Plates and Shells

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Module-1
Lecture-01
Introduction, Classification of Plates and Some Useful Relations

Hello everybody, welcome to you all in massive open online course under NPTEL on plates and shells. In my introductory video, I have given different topics that have to be covered in the course plates and shell. You have seen that the course is distinctly divided into 2 parts, first is about the plate and the second part will be about the shell. First, I will cover the plate and then I will take up the chapters on shells.

In plates, I will cover 7 modules and, in the shell, I will cover 5 modules. After completion of each module, you will receive the assignment based on the lectures given in the modules and there will be a final examination also in this course. And I encourage all of you to appear the exam and you can test whether you have learnt the subject or not. Also, I will have satisfaction, if you learn the subject and able to apply in practical problems.

Now today, it is my first lecture on this subject plates and shells and first I will start the chapters on plates and this is the first lecture of the module 1. So, in this lecture, I will give you the introductory topics on the theory of plates and how the topics can be utilized to formulate the problems in plates, that will be discussed. So, the plate that I want to cover in this subject will be introduced gradually. (refer time: 02:44)

And first the introduction to theory of plate that I will cover today will consist of 5 sections that you are seeing here. So, first section we will deal with the classification of plate theories, second section we will deal with the fundamentals of plane elasticity, third section is on deflection, slope and curvature of the plate, fourth section displacement-strain relations and then stress-strain relations. So, all the topics that I have listed here are actually the background topic you can call it.

The relations and theory that will be discussed in this class will be later used in formulation of the problems in plates ok. (refer time: 03:37) Now, let us see what is the plate? You have seen that in my introductory video, I emphasized that plate is a thin structural element and it is used in various fields of engineering, civil, mechanical, aerospace, marine, everywhere you get the structural component with a thin element and this behaves as a plate. So, plate is a thin structural element whose thickness is very small compared to other dimensions.

So, if I see the major dimensions of any structure that is length, breadth and thickness, then you will see that thickness is very small compared to length and breath. Therefore, these type of structures gives advantage in reduction of weight and also it has high rigidity in flexure (04:31) as well as in axial deformations. So, it is bounded by 2 flat surfaces, the surfaces of the plate are flat before the application of the load. After application of the load, the bending will take place.

But before the application of the load initially, it is bounded by 2 flat surfaces and with straight or curved edges, plate may have straight edges or it may have curved edges. The distance between the two-flat surface, flat surface is known as thickness. So, in a plate where the 2 top and bottom surface that you are seeing here are parallel everywhere means that the thickness is constant, but there may be situation where thickness varies along the plate dimension ok.

Then a term that will be encountering in theories of plate that is known as middle surface, which is a plane parallel to the plate surfaces that bisects the thickness of the plate. So, this is the middle surface that I labelled here, you can see and this is the thickness at this point ok. The plates are found in different shapes. Most common application of the plate is rectangular plate and circular plates are also found in applications, elliptical shape in any other shape triangular also frequently encountered or encountered in some cases that we have to analyze it. (refer time: 06:20)

So, different shapes are common, ok. Now first let us discuss what are the different theories based on which we will proceed to formulate the problems on plate. Three theories are developed for studying the plate. First theory is a thin plate with small deflection, second theory is thin plate with large deflection and third theory is thick plate theory ok. Now in the first theory, the deflection of the plate is small in comparison to the thickness.

And generally, this deflection is assumed to be less than one fifth of the thickness. So, in this type of plate the bending moments, twisting moment and shearing forces are produced in response to the external loaded load. Then the second theory says that the plate is thin, but the deflection may be large, so this theory is known as thin plate with large deflection. So, what happens here let us see. In this case, deflections are not small as compared to the plate thickness.

So, that means where we define the limit here in the thin plate with small deflection this limit may be exceeded here with thin plate with large deflection. So, therefore due to large deflection, the geometric non-linearity will be produced. And as a result of a large deflection the “in plane stresses” are also produced in the plate in addition to bending moment and twisting moment. So, in plane stresses are produced.

If we alter the nature of the problem and the governing differential equations of equilibrium. The solution techniques also will be different compared to the thin plate with small deflection. Then third theory is thick plate theory, in this theory the thickness of the plate is larger and it is generally assumed that it is greater than one tenth of it is longer dimension. And therefore, due to its thickness the shear deformation contributes to the deflection.

So, thick plate theory considers the shear deformation in the deflection of the plate, whereas this thin plate with small deflection the shear deformation may be neglected and is insignificant. Similarly, in thin plate with large deflection, we need to consider the geometric non-linearity in the governing differential equation of equilibrium ok. So, in our course we will mainly focus on the thin plate with small deflection.

