Advanced Soil Mechanics
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Lecture – 8

Relationship Between Stress Invariants

In the last lecture, we have discussed about invariance. We have seen the stress invariants. That means invariants of stress tensor I_1, I_2, I_3 and invariants of deviatoric stress tensor J_1, J_2, J_3. In this session, we will try to see some of the important aspects of these invariants and its relationships okay.

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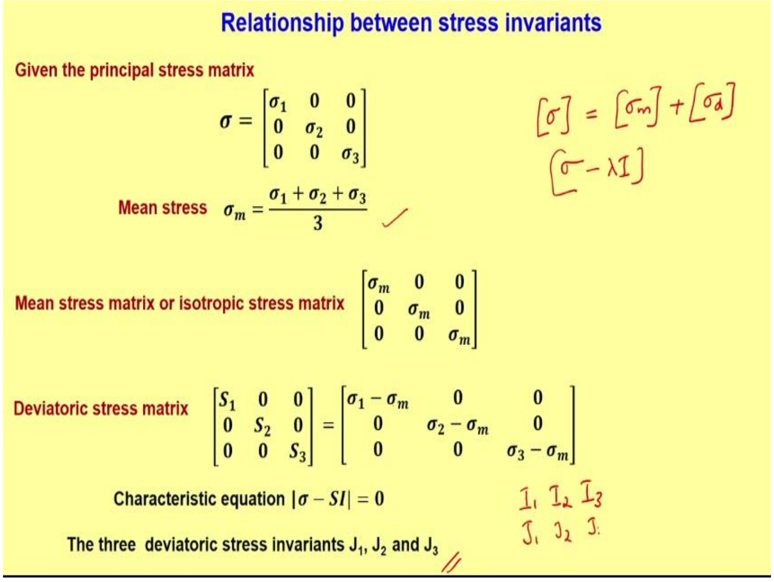

So we are now given a principal stress matrix where σ is equal to σ_1, σ_2, σ_3. We have seen that the principal stress matrix gives the same meaning and representation as like any other stress tensor in a more simplified form because we have only diagonal elements. We can easily write what is the mean stress which we know already it is $(\sigma_1 + \sigma_2 + \sigma_3)/3$.

So we can formulate the mean stress matrix or isotropic stress matrix which is given as $\begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$. We can always write deviatoric stress matrix because you know the given stress matrix is decomposed into σ_m and σ_d which can be written as $\sigma = \sigma_m + \sigma_d$. So, one can always write the deviatoric stress matrix which is given as

$$\begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 - \sigma_m & 0 & 0 \\ 0 & \sigma_2 - \sigma_m & 0 \\ 0 & 0 & \sigma_3 - \sigma_m \end{bmatrix}$$

which is this particular matrix.

So, the characteristic equation just like we have done in the previous case what was the characteristic equation? It $[\sigma - \lambda I]$ where I is the unit vector. So here characteristic equation is $[\sigma - SI] = 0$. So one can easily get J_1, J_2, J_3. So that is what we have done. So we have I_1, I_2, I_3 and then J_1, J_2, J_3.

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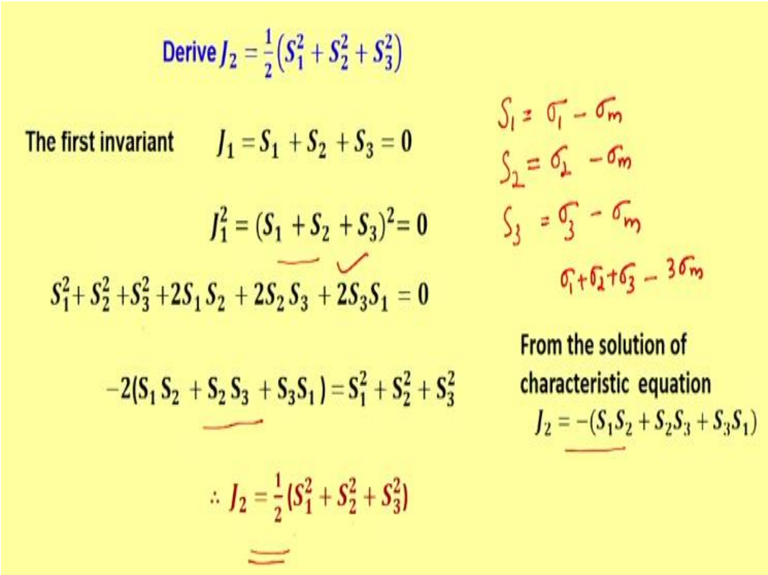

