Advanced Soil Mechanics

Prof. Sreedeep Sekharan
Department of Civil Engineering
Indian Institute of Technology – Guwahati

Lecture – 06
Transformation of Stress Tensor

In the last lecture we have discussed about stress acting on a plane as an example of what is the application of cauchy stress tensor. So in today's lecture we will see what is mean by transformation of stress tensor. Why this is important? Why this is important is we know stress tensor rather cauchy stress tensor it is a 3 by 3 matrix and there are 9 elements we have already seen that there are 6 independent stress components.

Now this corresponds to the axes σ_x, σ_y, σ_z. Now if I want to determine the stress tensor corresponding to any other orthogonal set of axes can I do that? So this is explained in today is lecture. And that is all about transformation of stress tensor or rather I can call it as transformation of coordinate axes. So one axes transformed to the other. So what is going to happen to the stress tensor the net effect of stress acting at a point is not going to change that means the stress tensor as such is not going to change, what is going to change?

The components within the stress tensor will change. Now how it will change that goes by some rule so that is what we will see in today is lecture.

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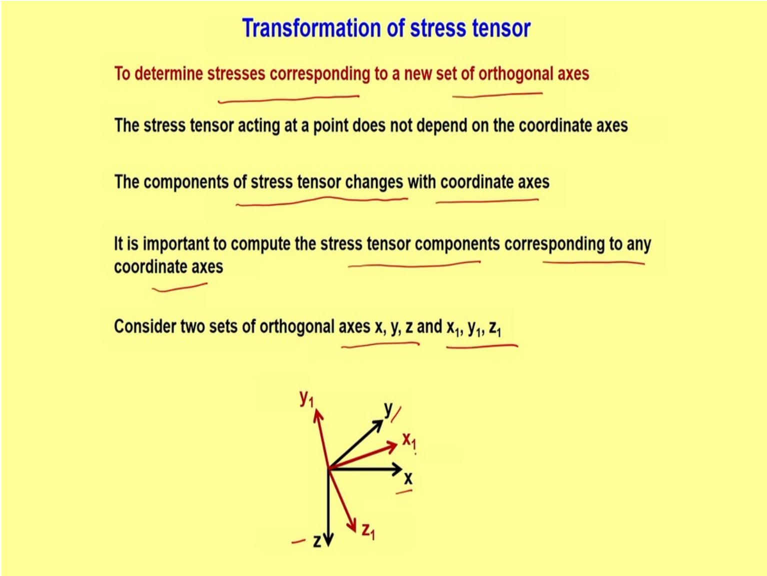

So the focus is to determine stresses corresponding to a new set of orthogonal axes, the stress tensor acting at a point does not depend on the coordinate axes I mean to say there are 2 things here stress tensor and the components within the stress tensor, stress tensor is not going to change depending upon the coordinate axes, I mean to say this some effect of some external force acting on a body represented by a stress tensor acting at a point that is the internal traction.

And the stress tensor acting at a point that is not going to change because nothing else has changed but the components would change, the components of stress tensor changes with the coordinate axes that is what we need to see today. So it is important to compute the stress tensor components corresponding to any coordinate axes you will understand the importance of this subsequently.

So there are 2 sets of orthogonal axes one is x, y, z which we already know the other one is x_1, y_1, z_1, this is what it is, so x, y and z this we already know now what will be the stress components corresponding to a new set of orthogonal axes x_1, y_1 and z_1 this is the task.

