2a)$, this term is common, so it is summed up. And it is written in a compact form taking the common term together. So, it is written $\{2m\pi/a \cosh(\alpha_m) + \alpha_m\, m\pi/a \sinh(\alpha_m) - \alpha_m\, m\pi/a \tanh(\alpha_m) \cosh(\alpha_m)\} \times \sin(m\pi x/a)$.

Now you can note that left hand side is also series of infinite number of terms, right hand side is also a series of infinite number of terms, equating the coefficient of like term that is here you are getting coefficient of B_1, and here you are getting another coefficient with when $m = 1$. So, that can be compared or equated because the series has to be equal when the coefficient of like terms are equal, otherwise it cannot be equal.

So, it is a sine series, so in the both sides you are getting sine series. So, each term of the sine series will be equal then only the series is equal. So, from that condition for any general term m, we get $B_m = 4M_0/Dm\pi$ into this. So, this can be easily verified from this expression, that coefficient of $\sin(m\pi x/a)$ that is the m^{th} term will be $4M_0/Dm\pi$, and this is the coefficient of B_m, so B_m will be this divided by this term that you are getting.

So, B_m = $4M_0/Dm\pi$ into this factor, this factor is coming as a reciprocal of this expression. So, $1/\{2m\pi/a \cosh(\alpha_m) + \alpha_m m\pi/a \sinh(\alpha_m) - \alpha_m m\pi/a \tanh(\alpha_m) \cosh(\alpha_m)\}$. So, very carefully we have to see the expression otherwise even if you miss one term it will give you wrong results. So, carefully you have to see the all terms are included in the expression.

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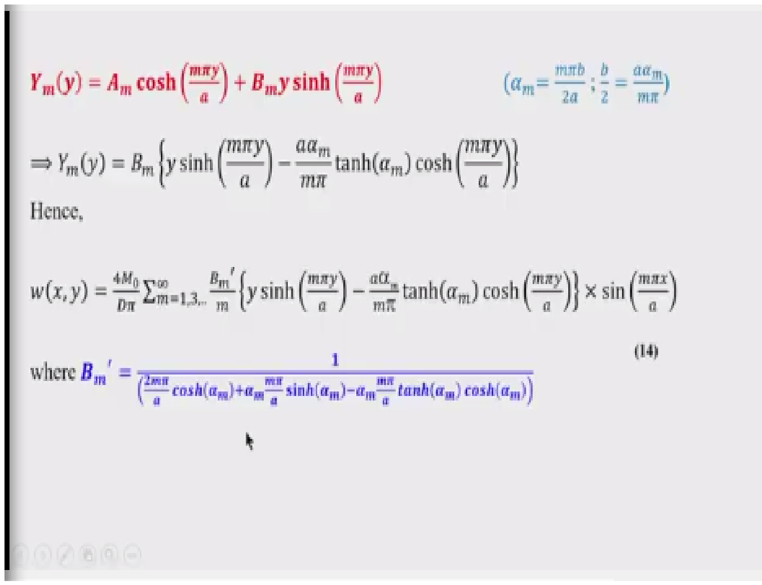

So, now A_m is found out earlier B_m is known, so A_m can be easily known. Because A_m is expressed in terms of B_m. So, now the solution can be written in terms of B_m because only one constant is required because A_m is now expressed in terms of B_m. So, we get this function as the solution of the homogeneous differential equation for this plate equation, homogeneous differential equation that resulted after substitution of the Levy's series in the partial differential equation.

That is actually the separation of variable technique that we adopted and we ultimately get this function Y_m equal to this, $B_m \times \{y \sinh(m\pi y/a) - a\alpha_m/m\pi \tanh(\alpha_m) \cosh(m\pi y/a)\}$. So, this is the series, B_m is known from that expression completely. So, now, Y_m is completely known, that means deflection is completely known.

So, deflection can be written as now, because we know that this B_m that we have found earlier contains $4M_0/Dm\pi$. So, $D\pi$ is the constant term, M_0 is also constant term. So, it is taken outside the summation term and inside the summation term all the quantities which are dependent on the wave number or you can tell the half wave number *m* is written because the sum has to be carried out with the respect to the half wave number.

So, therefore, this term I have separately written as B'_m term equal to this. So, deflection series is completely known, now for the edge moments which are symmetrically distributed on the 2 opposite edges. That means 2 opposite edges has equal magnitude of moment, if the moment is uniformly distributed or an equal nature. That means it is trying to sag the plate in this case, sometimes it may try to hog the plate also in the both the cases, then also the symmetrical distribution of moment has to be considered.

So, after getting the symmetrical distribution of function, we can get completely the deflected surface from which we can now find the deflection at any point on the plates. Of course, in this case for symmetrically distributed load, we will get this maximum deflection at the centre of the plate, so maximum deflection is found at $x = a/2$ and $y = 0$, so this is the centre of the plate. So, after substituting this value in this term, when M_0 is specified, we can completely find out the deflection after putting the numerical value of these material properties.

Because *D* contains the material properties, what is the parameter involving *D*? Parameters that is involved in *D* is Young's modulus of elasticity and poisson ratio of the plate. So, these 2 values are important to calculate the deflection of the plate as well as the dimension of the plate *a* and *b* should be known. Thickness of the plate is also important to find the flexural rigidity of the plate.

So, once you get the deflection of the plate, then you can find the shear force if it is required Q_x and Q_y, you can find the Kirchhoff's edge shear, that is a shear that is found after combining the vertical shear force with the shear force contributed by the twisting moment. So, these 2 effects are combined to give edge shear which is due to Kirchhoff and therefore it is known as Kirchhoff edge shear.

So, all these distributions you can find out, in addition you can find the corner reaction force that is developed at the corners due to twisting moment effect on the 2 adjacent edges. So, this corner force actually will be trying to leave the plate, if the plate is not properly anchored or held down then it will try to leave the plate. And therefore in case of simply supported plate generally the corners are provided with anchoring device in case of this RCC slab which is modeled as a plate. It is reinforcement of special nature is provided at the corner in the form of mesh top end bottom to prevent or to reduce the detrimental effect of the corner lifting force.