So now first we are asked to derive J2 which is the second invariant of deviatoric stress $J_2 = (S_1^2 + S_2^2 + S_3^2)/2$. So let us see how to derive this. We know the first invariant $J_1 = S_1 + S_2 + S_3$. What is S_1? $S_1 = \sigma_1 - \sigma_m$, $S_2 = \sigma_2 - \sigma_m$, $S_3 = \sigma_3 - \sigma_m$. Now if you take a summation of this you will get $\sigma_1 + \sigma_2 + \sigma_3 - 3\sigma_m$.

Now σ m = $(\sigma_1 + \sigma_2 + \sigma_3)/3$. So if you substitute it here this will be equal to 0. So that is why we know the first invariant $J_1 = S_1 + S_2 + S_3 = 0$. If you take the square of J_1 you can easily write $S_1 + S_2 + S_3$ the whole square which is equal to 0. So expand this. So once you expand this $S_1^2 + S_2^2 + S_3^2 + 2 S_1 S_2 + 2 S_2 S_3 + 2 S_1 S_3$.

So if you take 2 outside you can write –2, so if you bring S_1, S_2 , S_3 on the other side you can always write $-2 (S_1 S_2 + S_2 S_3 + S_1 S_3) = S_1^2 + S_2^2 + S_3^2$. We also know from the solution of the characteristic equation just like we had $I_2 = \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1$, we have $J_2 = -(S_1 S_2 + S_2 S_3 + S_1 S_3)$. By substituting it in the equation we finally get

$$J_2 = \frac{1}{2}(S_1^2 + S_2^2 + S_3^2)$$

So it is proved. Now these relationships become important towards the end of this lecture where you may have to use these equations for further proving certain other relationships. So please make a note of this.

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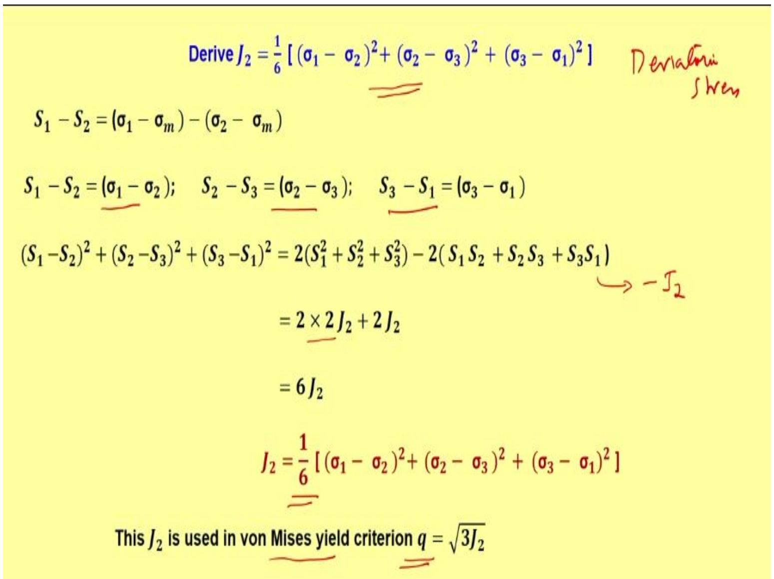

The next relationship is to derive

$$J_2 = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

You would have noticed by this time that this expression is very important because this gives us the notion of deviatoric stress. So we will see how to derive this. Let us first see what is $(S_1 - S_2)$, which is equal to $(\sigma_1 - \sigma_m) - (\sigma_2 - \sigma_m)$. So this will give us $(S_1 - S_2) = (\sigma_1 - \sigma_2)$; $(S_2 - S_3) = (\sigma_2 - \sigma_3)$; $(S_3 - S_1) = (\sigma_3 - \sigma_1)$.