And we will develop different theories to solve the bending and buckling of the plate using the classical method as well as some numerical and approximate technique, ok. (refer time: 09:31) So, to study the plate one requires to know fundamentals of elasticity. So, because we consider the plate as a continuum where the properties, physical properties of the system (that is you can call it is the weight or self-weight or you can call moment of inertia distributed in this space domain). Therefore all properties including it is stiffness, flexural rigidity or axial rigidity whatever you consider in the problem are generally function of space variable.

Now in general any structural element has 3 principle dimensions and first we should know the stress system or the background of the stress and strain that is produced in a structural element in general with a three-dimensional flavour. Then from this condition or this formulation we can simplify it to the plate because in plate the thickness is considered to be negligible and therefore the two-dimensional elasticity formulation will be considered mainly.

Now let us consider a small rectangular parallelepiped as shown here. So, you are seeing a parallelepiped whose length along x direction and along the x axis is dx and along y axis is dy and along z axis is dz ok. So, volume of this small element is dx, dy, dz. Now here I have shown the stresses on the faces of the parallelepiped. So, on the 6 faces the stresses are there but I have shown here the stresses in 4 faces and other faces can be similarly constructed or written ok.

Now in general in any elastic body there is normal stress as well as shear stress. Now consider this face, so this face is parallel to y, z plane. So, in this face the x axis is normal or the x is normal to these faces parallel to x axis. In Cartesian axis x, y, z, the normal stress is denoted by σ_x and there are 2 shear stresses that are acting tangential to the plane. So, that stresses are denoted as τ_{xy}, τ_{xz}. One interesting thing you are seeing here that normal stresses are denoted by σ accompanied by a single subscript.

Whereas, the shear stresses that is the tangential stress that are denoted by 2 subscripts with a symbol τ ok. Now there is some meaning of this subscript why it is given to? Why 2 subscripts are associated with the stresses? The first subscript denotes the direction of normal to the face. So, here in this face you can see the direction of the normal is x, so therefore, first subscript x is associated here.

Second subscript denotes the direction in which the stress acts. So, here you can see the stress x along the z direction, so therefore second subscript is given as z. Similarly, here you can see τ_{xy}, x denotes the normal to the face and y denotes the direction in which the stress x. In this face we see that normal stresses is σ_y and the shear stresses are τ_{yz} and τ_{zx} ok. First subscript you can see here is y, that clearly indicates that y is the normal to this face.

So, the first subscript is written as y and second subscript z denotes the direction in which the stress acts. Similarly, here also you can see that the τ_{yx}, because y is the normal directions direction of normal and x is the direction in which the stress acts ok. There are different sign convention possible, but generally on the positive faces stresses acting along the positive direction of axis are considered positive.

While on the negative face, the stresses acting along the negative direction of the axis are considered positive. Now in engineering application customarily the tensile stress is considered positive, but there is no harden first rule one can take the sign convention in other way also. You can see one thing that in this face, we have denoted the normal stress as σ_x, the shear shear stresses are τ_{xz} and τ_{xy}.

On the opposite face, if you see the normal stress is written as $\sigma_x +$ and shear stresses as written as $\tau_{xy}+$ and $\tau_{xz}+$. Why plus sign is given? The meaning of this is that on the opposite faces, the stress will change. So, the increment or decrement whatever takes place will be denoted by another quantity, this quantity can be found from Taylor series expansion. And if you consider the first term only, then the total stress here that I written only $\sigma_x +$ will be equal to $\sigma_x + \frac{\partial \sigma_x}{\partial x}$ dx

So, similarly other stress with increment can be written. Now, here you can see that I consider the partial derivative because the variables that is stresses or whatever you consider in this three-dimensional problem are functions of xyz. So, therefore partial derivatives have to be brought into picture. (refer time: 16:40) So, normal stresses are denoted generally by σ followed in our discussion and it is also denoted or symbolized with a single subscript.

And shear stresses will be generally denoted by "**τ**" symbol and it will be accompanied by two subscripts. If you see the resultant stress in any face say considered this face. Here the stresses are normal stresses are sigma x and the shear stresses are τ_{xz} and τ_{xy}. Now if I assign the unit vectors along x, y, z axis as i, j and k which is customary. These i, j, k are unit vectors that means, magnitude of i, j, k is 1. And its directions are, direction of i is along the x direction, direction of j is along the y direction and direction of k is along the z direction. So, in terms of vector I can write $\sigma = \sigma_x i + \tau_{xy} j + \tau_{xz} k$, where i, j, k are the unit vectors along the respective Cartesian axis.

The magnitude of the stress on this face can be found by the resultant of this quantity and this is nothing but $\sqrt{\sigma_x^2 + \tau_{xy}^2 + \tau_{xz}^2}$. So, the stresses can be written or expressed in vector form also in any face or any plane also. (refer time: 18:28) Now 3D state of stress at any point on elastic body can be represented by 9 components of second order tensor with pertinent matrix.