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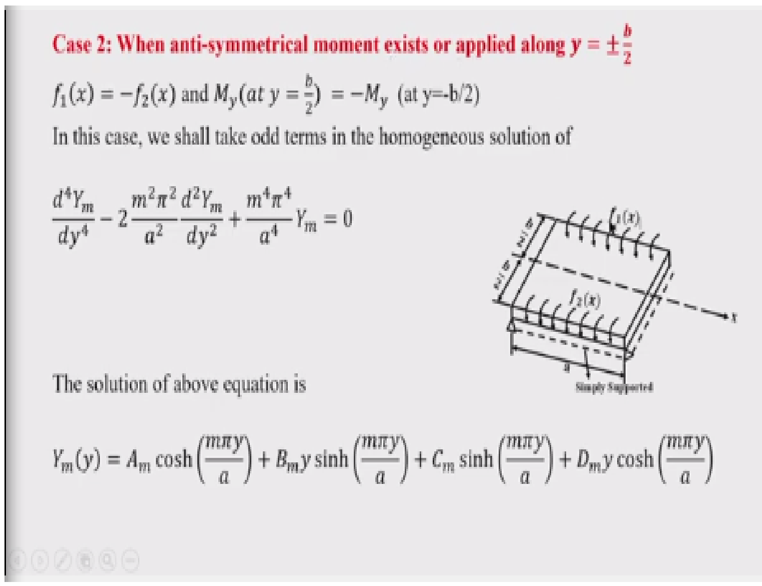

Now, we go for anti symmetrical moment at the edges. Now here we see that $f_1(x) = -f_2(x)$. So, this is the anti symmetric case, but magnitude is same in both the cases but they are of opposite nature. So, $f_1(x) = -f_2(x)$ and M_y at $y = b/2$ whatever distribution is there you will get opposite distribution or opposite nature in the other edge. So, at $y = -b/2$ it is $-M_y$ and at $y = b/2$ it is M_y. So, in this case we shall take odd terms in the homogeneous solution of the differential equation.

So, differential equation that you know earlier that I have shown you, it is obtained after substituting the Levy's series in the partial differential equation of the plate. And then separating the variables y, we get this differential equation. So, this differential equation is a 4^{th} order ordinary differential equation and this can be solved for homogeneous solution can be obtained by using the characteristic roots of this equation.

And it is shown that this solution contains the odd terms as well as even terms, that is found in terms of cos hyperbolic and sine hyperbolic terms. Originally the function was written in the form of exponential term, but exponential term is converted into hyperbolic terms. So, that we can easily understand the influence of this symmetrical and anti symmetrical term in the plate.

And importance of this term for case of symmetrical loading and anti symmetrical loading is now giving advantage for the solution of the plate problem. You can note here that this is the symmetric term cos hyperbolic, y sine hyperbolic is again symmetric term. Because y is anti symmetric, sine hyperbolic is anti symmetric, so product of 2 anti symmetric term is your symmetric term.

Then sine hyperbolic term is again this anti symmetric term and $D_m\, y \cosh(m\pi y/a)$ is again anti symmetric term. Now since the moments are anti symmetric, so we need to drop this term B_m and A_m coefficient we need to drop. So, from physical reasoning that the deflection of the plate should contain the anti symmetric term, that means the plate will deflect in anti symmetric manner.

Therefore, in accordance with that we have to take only the anti symmetric term that is $\sinh(m\pi y/a)\ C_m + D_m\, y \cosh(m\pi y/a)$. So, that means we can retain only the 2 constants in the solution C_m and D_m which are needed to be found out.

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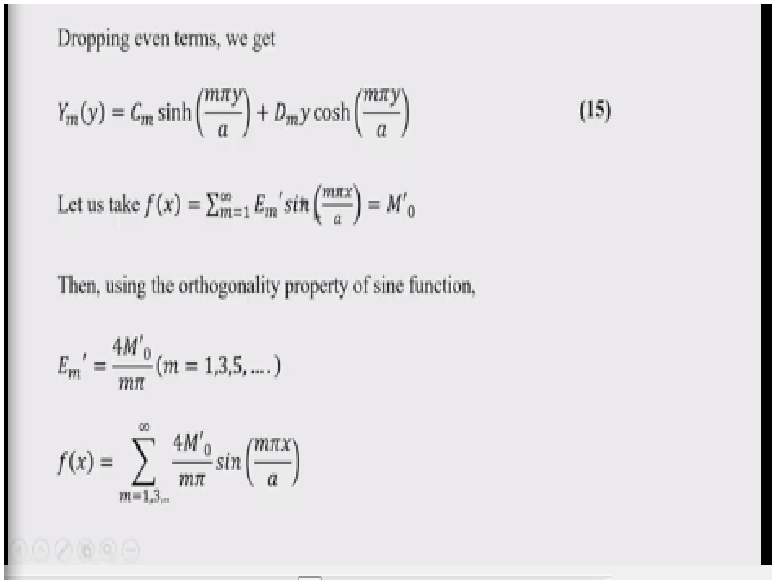

So, dropping the even terms, now we write the solution as $Y_m(y) = C_m \sinh(m\pi y/a) + D_m y \cosh(m\pi y/a)$, you can see both the terms are anti symmetric terms. Now, let us also express the moment anti symmetric moment in the form of Fourier series. So, here I am taking as a different constant E'_m, E'_m, previously we have taken E_m, now we are taking $E'_m \sin(m\pi x/a) = M'_0$.

So, this is the Fourier series and because this is actually one edge it is plus, in other edge it is minus. So, after multiplying this function with $\sin(m'\pi x/a)$, and integrating in the limit 0 to a, we get again you will get this function $4M'_0/m\pi$. So, this parameter will appear for the E'_m. So, the coefficient of Fourier series is now known. So, first term will be $4M'_0/\pi$.

Second term that is an odd term, so it will be $4M'_0/3\pi$, like that it will go. Only odd terms need to be taken because even term after integration vanishes. Because after integration the cos function will appear and when you put the limit 0 to a you will find that it will be 0 for even number of terms. Even number of terms it will be 1 - 1, that because whatever even function cos 0 is 1 and $\cos 2\pi$ is also 1, so 1 - 1 will be 0.