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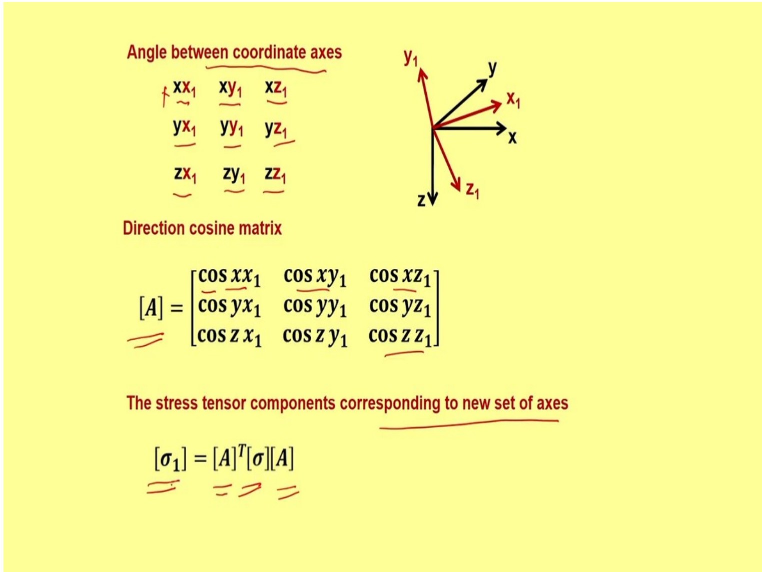

So to explain this, I will give you a clue that it has something to do with the direction cosines because one needs to be mapped on to the other. So it is quite apparent that it is the direction cosine which helps us doing this. So when we say direction cosine we should know what is the angle now that is what has been discussed here, the angle between coordinate axes need to be known so x, y, z.

So now what is the angle between x and x_1 that is given by xx_1 now what is the angle between x and y_1 that is given as xy_1 similarly xz_1, yx_1, yy_1, yz_1, zx_1, zy_1, zz_1. So all these angles are known this will be clockwise anti clockwise but the angles are known. So the direction cosine matrix can be written in this form and this is represented by matrix A. Now what is matrix A, Matrix A is the direction cosine matrix between x y z. And the new set of orthogonal axes x_1, y_1, z_1. Now this is given by cos xx_1. I have just translated from here to here the only thing is I have added cos. So cos xy_1, cos xz_1 similarly up to cos zz_1. So this is the direction cosine matrix rather the transformation matrix which will help us to transform the stress tensor components. So the stress tensor components corresponding to new set of axes is given as now what is the new stress tensor it is represented by σ_1 what is the existing stress tensor that is σ this we know.

Now the new set of axes has been given the direction cosine matrix A is defined so the stress components or the transformed components of stresses is given by

$$[\sigma_1] = [A]^T[\sigma][A]$$

So this is how we determine what is the new stress element. So by doing this transformation we will be left with new stress components which is given by σ_1.

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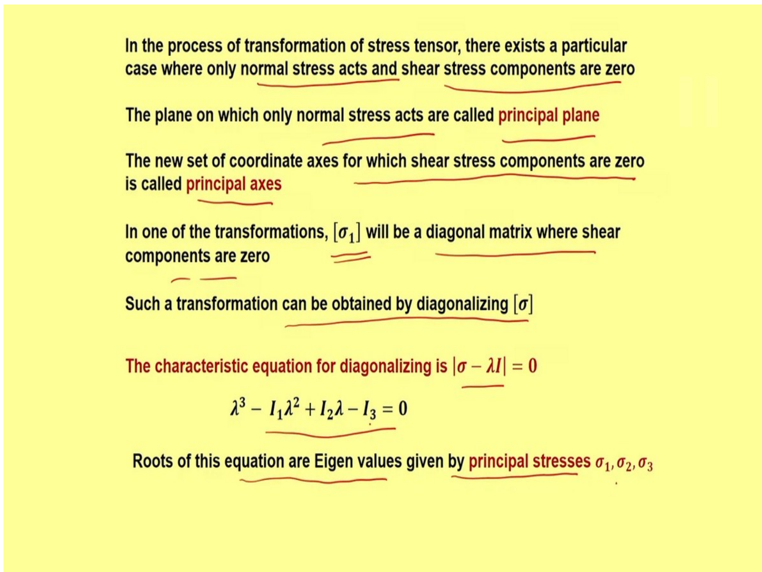

So in the process of transformation, there can be multiple transformations x_1, y_1, z_1 can be any. So there are different possibilities of transformation and accordingly the stress components in this stress tensor also would change. Now in the process of transformation of stress tensor there exists a particular case a unique case where only normal stress acts and the shear stress components are zero, probably you will get a hint of what I am trying to mean here.

