Advanced Soil Mechanics
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Lecture-53
Strain From MCCM

Welcome back all of you, in the last lecture we have completed prediction of stress states based on modified Cam Clay model. In that lecture we have seen how handy the information of yield curve is for interpreting, how the stress path moves beyond yielding and what will be it is implication in the volume change behavior? That is v ln p' plot; it has added more clarity we have seen.

We have also discussed about determination of soil stiffness from MCCM. Now stiffness means, it basically relates to the elastic behaviour. So, we have completed more or less these stress states, the stiffness. Now there is one more component which is left out which is strain. Now the question is, can we determine strain by knowing MCCM or by adopting MCCM? Now strain is an integral part when it comes to the defining the constitutive relationship.

That is a stress strain relationship, and when you consider a plastic model like MCCM we need to have both the ranges that is elastic as well as the plastic range after yielding, both are important. Now unfortunately we will not be touching upon the constitutive modelling using MCCM. But we will try to understand whether we can determine strain by considering MCCM model, and that is what we will see in today's lecture.

And when I say strain, there are 2 strains because we are considering everything in q p' plot, so deviator stress and mean stress. Hence we also have deviatoric strain and volumetric strain. We have also seen the yield curve, the plastic potential, plastic strain increment vector where it is defined in terms of ϵ_q and $\epsilon_{p'}$. So, when I say determination of strain, it pertains to ϵ_q and $\epsilon_{p'}$, that is deviatoric and volumetric strain.

Now this strain has got 2 components, one is the elastic part, the other one is the plastic part. Now if you are considering elastic part, it is quite easy. We just need to know only a single information based on which for different stress condition the response we can predict, because it is a linear response, but the moment it yields it becomes nonlinear. Now we cannot just simply substitute the value and get the result, rather since it is a nonlinear behaviour it has to be in an incremental form.

So, small, small increments, both for stiffness as well as for strain determination, so that is very important. So, whatever we are going to study in today's lecture is like incremental strain determination**.**

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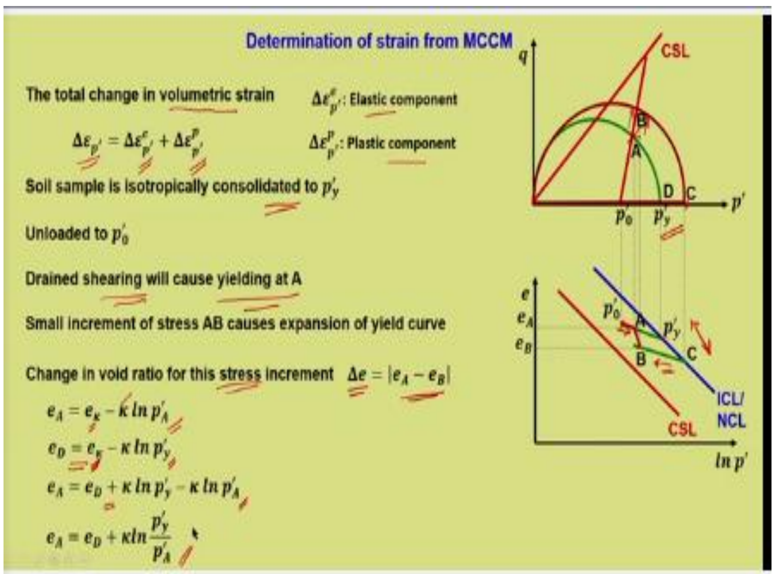

So, let us move on to the subject that is determination of strain from MCCM. So, as I told there are 2 parts, one is the volumetric strain and the other one is the deviatoric strain. So, the total change in volumetric strain can be represented as $\Delta\epsilon_{p\prime}$

$$\Delta\epsilon'_p = \Delta\epsilon^e_{p\prime} + \Delta\epsilon^p_{p\prime}$$

$\Delta\epsilon^e_{p\prime}$ is the elastic part, and $\Delta\epsilon^p_{p\prime}$ is the plastic part.

So, this is the elastic component, and this is the plastic component, summation of 2 gives the total change in volumetric strain. Soil sample is isotropically consolidated to p'_y that is our starting sentence, it is unloaded to p'_o a typical LOC. So, q p' critical state line, in the last lecture also when we discussed about stiffness, we told that either void ratio or specific volume either of the one will do.

So, here also for convenience I am considering void ratio because that is what we are familiar about, what specific volume we do not use it that frequently, what we use is void ratio. So, since we are discussing about stiffness in the last lecture and strain in today's lecture I would like to go ahead with the void ratio but everything other than that remains same. Now this is the yield curve which is related to p_y' , so the maximum yield stress is p_y' and it is unloaded to p_o'.

Drained shearing, because when we say volumetric change it is drained response. So, the drained shearing will cause yielding at A, so this is the point A, so this is the point where it yields. Small increment of stress AB, now what goes beyond is the expansion of yield curve and that is denoted by C. So, a small increment, we are talking about a small increment of stress, so that we are determining a small increment of strain as I told in the beginning.