So, therefore even terms will not contribute to the Fourier series. So, only odd terms will contribute to the Fourier series as it was shown in case of the symmetrical loading also. So, therefore distributed moment $f(x)$ can be expressed as a series $4M'_0/m\pi \times \sin(m\pi x/a)$. Now one thing you observe that this quantity M'_0 is a known quantity and in case of distributed moment, that is which is uniformly distributed will get the constant M_0 in the 2 edges.

But it will be opposite in other edges that is in nature, if in one case it is sagging in other case it will be trying to hog the plate.

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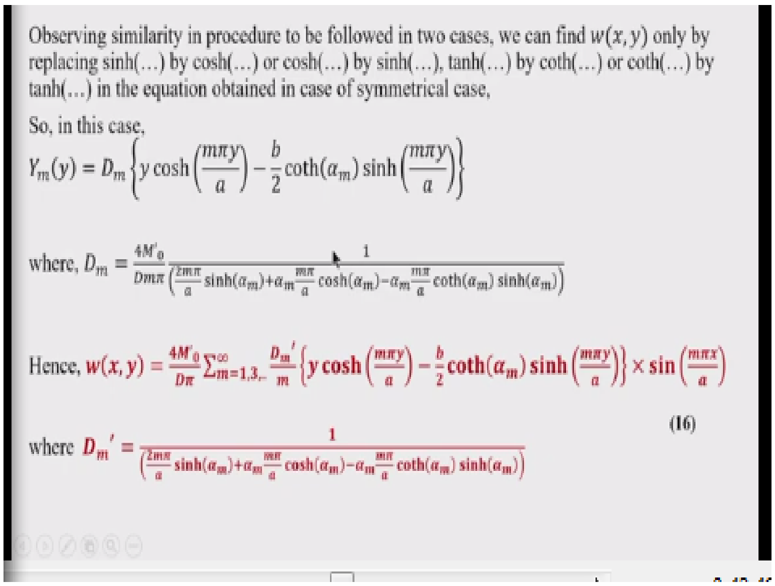

So, observing similarity in procedure to be followed in 2 cases. Now, we have observed one thing that procedure of finding the solution in symmetric case and anti symmetric case is similar, there is no drastic change in procedure. Because the same steps are followed that is you express the moment in terms of Fourier series and then Fourier series coefficient is found by using orthogonality condition of the sine function.

Then you can see that this boundary condition needs to applied in the earlier cases the condition because it was also simply supported and now also it is simply supported. So, at the edges y = + or - $b/2$, you will get the condition as deflection 0 and bending moment 0. But since it is anti symmetric problem, so all the terms in the deflection surface should involve the symmetric terms. So, there is no contribution with the anti symmetric terms in the deflected surface.

So, observing this fact we can now replace the several terms by symmetric term by their anti symmetric counterpart in the deflection series. So, deflection series is $w(x,y)$ and $w(x,y)$ you can see that it is nothing but $w(x,y)$ is your Levy's series in the past I have shown, this is the $w(x,y)$ and Y_m contains the symmetric and anti symmetric terms. In the present case, we are dealing with anti symmetric cases.

So in that case, our only anti symmetric terms are of importance. So, keeping the anti symmetric terms, that means using the earlier results, we can write the deflection that is Y_m, simply by replacing that sine hyperbolic by cos hyperbolic or cos hyperbolic by sine hyperbolic tan hyperbolic by cot hyperbolic or cot hyperbolic by tan hyperbolic, in the equation obtained in case of symmetrical cases.

Because we have these terms only in the symmetric cases, there are hyperbolic functions in the expression of y. So, replacing this term by appropriate anti symmetric counterparts, we can now write $Y_m(y) = D_m\{y \cosh(m\pi y/a) - b/2 \coth(\alpha_m) \sinh(m\pi y/a)\}$. In the earlier case, this term was your sine hyperbolic term, so that the product of 2 anti symmetric term was symmetric.

But now since this is anti symmetric problem with sine hyperbolic is replaced by cos hyperbolic and therefore product of one anti symmetric term and one symmetric term is anti symmetric. So, this is reflecting the true nature of the problem. Then here $\tanh(\alpha_m)$ was replaced by $\coth(\alpha_m)$, and this term was cos hyperbolic in the earlier case which was the symmetric term, now it is replaced by sine hyperbolic function.

So, all the symmetric terms are now replaced by anti symmetric terms, where D_m in the similar fashion, we now express the coefficient D_m. We know that this function, this is found after applying the second boundary condition that is due to bending moment at the edges. So, when we apply the second boundary conditions second derivative is of importance. So, equating second derivative to the given moment function we can now get this series in the left hand side and the right hand side.

So, left hand side also contains infinite number of terms, right hand side also contains infinite number of terms. So now, equating the coefficient of like terms, we can find the coefficients of each of the term in the series. For any general term m, we found the coefficient $D_m = 4M'_0/Dm\pi$, and is multiplied by a factor these $2m\pi/a$ $\sinh(\alpha_m)$ + α_m $m\pi/a$ $\cosh(\alpha_m)$ - α_m $m\pi/a$ $\coth(\alpha_m)$ $\sinh(\alpha_m)$.

One thing is observed here, that in earlier cases all the terms were just this counterpart, that is now only the symmetric terms are used. Earlier all terms were symmetric terms; now the anti symmetric terms are only used in the expression. So, we now write the $w(x,y)$, the deflection series as this, where D'_m is found with this, D'_m is this term that is found here. So, now 2 conditions are analyzed, symmetrical condition first and then second is anti symmetrical condition. So, you have understood the procedure. Now, if there is any general type of distribution.