These stresses are considered to be second order tensor and the matrix that I write will consist of 9 components in general. So, what are the components let us see? One is σ_x normal stress, τ_{xy} is the shear stress, τ_{xz} is the shear stresses which are acting on the plane that we have seen the plane is this plane, that is parallel to yz plane ok. Next τ_{yx}, σ_y, τ_{yz}, then τ_{zx}, τ_{zy} and σ_z, this shear stress is symbolized by **τ** ok.

Now you can see the normal stresses are σ_x, σ_y, σ_z and the shear stresses are τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{xz}, τ_{zy}. Now, it has been seen that due to reciprocal law in structural mechanics, that you know has been developed by Maxwell and later improved by Beti and then these stresses are found to be symmetrical. So, shear stresses are found to be symmetrical, that means

$$\tau_{xy} = \tau_{yx};\ \tau_{zx} = \tau_{xz};\ \tau_{zy} = \tau_{yz}$$

Now symmetry is due to reciprocal law of shearing stresses, and symmetry can also be established from moment equilibrium condition. So, if the symmetry is not there, the cubic element or any element will not be maintaining the moment equilibrium. That means, rotation is bound to take place ok. So, moment equilibrium is not maintained, if the shear stresses are not symmetrical. (refer time: 20:46)

Then for any body-forces, body forces are present in the system which is body forces due to self-weight, that is one primary body force. Then other type of body forces maybe your magnetic force or if the body is in motion, then inertia force. So, in presence of body forces, the equilibrium of these 3D elastic body can be established by 3 differential equations. Actually, there are 6 equations of equilibrium, 3 equations that I have written here as 1.a, 1.b, 1.c are derived from force equilibrium in x, y and z direction other 3 equations of equilibrium are from moment equilibrium.

And that we can see the symmetry of the shearing stress that is nothing but the result of moment equilibrium. So, **$\tau_{xy} = \tau_{yx}$ or $\tau_{zx} = \tau_{xz}$ or $\tau_{zy} = \tau_{yz}$**, these are actually moment equilibrium equation actually, it is obtained from moment equilibrium equation. And other 3 equation that you are seeing here 1.a, 1.b, 1.c are obtained from the equilibrium of forces. That I have shown you this parallelepiped and I have shown the stresses on each face.

Here the stress will be written fully including the incremental quantity that is $\sigma_x + \frac{\partial \sigma_x}{\partial x}$ dx like that. So, similarly here the incremental quantity on the other face for sigma y can be written as to $\sigma_y + \frac{\partial \sigma_y}{\partial y}$ dy. So, all the incremental quantity can be written and then if I sum up the forces in it is direction that is x, y, z direction, I will arrive at these 3-equilibrium equation.

So, first equation is obtained by considering the equilibrium of forces in the x direction. So, you can see the equilibrium of forces first one $\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0$, f x is the body force that is distributed in the element ok or in the body. So, all the forces or quantity σ_x, τ_{xy}, τ_{xz} even these f x are functions of x, y, z ok. So, in the second equation which is resulted from the equilibrium condition in y direction.

That is written as $\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0$, so this is second equilibrium condition. Third equilibrium condition is $\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0$, where $\frac{\partial \tau_{zx}}{\partial x}$ this is the partial derivative of τ_{zx} with respect to x plus $\frac{\partial \tau_{yz}}{\partial y}$. That means partial derivative of τ_{yz} with respect to y plus $\frac{\partial\sigma_z}{\partial z}$ + f $_z$, f_z is the body force component of the body force in the z direction. So, these are component of the body forces in the x, y, z direction.

So, 3 equations of equilibrium that I have written here are from force equilibrium condition ok. Other 3 equations of equilibrium are from moment equilibrium condition ok. (refer time: 25:15) Now the three-dimensional stresses are simplified in case of plates. So, the plates are actually is a two-dimensional case and we need to neglect, ignore some stresses because here the thickness is small compared to other dimensions. So, all stress components on the surface parallel to xy plane, this x, y, this is the x direction and in the y direction. So, xy plane is the plane of the plate.

So, here you can see that τ_{zx}, τ_{xz}, τ_{yz}, τ_{zy} and σ_z,where σ_z is the normal stress on the plate that is plane of the plate that is lying in the xy plane. So, if I consider a normal stress here σ_z is 0 in case of plate problem. And the shearing stresses that are produced here τ_{zx}, if you see τ_{zx}, z is the direction of normal. So, and x is the direction in which it acts, so this is neglected.

And similarly it is the symmetry part that is τ_{xz}, are also neglected τ_{yz} and τ_{zy} are also neglected, so these quantities are treated as 0. So, this is very important assumption or condition in case of plane problem that is plane elasticity problem. So, matrix representation becomes simpler. In case of 3D elements, we require 9 components of stress to write the matrix. Now here we will write the matrix in 2 by 2 form the elements are σ_x, σ_y, these are normal elements, diagonal elements you can see here.