So if you write $(S_1 - S_2)^2 + (S_2 - S_3)^2 + (S_3 - S_1)^2$ that will give us $2(S_1^2 + S_2^2 + S_3^2 - 2(S_1 S_2 + S_2 S_3 + S_3 S_1))$. What is this $S_1^2 + S_2^2 + S_3^2$? We have seen in the previous slide it is nothing but $2 \times J_2$ and $(S_1 S_2 + S_2 S_3 + S_3 S_1)$ is this is $-J_2$. Now on the RHS we get, $6 \times J_2$.

So you can easily write $J_2 = ((S_1 - S_2)^2 + (S_2 - S_3)^2 + (S_3 - S_1)^2)/6$, $(S_1 - S_2)$ is nothing but $(\sigma_1 - \sigma_2)$. So J_2 final expression is

$$J_2 = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

So this comes from because $(S_1 - S_2) = (\sigma_1 - \sigma_2)$. Now this equation has been used by various failure criterion or yield criterion for defining deviatoric stress.

And one we have already discussed that in von Mises yield criteria the deviatoric $q = \sqrt{3J_2}$, so that is what we have already evolved in the previous lecture.

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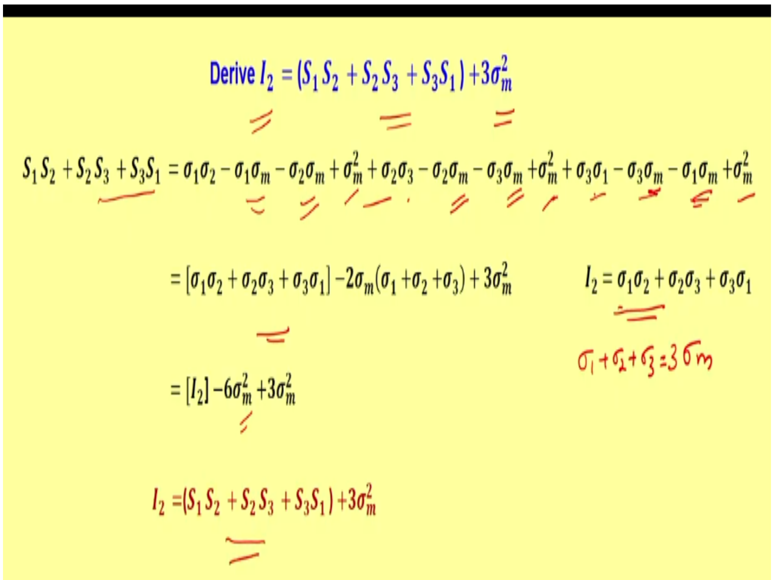

The next relationship is $I_2 = S_1 S_2 + S_2 S_3 + S_3 S_1 + 3 \sigma_m^2$. Now here it is the relationship between the second invariant of stress tensor with the deviatoric stress elements and σ_m^2. So let us see how to do this. We already know $S_1 S_2 + S_2 S_3 + S_3 S_1$. If you substitute that for S_1 is $\sigma_1 - \sigma_m$, if you substitute that and expand you will get

$S_1 S_2 + S_2 S_3 + S_3 S_1 = \sigma_1 \sigma_2 - \sigma_1 \sigma_m - \sigma_2 \sigma_m + \sigma_m^2 + \sigma_2 \sigma_3 - \sigma_2 \sigma_m - \sigma_3 \sigma_m + \sigma_m^2 + \sigma_3 \sigma_1 - \sigma_3 \sigma_m - \sigma_1 \sigma_m + \sigma_m^2$.

I request you to please work it out and see for yourself, it is just a very simple equation and rearrangement. If you simplify this you can get

$$[\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1] - 2\sigma_m(\sigma_1 + \sigma_2 + \sigma_3) + 3\sigma_m^2$$

Now we know from the characteristic equation of stress tensor Cauchy stress tensor $I_2 = \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1$

I_2 can be substituted. So this is replaced by I_2 and then what is $\sigma_1 + \sigma_2 + \sigma_3$, that can be written in terms of $3 \sigma_m$. So $\sigma_1 + \sigma_2 + \sigma_3 = 3 \sigma_m$. So if you substitute it here it will become $-6 \sigma_m^2 + 3 \sigma_m^2$. So you can write now this is $-3 \sigma_m^2$.

$$\text{So } I_2 = S_1 S_2 + S_2 S_3 + S_3 S_1 + 3 \sigma_m^2.$$

So this is also equally important, we will consider this equation further.