We are trying to say that only the normal stresses are acting and the shear components are all zero in cauchy stress tensor we know the diagonal elements of course the stress tensor these are basically the normal stresses acting on any point whereas all other elements other than diagonal elements they are all shear components. Now shear components are zero that means it is a diagonal matrix cauchy stress tensor becomes a diagonal matrix.

So one transformation is corresponding to a case where only diagonal elements are present or only the normal stresses are present and shear stresses are zero the plane on which only normal stress acts are called principal plane this all of us know this has been taught in undergraduate. So the plane on which only the normal stress is acting is called principal plane. So the new set of coordinate axes in the process of transformation which gives stress components all diagonal elements or only normal stresses these are nothing but principal axes.

So the new set of coordinate axes for which shear competence are zero is called principal axes. So one transformation gives us a coordinate axes corresponding to principal axes or the principal stresses and any plane which represents that is called principal plane. So we have seen in one of the transformations that means σ_1 which we have just discussed that is the new set of transform competence σ_1 will be a diagonal matrix where shear competence as zero.

Such as such a transformation can be obtained by diagonalizing σ. So it is one of the same whether you use the direction cosine matrix for transformation in this particular case it is quite easy because we say we know that only diagonal elements are present. So a given matrix is diagonalized and in the process only diagonal elements exists. So these are normal stress components. So such a transformation can be obtained by diagonalizing σ matrix. And the characteristic equation for diagonalizing is

$$|\sigma - \lambda I| = 0$$

the equation takes the form as follows

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0$$

Now this is the characteristic equation for diagonalizing the given matrix and here I_1, I_2, I_3 these are parameters of cubical equation λ is just representative of σ because ultimately you are solving for σ only aspect is instead of repeating σ here I have used λ.

Now when you solve this cubical equation so there will be 3 roots, roots of this equation they are known as Eigen values and these gives the principal stresses σ_1, σ_2 and σ_3. So in the process of diagonalizing we use characteristic equation it is a very common method of diagonalizing the matrix $|\sigma - \lambda I| = 0$. Now solving this it results in a cubical equation and solving this equation we get 3 roots of the equation these are called Eigen values.

And the Eigen values represents the principal stresses σ_1, σ_2, σ_3. So in essence what will be the transformed stress tensor so it will be

$$\begin{bmatrix} \sigma 1 & 0 & 0 \\ 0 & \sigma 2 & 0 \\ 0 & 0 & \sigma 3 \end{bmatrix}$$

so that will be the transformation that take place. Now coming to parameters I_1, I_2, I_3. What are these expressions for parameters I am not deriving this rather the expressions are given?

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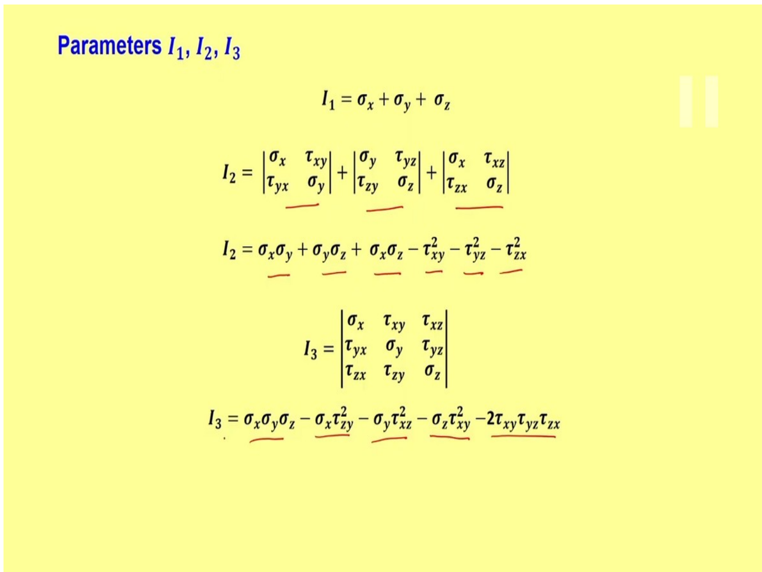