And τ_{xy}, τ_{yx} are the half diagonal elements and these are the shear stress. So, due to symmetry τ_{xy} can be taken as τ_{yx} and this is normal stresses σ_x and σ_y. So, in a plate problem the stress analysis will consist of evaluation of normal stresses as well as shear stress τ_{xy}. (refer time: 27:50)

The stresses that are written in the Cartesian coordinate system x, y, z can also be represented if the frame of reference rotated. So, here say for example, the frame of reference x, y, z original frame of reference is denoted by x, y, z, x is small letter, y is also small letter, z is small letter. So, if this frame is rotated about the z axis, then let us see what will be the component of stresses ok.

So, the frame is rotated counter clockwise with an angle Θ from the original configuration. So, now new axis becomes capital X, capital Y is the new axis. There are other directions say 1 and 2, I have marked here, and this direction is denoted as α measured from the original x axis. And obviously this other direction that is written as number 2 is orthogonal to the direction 1. So, capital X and capital Y are new frame of reference. (refer time: 29:30)

So, our intention will be to find the stresses, that is transform in the new coordinate system ok. So, if you see these stresses in an element, in the original frame of reference, this is the stress sigma x and this is normal σ_x and this is shear stress τ_{xy}. Similarly, on the other face that is normal stress is σ_y and the shear stress is τ_{xy}. So, here also you can see the normal stress and shear stress are also existing ok, I want to find the stress component in the new frame of reference.

So, new frame of reference is capital X and capital Y, whereas the new X axis is inclined at an angle of Θ with the original x axis, the angle of rotation is anticlockwise. So, with that given information, I can transform the stresses in the new direction capital X and capital Y. (refer time: 30:49) So, how to do this?

There is systematic procedure if I can find the transformation matrix, transformation matrix is given as $\cos\theta \quad \sin\theta$, - $\sin\theta \quad \cos\theta$. So, it can be easily verified that transformation matrix based on the angular rotation. That is the angle in which the frame is rotated, can be established by writing the equations of equilibrium of all forces along the inclined plane.

So, that is also another way that not using the matrix method or matrix product that I will show. One can also find its transformation by establishing the equilibrium of the forces along the inclined faces. So, both the methods if you applied the result will be same. So, here the transformation matrix that I show is 2 by 2 that is cos theta sine theta, -sine theta cos theta. Using these transformation matrix the transform stress can be written as the product of 3 matrixes.

So, what are the 3 matrices? First matrix is transformation matrix T, second matrix is [σ] that is the original stress matrix, and third matrix is transpose of this transformation matrix [T^T], this superscript T denotes the transpose of this matrix. So, superscript T can be written as $\cos\theta \quad - \quad \sin\theta$, then here it will be the $\sin\theta \quad \cos\theta$. So, transpose can be done for these matrixes and the last matrix is transpose of transformation matrix.

So, if you perform the triple product, the T matrix is shown here and σ matrix is shown earlier, σ matrix is shown here as $\left[\sigma_x\ \tau_{xy}\ \tau_{xy}\ \sigma_y\right]$

So, if you perform the matrix product triple product. (refer time: 33:09) Then you will get the transform quantity. So, transform quantities are σ_X that is with reference to new axis, capital X, σ_X equal to, this is original stress that is stress with reference to original axis, So, $\sigma_X = \sigma_x cos^2\theta + \sigma_y sin^2\theta + 2\tau_{xy} sin\theta \cos cos\,\theta$

Similarly, this second transform stress that is

$$\sigma_Y = \sigma_x sin^2\theta + \sigma_y cos^2\theta - 2\tau_{xy} sin\theta \cos cos\,\theta \quad \text{and}$$

$$\tau_{XY} =- (\sigma_x - \sigma_y)\, sin\theta cos\theta + \tau_{xy}\left(cos^2\theta - sin^2\theta\right)$$

So, the stresses in the two directions are available now. The 2 direction normal stresses and shear stresses are available that is σ_X, σ_Y and τ_{xy} in the new frame of reference are now available with the help of transformation. Now in this figure, I have shown other two directions 1 and 2 with an intention that there may be direction particular direction in which there is no shear stress.

So, we are interested to find that direction. So, in order to find that direction we equate this quantity to be 0. So, equating this equation 7 that is

$$\tau_{XY} = -(\sigma_x - \sigma_y)\, sin\theta cos\theta + \tau_{xy}\left(cos^2\theta - sin^2\theta\right) = 0.$$ (refer time: 35:18)

We get $tan2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$

So, that is the value of particular angle in which there will be no shear stress. So, that direction is known as principal directions and it gives the direction of principal plane on which only the normal stresses act and there is no shear stress in the principal plane. Other, once we find the theta other direction can be found as $(\theta + \pi/2)$ and principal stresses are obtained which are the extreme stresses. So, once you find the value of Θ that means tan2Θ is found here and then we can utilize the value of Θ in this expression to find the principal stresses.