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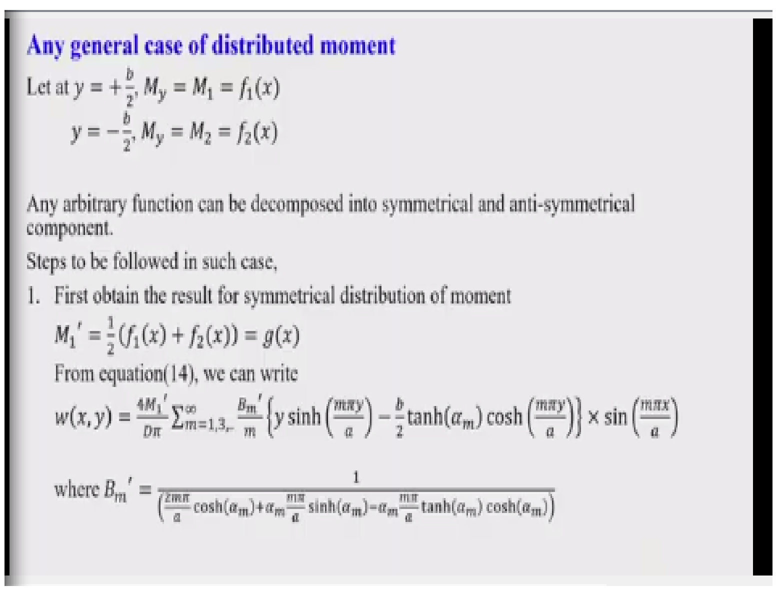

So, general type of distribution I mean that in one case it may be symmetric in another case it may be anti symmetric or in both cases there may be similar nature of moment but magnitudes are different. So, it is known that any arbitrary function can be decomposed into symmetrical and anti symmetrical component, that is well known fact in mathematical physics or you can say that in mathematics.

So, any arbitrary function is decomposed into symmetrical and anti asymmetrical term, that means here this function in the one case it is a $f_1(x)$, in the other edge it is $f_2(x)$. So, we write the components, M_1' will be $1/2\{f_1(x) + f_2(x)\}$ is a function say $g(x)$. So, now we find the deflection of the symmetrical distribution of function with this component. So, if this component of moments is acting on the 2 opposite edges, this is taken as a symmetrical part.

We can use this symmetrical expression for the deflection that we have derived earlier. Now, in the second case, the anti symmetrical term.

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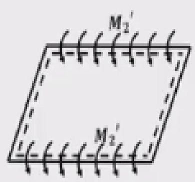
2. Obtain the result for anti-symmetrical distribution of moment also,

$$M_2' = \frac{1}{2}(f_1(x) - f_2(x))$$



From equation(16),

$$w(x,y) = \frac{4M_2'}{D\pi}\sum_{m=1,3,..}^{\infty} \frac{D_m'}{m}\left\{y\cosh\left(\frac{m\pi y}{a}\right) - \frac{b}{2}\coth(\alpha_m)\sinh\left(\frac{m\pi y}{a}\right)\right\} \times \sin\left(\frac{m\pi x}{a}\right) \quad (15)$$

$$\text{where } D_m' = \frac{1}{\left(\frac{2m\pi}{a}\sinh(\alpha_m) + \alpha_m\frac{m\pi}{a}\cosh(\alpha_m) - \alpha_m\frac{m\pi}{a}\coth(\alpha_m)\sinh(\alpha_m)\right)}$$

3. Add the result of the steps(1) and (2) to get the deflection for any arbitrary moment.

Anti symmetrical term will be $1/2\{f_1(x) - f_2(x)\}$ x, so that has to be used now with the anti asymmetrical term in the deflection that we have just analyzed it for anti symmetrical moment. So, 2 conditions are obtained separately, one is symmetrical moment condition, and another is anti symmetrical moment condition. But given the moment distribution in the 2 opposite edges any moment distribution will be first decomposed into 2 functions.

One is symmetrical function that is if the edge moment in one edge is f_1, and the edge moment in another edge is f_2. Then symmetrical distribution of function will be $(f_1 + f_2)/2$. Anti symmetrical distribution of functions that will take, in the second case will be $(f_1 - f_2)/2$. So, 2 cases are analyzed separately with the help of our known results and then we superimpose the 2 results of deflection to get the deflection for any arbitrary moment.

For example, this edge of the plate is subjected to a moment that is shown here is the moment that is acting in the clockwise direction, here, say 10 kNm/m. And in that case, it is also acting in the clockwise manner but the nature of the moment is such that it will produce the anti symmetric deflection. So, therefore these are any general case, so if this is a 10 kNm/m.

And here for example, this is another moment of nature symmetrical, so it is say 5 kNm/m. So in that case, in the first case we have to obtain the deflection of the plate using this symmetrical moment in the 2 edges as {10 + 5}/2, that is 7.5 kNm/m. So, in the second step we will find the deflection of the plate for anti symmetrical moment, which is imposed by subtracting the result {10 - 5}/2. By using the moment value as {10 - 5}/2, that is 2.5. So, in one case, the result will be obtained by using the edge moment as 2.5. In another case, the result will be obtained with the edge moment 7.5. Now, if you add these 2 results, you will get the actual distribution that you are requiring here. So, in this way by method of super imposition, we can obtain the deflection of the plate for any general distribution of moments.

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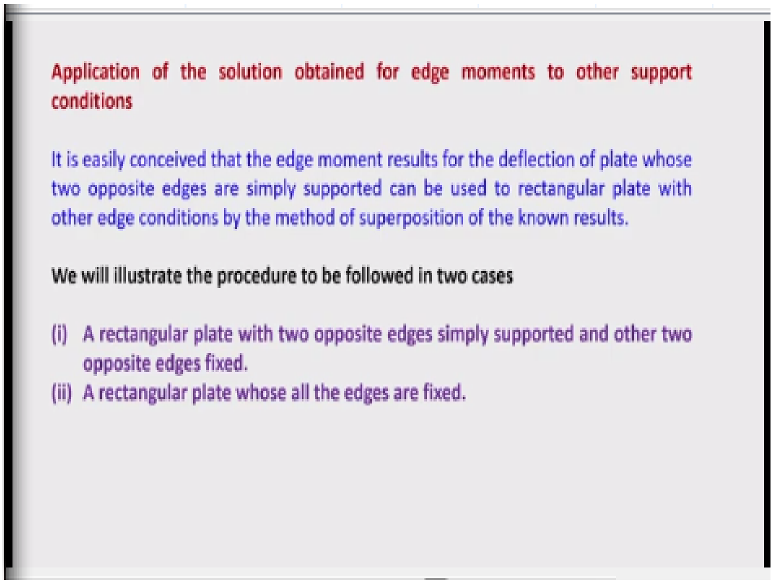

Now let us see the application of these results, for further analysis of the plate. Now here we are dealing with the closed form expression of the deflection. That is the exact solution of the differential equation is obtained in Levy's method as well as in Navier's method. Of course, here I am discussing the Levy's method. So, we will use the Levy's method for edge moments to other support condition.