So the first parameter $I_1 = \sigma_x + \sigma_y + \sigma_z$ is the summation of the diagonal elements of the original stress matrix I_2 is expressed in this form. So the determinant you can see the summation of the determinants which will give $\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_x \sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$. So this is I_2 and I_3 is determinant of the same matrix which will give the expression

$$= \sigma_x \sigma_y \sigma_z - \sigma_x \tau_{zy}^2 - \sigma_y \tau_{xz}^2 - \sigma_z \tau_{xy}^2 - 2\tau_{xy}\tau_{yz}\tau_{zx}$$

So that these are the parameters I_1 I_2 and I_3 in terms of the original stress tensor okay. So I_1 is explained I_2 is explained and I_3 is explained and there are some meaning associated with these which will be discussed later.

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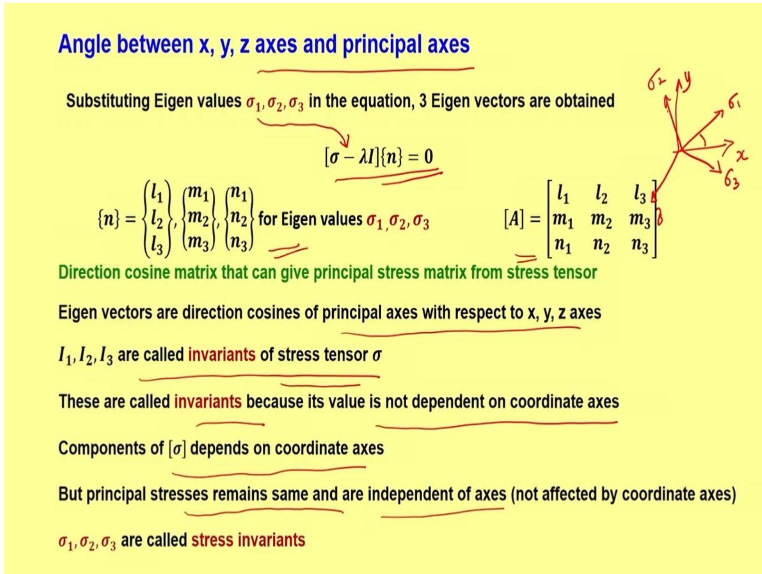

Now what we have done we have diagonalize the given stress tensor for what for finding out principal stresses and that is nothing but a process of transformation. Now we have 2 sets of axes one is the original x, y, and z again the next set is σ_1, σ_2, and σ_3 you remember where we started with now actually according to our earlier discussion σ_{1x} this angle all the angles should be known and the A matrix is formed.

And then we go for transformation but here we have done it straight forward diagonalization of the given stress tensor. Now but this is also equally important what is the angle of inclination between x and the new set of axes which is σ_1, σ_2, σ_3 the principal axes what is the angle between. So angle between x y z axes and principal axes how to find that? Now we know from the diagonalization or based on the characteristic solution of the characteristic equation we know we have obtained the Eigen value σ_1, σ_2, σ_3.

Now if we substitute these Eigen values in the equation

$$|\sigma - \lambda I|\{n\} = 0$$

by solving this equation what we will be left out with because we substitute σ_1 in this equation first solve this equation we will get 3 roots again. That means n_1, n_2, n_3 which is represented by l_1, l_2, l_3. So substitute σ_1 solve this equation we will get an n vector that n vector corresponds to l_1, l_2, l_3 in the same manner substitute σ_2 in this equation, σ_3 in this equation will get m_1, m_2, m_3, n_1, n_2, n_3.

And these are Eigen vectors for Eigen values σ_1, σ_2, and σ_3. So this n represents the Eigen vectors. So the direction cosine matrix that can give principal stress metrics from stress tensor can be written in this manner. So it is one of the same thing. So the transformation matrix A can be written in this form l_1, l_2, l_3, m_1, m_2, m_3, n_1, n_2, n_3 which we get from the solution of this equation and by knowing the Eigen vectors.