So, in that cases for particular value of $tan2\theta$ that I have shown here $2\tau_{xy}$ divided by $\sigma_x - \sigma_y$ will result 0 shear stress. So, τ_{XY} is 0 for that particular orientation. So, only these two normal stresses will exist that is σ_x *and* σ_y. And these will give you the principal stresses one will be maximum and other will be minimum. Now here question and is that the cos square theta, sine square theta is there, but the actual result or expression is obtained in terms of this $tan2\theta$. (refer time: 37:04)

So, we can convert these $tan2\theta$ as $sin2\theta$ and $cos2\theta$ simply by using the Pythagoras theorem. So, $sin2\theta$ is found to be

$$sin2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

Similarly,

$$cos2\theta = (\frac{\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

Now in earlier equation the cos square theta, sine square theta can be written using the trigonometrical identity.

Say you know that $2\cos^2\Theta = 1 + \cos 2\Theta$, and $2\sin^2\Theta = 1 - \cos 2\Theta$ and $2\sin\Theta\cos\Theta = \text{sine } 2\Theta$. So, if I use this trigonometrical identity and utilizing this expression of sine 2Θ cos 2Θ, we ultimately arrived at the principal stresses. So, principal stresses are found as

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

One thing you can note here that, 2 values are obtained from this equation, one value will give the maximum stress and another value will be given will be given as a minimum stress. So, maximum stress is called as measure principal stress and minimum stress is called as minor principal stress. The maximum shear stresses, the absolute value of maximum shear stress is found as

$\tau_{max} = \frac{1}{2}\left|\sigma_1 - \sigma_2\right|$ (refer time: 39:09)

So, one interesting thing you can see here that is

$$\sigma_X = \sigma_x cos^2\theta + \sigma_y sin^2\theta + 2\tau_{xy} sin\theta \cos cos\,\theta$$

And another expression you can see here that

$$\sigma_Y = \sigma_x sin^2\theta + \sigma_y cos^2\theta - 2\tau_{xy} sin\theta \cos cos\,\theta$$

If we add these 2 quantities, then very interesting relationship is obtained

$$\sigma_X + \sigma_Y = \sigma_x + \sigma_y$$

That is with reference to transformed axis is equal to some would be normal stresses with reference to original axis.
So, this quantity that is some of the two normal stresses that is some of the diagonal elements in the stress matrix is known as first stress invariants I_1. So, the conclusion is that the I_1 does not change, I_1 is what? The summation of the diagonal elements of the stress matrix that is the sum of the normal stresses does not change with the rotation of the axis and it remains invariance of the reference frame. So, therefore this quantity is known as stress invariant and it is known as first stress invariance. (refer time: 40:46)

So, there is other stress invariant, it can be proved that

$$\sigma_X\sigma_Y - \tau_{XY}^2 = \sigma_x\sigma_y - \tau_{xy}^2$$

So, you can note here this is with reference to the transformed axis capital X and capital Y and these quantities are with reference to original axis that is small x and small y. And again, these quantities are equal which signifies that it is this quantity is also invariant of the rotation of the axis.

So, this quantity is known as second stress invariant and it is denoted by I_2, mathematically it

$$I_2 = \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{vmatrix}$$

can be written with the determinant
So, if we expand this determinant I will get this quantity ok. (refer time: 41:49) Now let us discuss about the deflections, slope and curvature of the plate. Now in a slightly bent plate we will neglect deformation due to in plane forces because we are considering the theory of thin plate with small deflection.

And as a result, the deflection and slope both are small quantities, the slope in x direction is given as $\theta_x = \frac{\partial w}{\partial x}$, $\theta_y = \frac{\partial w}{\partial y}$. Let us find the slope in any arbitrary direction aa', aa' is this direction and this direction we call it n direction, this is our x direction, this is y direction. And length of this side is dx length of this side is dy. (refer time: 42:41)

So, the difference between the deflection in a and aa' is due to slope, in x and y direction both. So, let the difference be dw, you can see here the deflected line is shown in the x direction.

So, the difference of deflection you can note here it is nothing but this $w+\frac{\partial w}{\partial x}dx-w$ So, we can write now, the difference of deflection at a point which is due to contribution of slope due to both directions.