So, Levy's method is used for getting the results of the deflection of the plate subjected to edge moment symmetric as well as anti symmetric. And that condition we will now use to a plate, which has other support condition, not directly following the Levy's condition. So, let us see how it can be done. It is easily conceived that the edge moment results of the deflection of the plate whose 2 opposite edges are simply supported can be used to rectangular plate with other edge conditions by the method of super imposition.

That you know say from your elementary structural analysis, you know that a fixed beam can be analyzed by first using a simply supported beam. And then using a beam with the end moment and then superimposing the result such that the moment at the end that is applied will make the slope at the end to be zero. So, that gives the fixed end condition of the beam. So, this principle will be applied here in case of plate also.

So, we will illustrate the procedure to be followed in 2 cases, a rectangular plate with 2 opposite edges simply supported and other 2 opposite edges fixed. Then second case a rectangular plate whose all the edges are fixed, that is a clamped plate.

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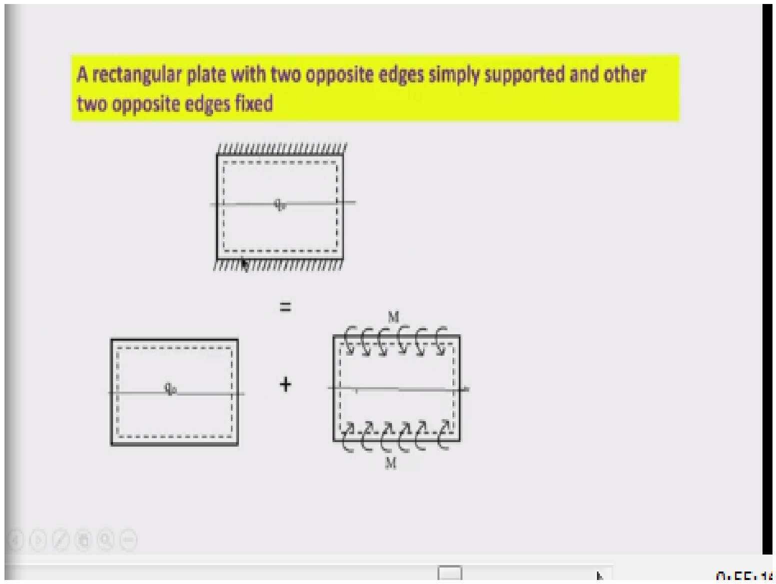

Let us see in the first case, we have to deal with this condition, that of course already obtained in my earlier classes, that 2 edges are simply supported $x = 0$ and $x =$ a are simply supported, if this length is a and this is b. And this is clamped $y = 0$ and $y = b$ is clamped, but of course here we are taking the x axis running through the middle of the plate. So, naturally the boundary condition will specify at $y = +b/2$ and $y = -b/2$.

So, at $y = +b/2$, or $y = -b/2$, we have the fixed boundary condition. That condition we have already obtained, solution already we obtain using imposing the boundary condition at the 2 opposite edges, with the help of the Levy's series. But now we shall try how we can obtain the similar result with the help of the results obtained in today's class. Today's class, we are focusing on the deflection of the plate, subjected to edge moment.

So, now this condition can be written as an equivalent of the simply supported plates subjected to UDL. Of course, we are analyzing the plate for UDL. So, simply supported plate subjected to UDL plus simply supported plate subjected to edge moment which is symmetrical. So, if we superimpose these 2 conditions, this will reflect this condition provided the *M* has to be found in such a way that it will make the slope along the y direction, as zero. So, that condition must be satisfied by the particular value of *M*. So, let us see how we can proceed.

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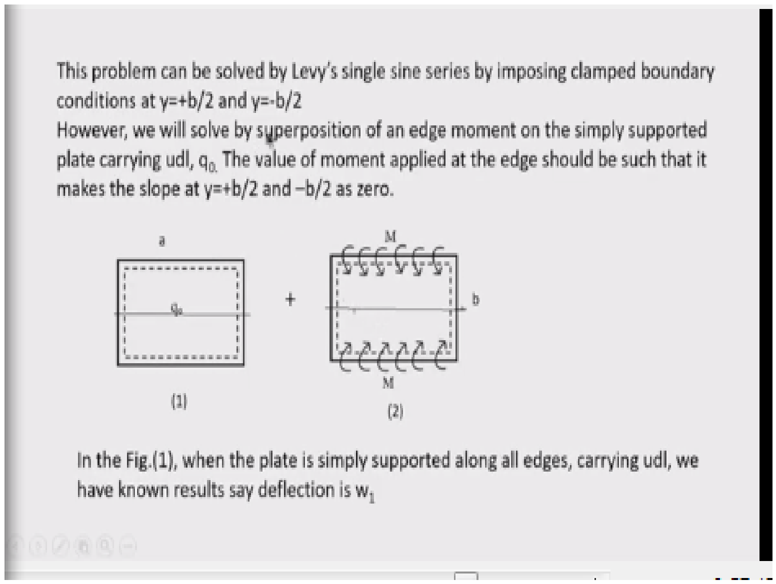

This problem can be solved by Levy's single series that is obvious by imposing clamped boundary condition that we have already obtained it. But now, in this method we are discussing the use of the results obtained for edge moment. So, in figure 1, the plate is simply supported along the edges carrying the UDL whose results are already known by us. And this is the plate, in which the 2 opposite edges are simply supported, but 2 opposite edges are subjected to this same moment this is symmetrical type of moment.