So Eigen vectors are direction cosines of principal axes with respect to xyz axes where I_1, I_2, I_3 are called invariants of stress tensor. Now I told that the parameters I_1, I_2, I_3 has got some significance now what is the significance the significance is I_1, I_2, I_3 these are called invariants of stress tensor σ. Now what is invariants? These are called invariants because its value is not dependent on coordinate axes for that matter σ_1, σ_2, σ_3 is also now independent of any axes it corresponds to only principal axes.

And here I_1, I_2, I_3 which are the parameters of the characteristic equation these are called invariants of stress tensor σ please make a note it is specified as invariance of stress tensor σ because later we will have another set of invariants these are called invariants why because the value of I_1, I_2, I_3 these are not dependent on coordinate axes, components of σ depends on coordinate axes.

We have already told several number of times why I am repeating this is that particular aspect need to be very clear the components of cauchy stress tensor which is in terms of x y z these are dependent on coordinate axes but principal stresses remain same whatever be the kind of stress tensor in terms of x y z the principal stresses are going to be the same and they are independent of axes not affected by coordinate axes. Hence σ_1, σ_2, and σ_3 can also be called stress invariance. So we have invariants of stress tensor sigma I_1, I_2, I_3 and we have stress invariants σ_1, σ_2, σ_3.

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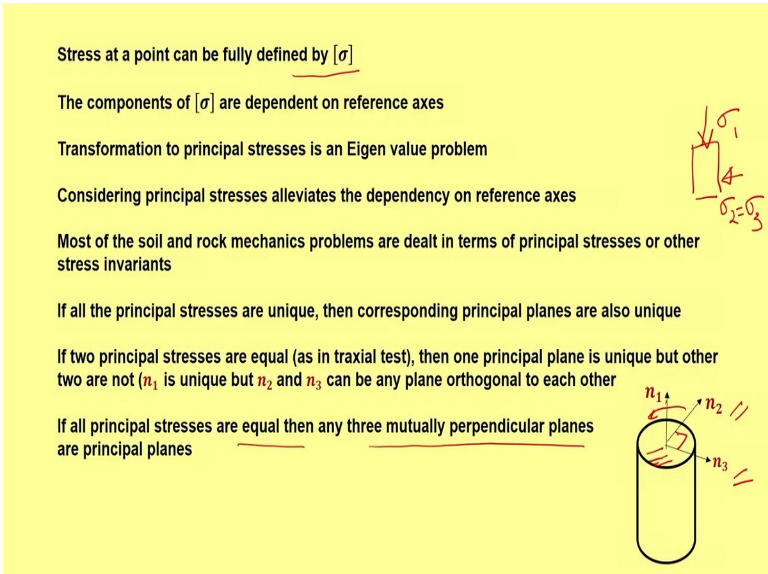

So stress at a point can be fully defined by this cauchy stress tensor σ. The components of σ are dependent on reference axes transformation to principal stresses is an Eigen value problem we have already seen that considering principal stresses alleviates the dependency on reference axes now probably you will understand why most of the problems in soil mechanics geo mechanics or rock mechanics they are all in terms of principal stresses.

Because you need to handle only 3 stresses and other stresses are 0. But this these 3 stresses σ_1, σ_2, σ_3 represents the same aspect of what a normal cauchy stress tensor presence that is the convenience most of the soil and rock mechanics problems they are dealt in terms of principal stresses or any other stress invariants because I_1, I_2, I_3 they are also invariants. Similarly, there are several other definitions for invariants which you can see in the literature.

So, any of these invariants are used in defining the failure criterion. Failure criterion is very much essential for solving any sort of mechanical problems in geo mechanics or any other mechanics. Most of the soil and rock mechanics problem they are dealt in terms of principal stresses. So if all the principal stresses are unique then corresponding principle planes are also unique please understand the sentence a but carefully.