Here I have shown the deflected line only in x direction, but there is also deflected line in the y direction. So, considering the slope in both direction we can write the change in deflection dw is nothing but $\frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy$, so this equation is written. We are interested to find the slope along this direction that is n direction ok. So, slope and n direction that is aa$^{/}$ direction is written as $\frac{\partial w}{\partial n} = \frac{\partial w}{\partial x}\frac{dx}{dn} + \frac{\partial w}{\partial y}\frac{dy}{dn}$ So, this follows the chain rule of calculus ok. (refer time: 45:00)

From figure, now if you consider this figure where this is dx, this is dy and this hypotenuse (44:30) which you can symbolize it as dn, I have symbolized this as dn. This angle is α, so from this triangle it is obvious that

$$\frac{dx}{dn} = cos\,\alpha$$

and $$\frac{dy}{dn} = sin\,\alpha.$$

So, now this equation that we have got is in the the rule of calculus. Then we can write now $\frac{\partial w}{\partial n} = \frac{\partial w}{\partial x}$, this quantity remains as it is and here we have written $cos\,\alpha$.

Similarly, here $\frac{\partial w}{\partial y}$ and $sin\,\alpha$ is written for $\frac{dy}{dn}$. So, we clearly distinguish this function or this expression. And we can now assign an operator, this is operator who is relates to the normal direction. So, the here actually the equation number is 13 actually it is the slope along the normal direction at the edges or at the boundary. So, here this operator $\frac{d}{dn}$. can be written like $$\frac{\partial}{\partial n} = \cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}$$

So, this operator we will use in later. (refer time: 46:00)

Now let us find the direction of maximum slope in a slightly bent plane. In any arbitrary direction say 'n', this slope is given by $\frac{\partial w}{\partial n} = cos\alpha\frac{\partial w}{\partial x} + sin\alpha\frac{\partial w}{\partial y}$

For maximum value of slope, we differentiate this quantity with respect to α , and say this α is α_1. So, we find the derivative of this $\frac{\partial}{\partial \alpha}(\frac{\partial w}{\partial n})\,\Big|_{\alpha=\alpha_1} = -\sin\alpha_1\frac{\partial w}{\partial x} + \cos\alpha_1\frac{\partial w}{\partial y} = 0$ for finding the maximum value.

After simplification, you will get $\tan\alpha_1 = (\frac{\partial w}{\partial y})\Big/(\frac{\partial w}{\partial x})$. So, this is the condition for getting the maximum slope. (refer time: 47:03) For finding the 0 slope, let $\alpha = \alpha_1$, and then equating the original expression of the slope, this is the expression of the slope in any arbitrary direction. We could this expression to 0, so therefore if this direction is alpha 2, then we can find $\tan\alpha_2 = -\frac{\partial w}{\partial x}\Big/\frac{\partial w}{\partial y}$

So, at a particular angle say $\alpha = \alpha_2$ the slope value is 0. One interesting thing, you can note here that $\tan\alpha_1 = (\frac{\partial w}{\partial y})\Big/(\frac{\partial w}{\partial x})$ and $\tan\alpha_2 = -\frac{\partial w}{\partial x}\Big/\frac{\partial w}{\partial y}$

So, if I multiply $\tan\alpha_1 \times \tan\alpha_2$, the quantity will be -1. So, from this expression we can conclude that the direction of maximum and minimum slope in a slightly bent plate is mutually perpendicular. So, this is an important conclusion in our slightly or small deflection theory of the plate ok. (refer time: 48:19)

Now let us see what is the curvature? Curvature of the bend surface is numerically equal to the rate of change of slope. If the curvature is positive, the plate is sagging therefore we have the curvature in the x direction written as $\frac{1}{r_x} = -\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = -\frac{\partial^2 w}{\partial x^2}$

Similarly, curvature in y direction is written as $\frac{1}{r_y} = -\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = -\frac{\partial^2 w}{\partial y^2}$ (refer time: 49:01)

We are interested to find the curvature along n direction that is any arbitrary direction, a-a$'$. So, again we find the derivative of this quantity $\frac{\partial w}{\partial n}$ with respect to m. So, this will give the curvature along the direction a-a$'$ dash. Now using the earlier relation that we have found the operator $\frac{\partial}{\partial n}$ and this operator is written here $-\left(\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\ \frac{\partial}{\partial y}\right)$

And this slope we have also found out $\frac{\partial}{\partial n}$ which is written as $\left(\cos\alpha\frac{\partial w}{\partial x} + \sin\alpha\frac{\partial w}{\partial y}\right)$. So, if I carry out the term by term multiplication, then I will get $\frac{1}{r_n} = -\left(\cos^2\alpha\frac{\partial^2 w}{\partial x^2} + 2\sin\alpha\cos\alpha\frac{\partial^2 w}{\partial x\partial y} + \sin^2\alpha\frac{\partial^2 w}{\partial y^2}\right)$ (refer time: 50:11)

Let us introduce a term which is known as twisting curvature. So, twisting curvature is nothing but $1/r_{xy} = \partial^2 w/\partial x\,\partial y$. So, $\frac{1}{r_n} = \cos^2\alpha\frac{1}{r_x} - 2\sin\alpha\cos\alpha\frac{1}{r_{xy}} + \sin^2\alpha\frac{1}{r_y}$

To find the curvature in any direction perpendicular to a-a$'$ line, so what do we got actually earlier. This is our a-a$'$ line, so we have found the curvature along the a-a$'$ line.