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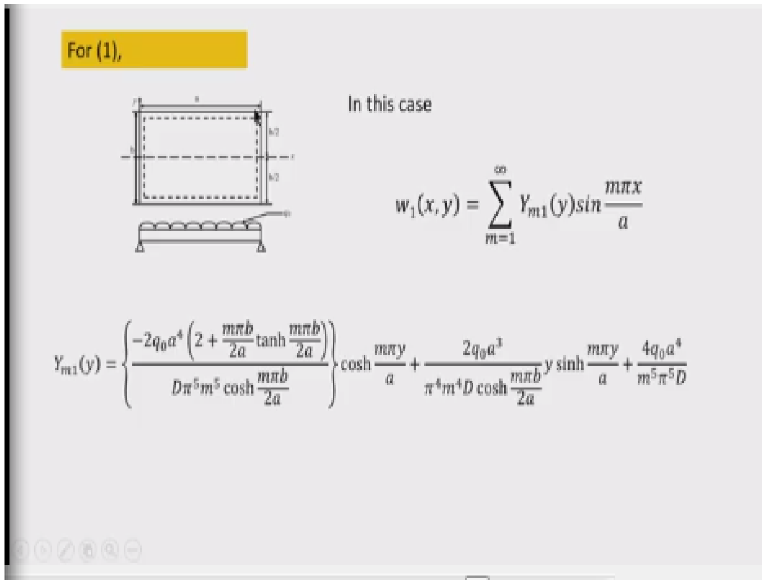

So, for this condition 1, the deflected series is known. From our earlier analysis I have written this result directly, you will get in my earlier note this result is given. So, this I call it as w_1, because this is a deflection due to UDL. So, once of the deflection is obtained, that means Y_{m1} is already known due to this UDL and we write the deflection series as it is. Because 2 opposite edges, $x = 0$, $x =$ a are simply supported.

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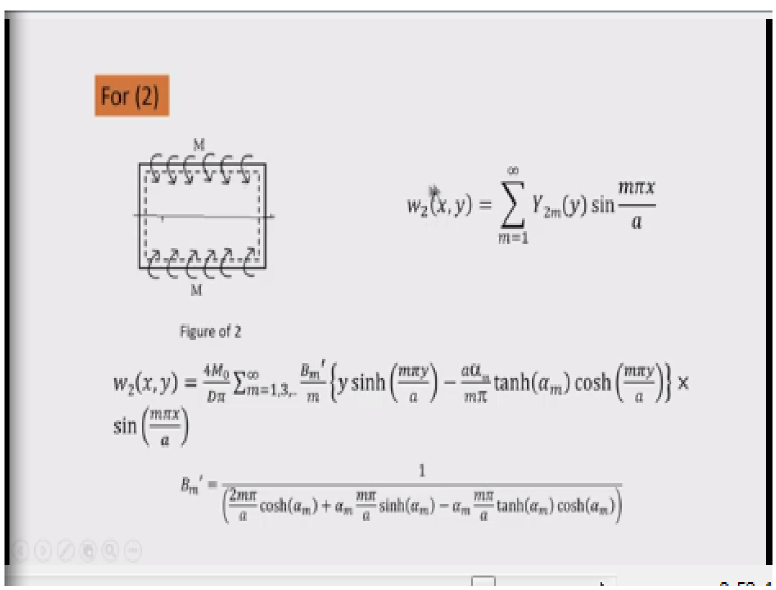

So, for 2, second condition, that is the edge moment applied condition; we call the deflected series as w_2. So, w_2 is now obtained as $Y_{2m}(y)$ $\sin(m\pi x/a)$, and $w(x,y)$ that just now we have obtained in terms of M_0 and the coefficient B'_m term etcetera, all are mentioned in the earlier slides, so this w_2 is written as this. Superimposing these 2 conditions will resolve the actual deflection of the plate subjected to uniformly distributed load, with the edge condition that 2 opposite edges are simply supported and 2 opposite edges are fixed only will be true if the value of M is such that it will make the slope as 0.

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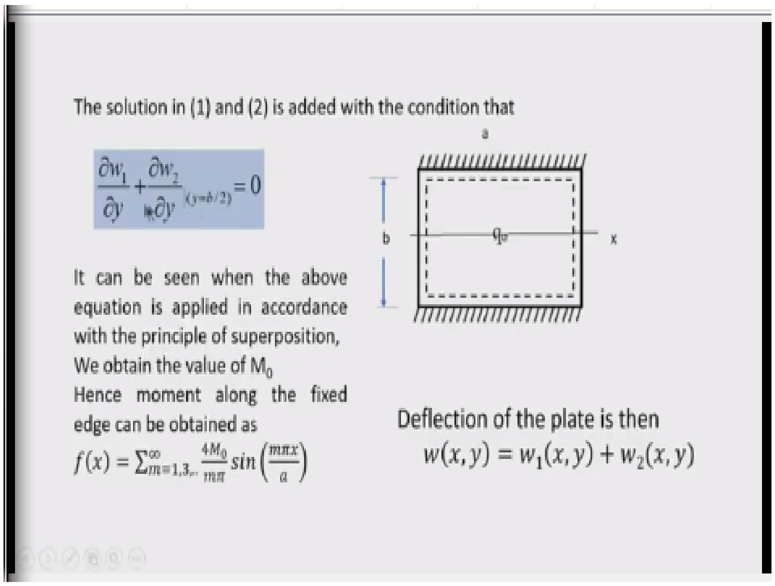

For that condition we will impose this, the slope along the y direction for w_1 plus slope along the y direction for w_2, at y = $b/2$ = 0. The above equation is applied in accordance with the principle of superposition; we obtain the value of M_0. Once you know the value of M_0 the fixed end moment along these edges are known as the series $f(x) = 4M_0/m\pi \times \sin(m\pi x/a)$. After summation with the infinite number of terms, it will represent the uniformly distributed moment. So, deflection of the plate is then $w(x,y) = w_1 + w_2$, all are functions of x and y.