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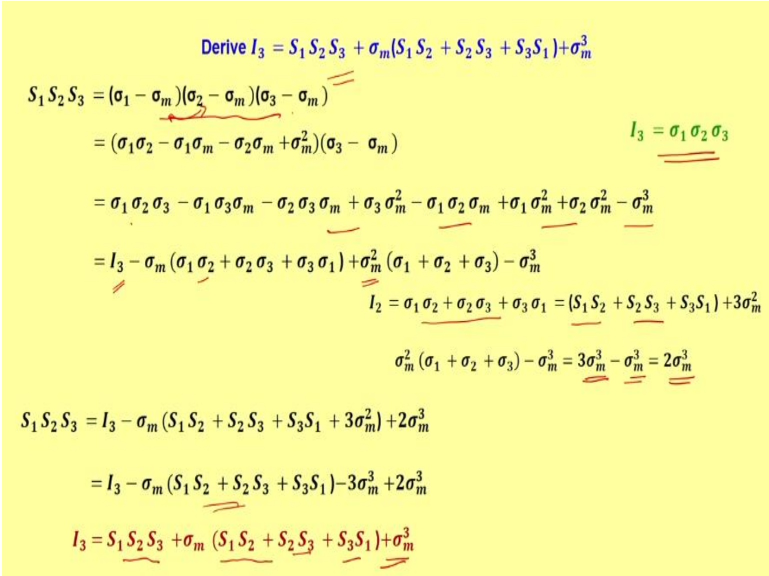

The next relationship is between the third invariant of stress tensor associated with the deviatoric stress elements. So $I_3 = S_1 S_2 S_3 + \sigma_m (S_1 S_2 + S_2 S_3 + S_3 S_1) + \sigma_m^3$. So we will first start with $S_1 S_2 S_3$ whose expression is

$$(\sigma_1 - \sigma_m)(\sigma_2 - \sigma_m)(\sigma_3 - \sigma_m)$$

So by expanding we get

$$I_3 - \sigma_m(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) + \sigma_m^2(\sigma_1 + \sigma_2 + \sigma_3) - \sigma_m^3$$

Now by replacing it by suitable expression. We know

$I_2 = \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1 = S_1 S_2 + S_2 S_3 + S_3 S_1 + \sigma_m^3$ which we have already derived in the previous slide. So this also gives σm^2.

Now the next $\sigma m^2 = \sigma_1 + \sigma_2 + \sigma_3 - \sigma m^3$ can be written as $3 \sigma m^3$ because $\sigma_1 + \sigma_2 + \sigma_3$ can be replace it by $3 \sigma m$. So $3 \sigma m \times \sigma^2 m$ gives $3 \sigma m^3 - \sigma m^3$ that is $2 \sigma m^3$. So that will be the relationship. So then you substitute back all these information, you will get

$S_1 S_2 S_3 = I_3 - \sigma_m (S_1 S_2 + S_2 S_3 + S_3 S_1 + 3 \sigma_m^2) + 2 \sigma_m^3$

$S_1 S_2 S_3 = I_3 - \sigma_m (S_1 S_2 + S_2 S_3 + S_3 S_1) - 3 \sigma_m^3 + 2 \sigma_m^3$

By rearranging the terms,

$I_3 = S_1 S_2 S_3 + \sigma_m (S_1 S_2 + S_2 S_3 + S_3 S_1) + \sigma_m^3$.