Now if I want to find the curvature along another direction which is normal to a-a$'$ line say this direction is n direction and this direction is t direction. So, curvature in this direction is written as $1/r_t$.(refer time: 51:09) And to find this curvature, we simply substitute by $(\alpha+\pi/2)$.

So, substituting this $\alpha = (\alpha+\pi/2)$. in this expression, expression 19, we will get $1/r_t$, that is curvature in another orthogonal direction is $\frac{1}{r_t} = \sin^2\alpha\frac{1}{r_x} + 2\sin\alpha\cos\alpha\frac{1}{r_{xy}} + \cos^2\alpha\frac{1}{r_y}$

One interesting thing you can note here, if I add these 2 quantities that is 19 and 20, you can see this term will get cancelled, the middle term will get cancelled and $\frac{1}{r_n} + \frac{1}{r_t} = (\cos^2\alpha + \sin^2\alpha)\frac{1}{r_x} + (\cos^2\alpha + \sin^2\alpha)\frac{1}{r_y}$.(refer time: 52:25) So, this quantity is 1, so therefore we can write $\frac{1}{r_x} + \frac{1}{r_y}$. So, this expression also gives a very important conclusion, the sum of the curvature in any 2 orthogonal directions in a slightly bent plate is constant ok. Twist curvature, with reference to n and t direction can be obtained as $\frac{1}{r_{nt}} = (\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y})\left(-\sin\alpha\frac{\partial w}{\partial x} + \cos\alpha\frac{\partial w}{\partial y}\right)$

And after simplification, the final expression becomes $\frac{1}{r_{nt}} = \left(\frac{1}{r_y} - \frac{1}{r_x}\right)\frac{\sin 2\alpha}{2} + \frac{1}{r_{xy}}\cos 2\alpha$

This you can see, this is the operator. And the operator $\frac{\partial}{\partial n}$ i.e. ($\left(\cos\alpha\frac{\partial w}{\partial x} + \sin\alpha\frac{\partial w}{\partial y}\right)$)and this is $\frac{\partial w}{\partial t}$ i.e $\left(-\sin\alpha\frac{\partial w}{\partial x} + \cos\alpha\frac{\partial w}{\partial y}\right)$ ok. So, we now get the twist curvature $\frac{\partial^2 w}{\partial t \partial n}$ So, term by term multiplication again results this quantity and using the trigonometrical identity $2\cos^2\alpha = 1 + \cos 2\alpha$ and $2\sin^2\alpha = 1 - \cos 2\alpha$.

And $2\cos\alpha\sin\alpha = \sin 2\alpha$, ultimately we get the twisting curvature with reference to n and t direction as this expression, the equation number 21. So, it is nothing but $\left(\frac{1}{r_y} - \frac{1}{r_x}\right)\frac{\sin 2\alpha}{2} + \frac{1}{r_{xy}}\cos 2\alpha$ ok.(refer time: 54:34)

Let us now come to the strain displacement relationship. So, a plate here whose length is dx and width is dy is deformed to this shape.

Original this plate was A, B, C, D, now it is deformed to a', b', c' and d' dash. So, the displacement of the point b, you can see two components of displacement, we are writing here, $u + \frac{\partial u}{\partial x}dx$, this is the x component and y component is $v + \frac{\partial v}{\partial x}dx$ ok. Similarly if I show the displacement of the element the corner point c, the 2 components, one is along the x direction, it is written as $u + \frac{\partial u}{\partial y}dy$.

And another component that is in the vertical direction, it is $v+\frac{\partial v}{\partial y}dy$. The component of displacement in x and y direction of the point a$^{/}$ is u and v. So, strain in x direction can be written as the change in length in x direction as $\varepsilon_x=\frac{\left(u+\frac{\partial u}{\partial x}dx\right)-u}{dx}$. We will get this $\frac{\partial u}{\partial x}$ as the normal strain in the x direction, so this is the normal strain. Strain in y direction, that is also normal strain is similarly obtained as epsilon $\varepsilon_y=\frac{\partial v}{\partial y}$.(refer time: 56:17)

Let us obtain the shear strain. So, shear strain is calculated as the sum of the angle, this φ_2 and φ_1. So, if you see this deformation, I have separated this here in the triangle, you can see the vertical displacement that is the change of vertical displacement is $\frac{\partial v}{\partial x}dx$ in a length of dx. So, tan φ_2 will be $\frac{\frac{\partial v}{\partial x}dx}{dx}$. And since the deformation is or rotation is small, so therefore we can take tan $\varphi_2 = \varphi_2$.