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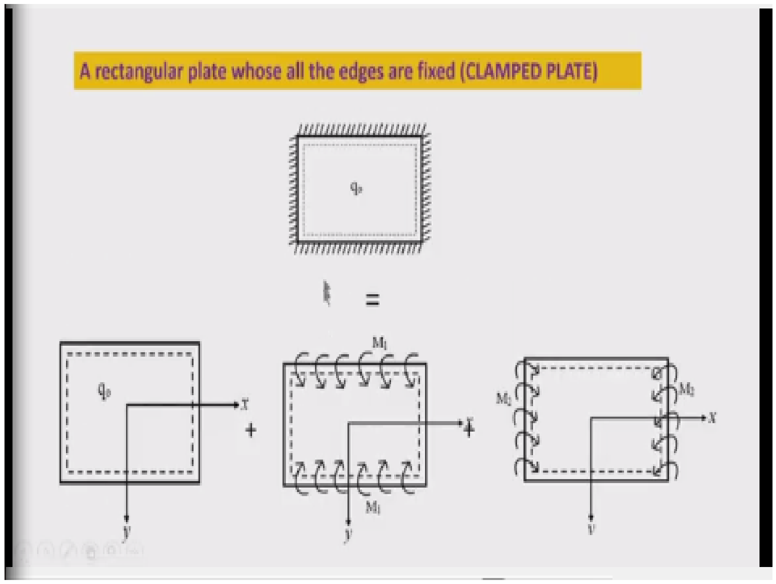

Now let us see the second case, that is the most important condition in practical purpose, the all edges are fixed in most of the cases for better rigidity and for better stability of the structure. So, here you can see that because of the clamped condition along 2 edges neither Navier condition nor Levy's condition will yield the closed form expression. So, this is decomposed into 3 parts that we need to superimpose.

One is plate with all the edges are simply supported, that is given as case 1 with q_0 is the uniformly distributed load. Then the moment applied along the edges parallel to x axis, that is y = $b/2$. Here of course you can know the differences there, because the plate is symmetrical with respect to x and y axis. So, we have taken the origin at the centre of the plate. So, with respect to origin at the centre of the plate that y = $b/2$ and y = - $b/2$, the condition that M_1 is applied here.

Thirdly, this simply supported boundary condition but $x = -a/2$, and $x = +a/2$, the moments are applied if the a is the length of the plate. So, you can see the 3 condition when superimposed will reflect the actual condition, provided the M_1 and M_2 should be evaluated in such a manner that it will make the slope at the edges along the y direction and along the x direction as appropriate maybe should be zero, in the appropriate cases it should be zero.

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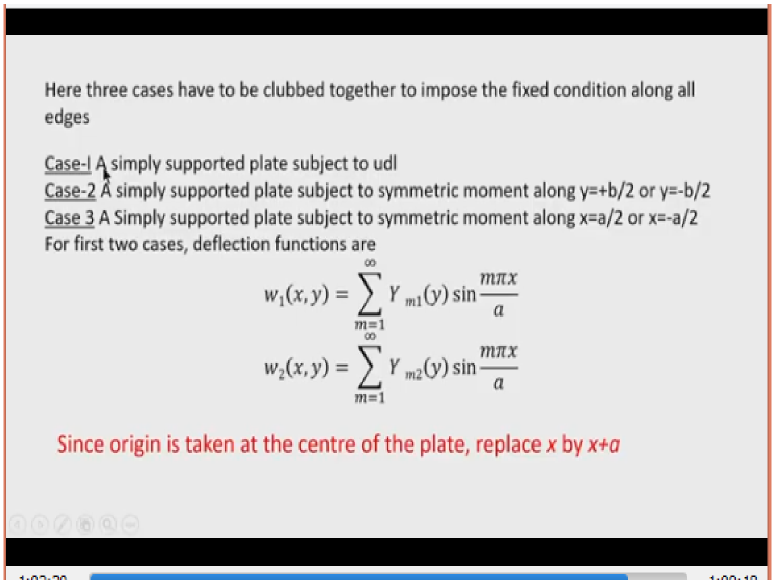

So, accordingly we divide the case into 3 types, we have divided the problem into 3 cases, a simply supported plate subjected to UDL. Case 2, a simply supported plate subject to symmetric moment along y = + *b*/2 or - *b*/2. Case 3, a simply supported plate subject to symmetric moment along *x* = *a*/2, and *x* = - *a*/2. For first 2 cases deflection functions are given, because these 2 cases require that the plate is symmetrical about the *x* axis, so that we have got.

But in the origin is taken of course in the centre of the plate, and therefore the *x* has to be replaced here by *x* + *a*, so that should be noted very carefully that *x* has to be replaced by *x* + *a*.

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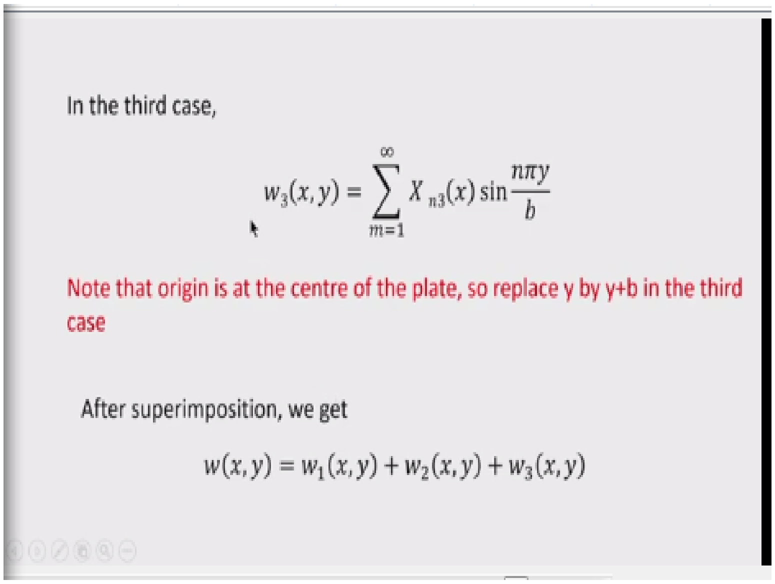

In the second case, that is the third case, in this case the third condition, that is the moments are applied along the edges, $x = 0$ and $x =$ a. For that the Levy's series can be written as $X(x) \times \sin(n\pi y/b)$. The Levy's series is now interchange that is the sine function previously it was the summation with the m that is half wave number along the x direction. Now, with respect to n, that is half wave number along the y direction.