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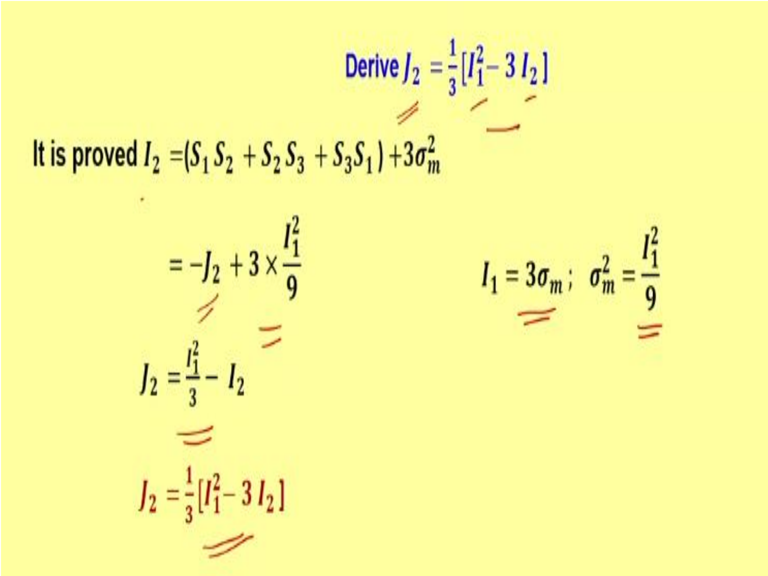

Now we have a relationship between the second invariant of deviatoric stress tensor and the first and the second invariant of Cauchy stress tensor. So let us see how to do this, it is already proved in the previous slide I2 = S_1 S_2 + S_2 S_3 + S_3S_1 + 3 σ_m^2. So it is already known to us. We also know $I1$ = 3 σ m and σ m^2 = I_1^2 /9. So it is a simple information.

What is S_1 S_2 + S_2 S_3 + S_3 S_1 that is nothing but $-J_2$ and if we substitute for σ m^2 = I_1^2 /9. So we will get – J_2 +3×(I_1^2/9). Simplifying further we get

$$J_2 = \frac{1}{3}[I_1^2 - 3I_2]$$

So this is one relationship between the both invariants.

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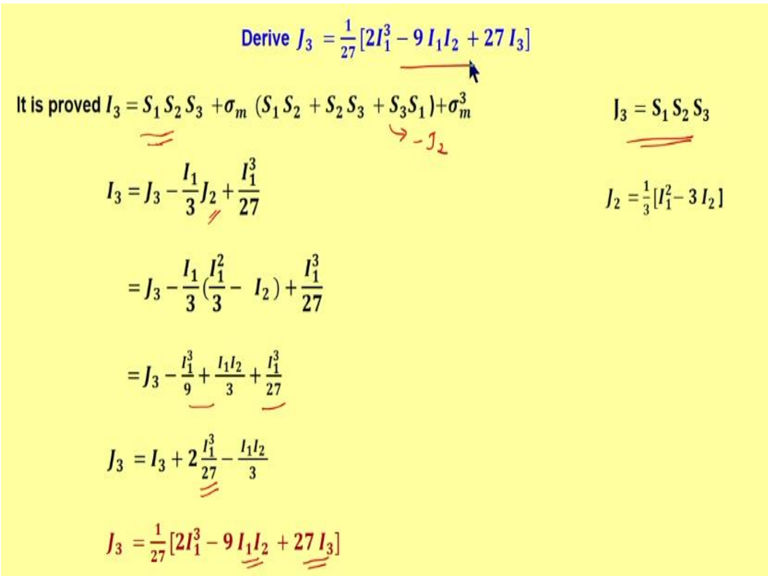

The next expression and the last one is

$$J_3 = \frac{1}{27}[2I_1^3 - 9I_1 I_2 + 27I_3]$$

Let us see where to start from. This we have already proved $I_3 = S_1 S_2 S_3 + \sigma_m (S_1 S_2 + S_2 S_3 + S_3 S_1) + \sigma_m^3$. We know that $J_3 = S_1 S_2 S_3$, $-J2 = S_1 S_2 + S_2 S_3 + S_3 S_1$ and $\sigma_m = I1/3$. So substituting we get

$$I_3 = J_3 - \frac{I_1}{3} J_2 + \frac{I_1^3}{27}$$

Now substituting J_2 expression we get

$$I_3 = J_3 - \frac{I_1}{3}[\frac{I_1^2}{3} - I_2] + \frac{I_1^3}{27}$$

Simplifying further we get

$$J_3 = \frac{1}{27}[2I_1^3 - 9I_1 I_2 + 27I_3]$$

So that is the required expression. So what we have done is we have discussed about the relationship between different invariants both for stress invariant and for deviatoric stress invariant okay. That is all.