So, φ_2 we have written $\frac{\frac{\partial v}{\partial x}dx}{dx}$. Similarly if I see here the component of displacement the original component of displacement of the point c, the change of displacement that is $\frac{\partial u}{\partial y}dy$ in length of dy. So, tan phi 1 if I calculate, so tan $\varphi_1 = \varphi_1$, because the φ_1 is small can be written as $\frac{\frac{\partial u}{\partial y}dy}{dy}$. So, shear strain is nothing but $\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}$.(refer time: 57:42)

Now in a strain, when we are in a three dimensional or two dimensional object, when we calculate the strain it should obey some condition, that is known as strain computability condition. So, before going to mathematical equation, first let us see what is the physical meaning of compatibility conditions? Compatibility condition, actually it is needing a continuous single value displacement.

So, in a structure, the displacement should be continuous and single value. If this is not obtained by analysis then we find that our assumption is wrong or this numerical technique or whatever solution method we adopt is wrong, so stain must satisfy the compatibility condition. Now here you can see laminar plate is taken, and it is discretized or divided into number of elements.

So, consider only 4 elements here 1, 2, 3, 4 I have marked it. And these elements are shown in undeformed states. So, you can see the elements fit together, there is no discontinuity and there is the displacements are single value, that is as corner you can get the single displacement ok. Now if the plate is strain or under the action of load and it is getting deformed.(refer time: 59:15)

Then, in the deform consideration, if the deformation is carefully done or plate is carefully strain. Then in the deformed position, we will also see that elements fits together, taking consideration of the neighbouring elements. That means there is no separation from the neighbouring elements. So, this type of deformation gives a continuous single value displacement function. (refer time: 59:53)

Now if the compatibility is lost or continuity of the deflection is lost, you will find that the individually elements are displaced without any concern for the neighbouring elements. So, there is a formation of gap and voids in the domain or in the continuum. So, that type of situation gives this does not obey the compatibility condition of the displacement. So, therefore deformed consideration, discontinuous displacement is seen here. And in the analysis, we should always see that continuous single value displacement is obtained, which is satisfied by the strain compatibility equation. (refer time: 1:00:37)

So, strain compatibility equation can be obtained very easily from the strain expression that we have derived the normal strain as well as shear strain. And compatibility conditions for strain is given as $\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$, where γ_{xy} is that shear strain ok. So, in a mechanics problem, when you solve for plate or any other structure adopting any method, numerical techniques or analytical techniques or finite element method, finite difference method. We should see that equilibrium conditions as well as compatibility condition must be satisfied ok.(refer time: 1:01:33)

Stress strain relationship for the plate. In normal direction if stress is ε_x, then it is given by $\frac{1}{E}\left(\sigma_x - \upsilon\sigma_y\right)$, where E is the modulus of elasticity and υ is the Poisson ratio of the material. Similarly, in other direction, that is y direction ε_y is given by $\frac{1}{E}\left(\sigma_y - \upsilon\sigma_x\right)$. And then shear strain $\gamma_{xy} = \frac{\tau_{xy}}{G}$, where G is the shear modulus and it is given by $G = \frac{E}{2(1+\upsilon)}$.

You may clearly note that that G is dependent on E and υ, so there are only 2 independent material constants that is modulus of elasticity and Poisson ratio.(refer time: 1:02:28) Now if I solve the equation 26 and 27, because these are actually simultaneous equation with 2 unknowns. Say if I consider sigma x and sigma y are the unknown in this equation, whereas epsilon x and epsilon y are known for example.

Then solving 26 and 27, we can write

$$\sigma_x = \frac{E}{1-\upsilon^2}\left(\varepsilon_x + \upsilon\varepsilon_y\right) \quad \text{and} \quad \sigma_y = \frac{E}{1-\upsilon^2}\left(\varepsilon_y + \upsilon\varepsilon_x\right)$$

So, these are the expression 4 stresses, normal stresses σ_x and σ_y which will be concerned in our plate problem. Because the other stresses σ_z is treated as 0 or insignificant in case of the analysis of thin plate. And shear stress τ_{xy} is now given as $\tau_{xy} = G\gamma_{xy}$. So, G is the shear modulus of the material G which is obtained from E and υ. (refer time: 1:03:43)

So, let us summarize today's lecture which is the introductory lecture, that is the introduction to the plate theory. And the background materials required for formulating the plate problem I have discussed in this lecture. So, in this lecture discussion was done on the classification of plate theories and fundamental relations of plane elasticity required to study the plate theories. Deflection, slope and curvature of the plate of slightly bent plate have been derived and conditions for maximum, minimum slope, characteristics of principal curvature of slightly bent plate were derived.

Displacement strain relations and compatibility conditions were given. Stress strain relation for plane elasticity and material constants are introduced. So, with this background material we will now proceed to formulate different problems in plate, and to solve it to find the deflection and stress resultants, thank you very much.