Here origin of the plate is again at the centre, so y has to be replaced by $y + b$ in the expression. So, after superimposition of these 3 results, 3 cases, we will get the w as w_1, w_2, w_3.

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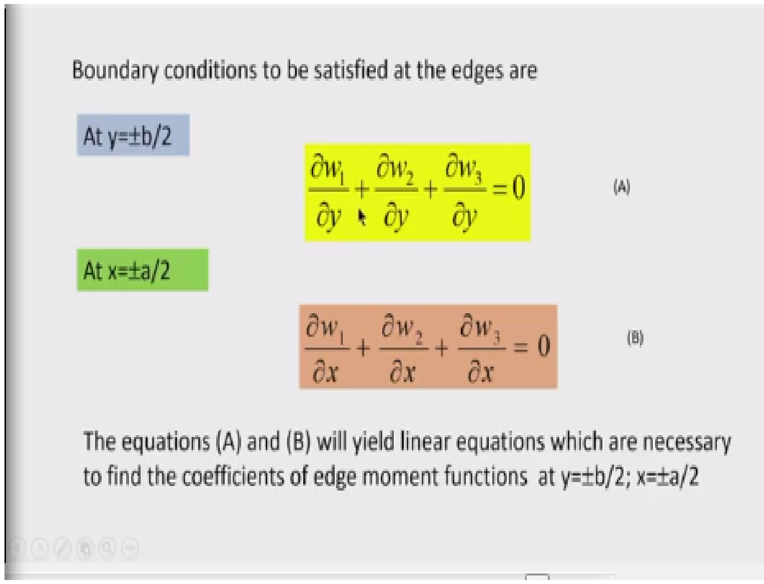

Now, the condition for fixity along all edges have to be satisfied only when at y = + or – b/2, the slope is 0, that is $\partial w_1/\partial y+\partial w_2/\partial y+\partial w_3/\partial y=0$ in y direction. Similarly, the slope in x direction should be zero, that means at the edges, so x = + or - a/2 $\partial w_1/\partial x+\partial w_2/\partial x+\partial w_3/\partial x=0$. So, we are getting 2 equations and these 2 equation will yield linear equation which involves the coefficient of the moment series.

The moment is expressed as E_1, E_2 etcetera in the Fourier series form, so we will get the coefficients E_1, E_2 etcetera and general term of the coefficient, E_m or E_n in the 2 cases. Because the moments are applied in one case along the x axis, and in another case the moment is applied along the y axis. So, you will get the 2 coefficients of this moment function and therefore the fixed end moments are determined.

Once the fixed end moments are determined, then you will be able to find the deflection completely. So, you have noted here that with the condition of edge moments that results can be known for the edge moment cases in a simply supported plate. And with the results known for the simplest supported plate with UDL, we can now analyze the plate whose all edges are clamped and subjected to UDL. So, the procedure can be applied to other types of loadings.

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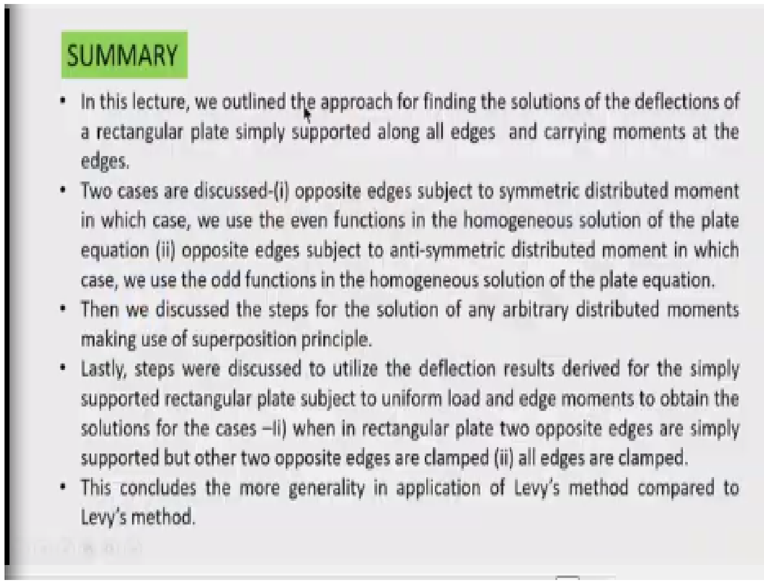

So, let us see what is the summary of this lecture that I have delivered today. In this lecture, we outlined the approach for finding the solution of the deflections of a rectangular plate. Of course, simply supported along all edges and caring moments at the edges. Say edge moment condition we have analyzed in today's class. Two cases are discussed; one is opposite edges subject to symmetric distribution of moments, in which case we use the even functions in the homogeneous solution of the plate equation.

In the second case, opposite edges subjected to anti symmetric distribution of moment, in which case we use the odd functions in the homogeneous solution of the plate equation. Then we discuss the steps for the solution of any arbitrary distributed moments, making use of superimposition principle. Lastly, steps were discussed to utilize the deflection results derived for the simply supported rectangular plate subjected to uniformly distributed load.

And edge moments to obtain the solution for the cases when the rectangular plate has 2 opposite edges as simply supported, but other 2 opposite edges are clamped and all edges are clamped. So, these conditions are analyzed, that means we actually obtain the exact solution using the Levy's condition by superimposing different results. This concludes that more generality in application of Levy's method compared to Navier's method.

So, Navier's method is only restricted to plate which has all edges simply supported. But in Levy's condition, Levy's method you will find that this condition can be relaxed by using the superimposition principle even we have gone for a plate which has all the edges clamped. But originally the Levy's equation is applied, Levy's series is applicable when 2 opposite edges are simply supported and other 2 edges maybe of any boundary conditions.

But using different results for different loading conditions, loading and edge moments we have now arrived the results of clamp plate, that is all edges are clamped using the Levy's method, which is actually the exact solution of the differential equation. So, thank you, next class we will see the other problems in the plate, thank you very much.