If all the principal stresses are unique that means I have σ_1, σ_2, and σ_3 they are all different if that is the case the principal planes also will be unique this is the normal case that we discuss. Now the next case is if 2 principal stresses are equal. Now this is a very typical case in soil mechanics in traxial testing where we have a traxial sample this is σ_1 and the confining stress is σ_2 equal to σ_3.

So that is how we take. So if 2 principal stresses are equal as in traxial tests then one principal plane is unique but other two are not. So this will be made clear now, consider this cylindrical figure you have n_1 which is the normal to this plane and n_2 and n_3. So these are the 3 planes on which are the these are the 3 axes corresponding to a given plane based on which we are explaining this. So if 2 principal stresses are equal if σ_2 equal to σ_3 which corresponds to the n_2 n_3.

That is the plane is always defined in terms of normal so that is why when I say plane the normal has been introduced. So, n_1 is a plane corresponding to this so σ_1. So in this particular case σ_1 is unique whereas σ_2 equal to σ_3 what it means is that the plane n_1 plane n_1 means the plane for which the normal is n_1. So the plane n_1 is unique but n_2 and n_3 these can be any plane which is orthogonal to each other only thing is this orthogonality has to be maintain it can be any plane in this which is orthogonal to each other.

So when σ_2 equal to σ_3 or when any 2 principal stresses are equal then you will have only one unique plane the other 2 can be any orthogonal plane if all principal stresses are equal that means $\sigma_1 = \sigma_2 = \sigma_3$ is a typical case of a mean stress that we will be explaining a bit later if all principal stresses are equal then any 3 mutually perpendicular planes are principle planes there is no uniqueness in that So all are equal. So any set of orthogonal planes mutually perpendicular planes are principal planes.

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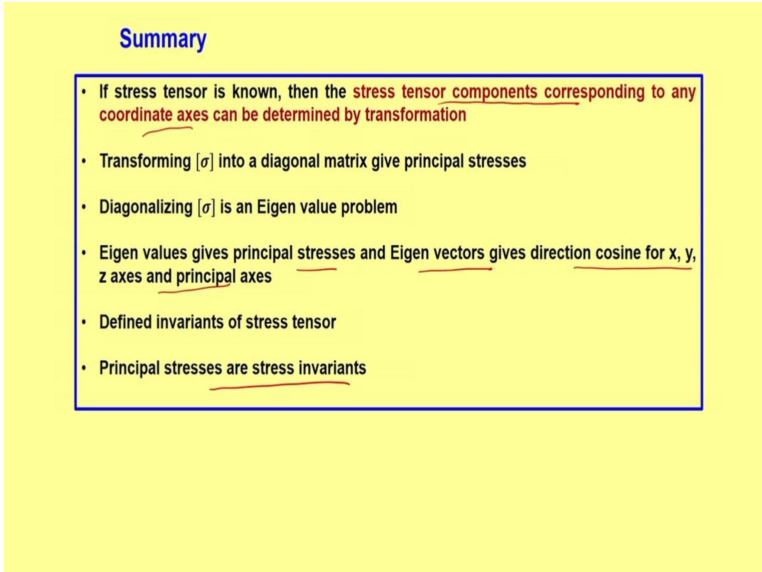

So the summary of this particular part is if stress tensor is known let us say cauchy stress tensor then the stress tensor components corresponding to any coordinate axes can be determined by the process of transformation when we know the transformation matrix which is the direction cosine matrix one can transform the components here please understand it is components stress tensor components gets transformed.

Transforming σ into a diagonal matrix give principal stresses that means diagonalizing σ gives principal stresses diagonalizing σ is an Eigen value problem Eigen values gives principal stresses that means when you solve for it you get the Eigen values which are nothing but σ_1, σ_2, σ_3 these are principal stresses and substituting Eigen values are the principal stresses in the equation we get Eigen vectors.

And that gives the direction cosine for x y z and principal axes we have also defined invariants of stress tensor and this stress invariants σ_1, σ_2, σ_3 principal stresses are stress invariants. So that is all for our transformation of stress stresses or stress components or transformation of coordinates and what is the importance of transformation of coordinates?