So, it is all in incremental form beyond yielding, so that is the point B. So, let us take it downwards, so this is p_y' which is isotropically consolidated, then there is unloading to p_o'**.** So, this point is the unloaded point then the point where it yields is given by A, so up to yield p_o' to A, we know that it will be on the reloading line. So, then the expanded yield curve the moment it crosses the point A, then it becomes expanded yield curve C so that is again plotted as C in e ln p' plot, we also have an unloading line at C.

For this particular condition this is not important but you will see why we have drawn this unloading line. So, this point B ideally can be represented on the unloading line of C. Now you may question like why it is not there on the unloading line of p_y' p_o'. Now you need to see here there is a plastic hardening that has taken place when the ESP moves from A to B. So, a plastic hardening has already taken place means the compression has happened.

That compression is given by this, so this part p_y' to C it has already happened; now it is on the C unloading line. So, that unloading line is marked here, so that is why B will lie on this particular unloading line. So, this is represented p_o' A is from p_o' A then as it moves from A to B that hardening is shown here, that is AB. So, the void ratio corresponding to A is given as e_A, void ratio corresponding to B is given as e_B.

Now e_A to e_B is a plastic response that is plastic volumetric strain, and p_0' to A is an elastic strain. So, change in void ratio for this stress increment, that is when the stress changes from A to B is given as $\Delta e = |e_A - e_b|$ Now I can ideally we have to write e_A - e_B because e_A is higher than e_B. But it hardly matters you can write e_B - e_A also provided you are putting modulus, because we are not interested in the negative sign. So, e modulus of $e_A - e_b$ in general is the change in void ratio when there is an expansion of yield curve from A to B. So, now we know that it is MCCM, hence we know all the relationships related to critical state. So, we can conveniently write

$$e_A = e_\kappa - \kappa \ln p_A'$$

so it is on this unloading-reloading line and hence the void ratio at A can be obtained by this expression by knowing the critical state parameter κ. For D, that is for this point which is the maximum yield stress corresponding to the starting point or the starting envelope, that can be obtained as e_D. It again it is on the same unloading-reloading line, so it is

$$e_D = e_\kappa - \kappa \ln p_y'$$

because D corresponding to the maximum that is p_y'. So, this particular point what we are talking about, that is what is the void ratio at this? This particular point, so that is given by e_D. Now for e_κ we can substitute, because this lie on the same unloading-reloading line.

$$e_A = e_D + \kappa \ln p_y' - \kappa \ln p_A'$$

$$e_A = e_D + \kappa \ln \frac{p_y'}{p_A'}$$

Similarly you can also get the expression for e_B.

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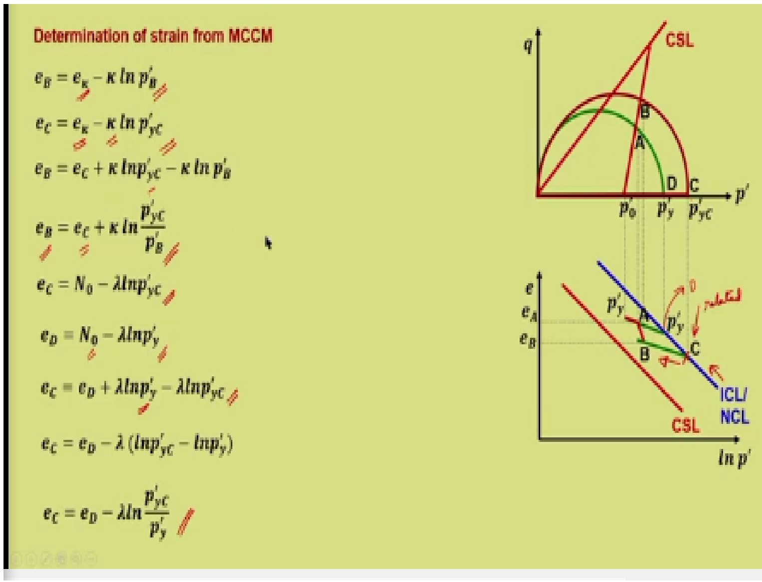

Now same way we have

$$e_B = e_\kappa - \kappa \ln p'_a \ ,$$

Please remember even though I have written e κ here this would change, why? Now the point B is on another unloading-reloading line and this point we have specified when we discussed about the critical state parameters. Like e_κ or v_κ keeps changing depending upon the point where it is unloaded. So, accordingly this e_κ is different from the previous e_κ.

But since this e_κ is not going to come into the equation, I have not used a separate terminology. So, but then you need to understand we are now discussing about the line CB, so accordingly $e_B = e_\kappa - \kappa \ln p'_a$, just like what we have got for e_A. And this point e_c can be written as

$$e_C = e_\kappa - \kappa \ln p'_{yC}$$

So, e_κ can be substituted.

$$e_B = e_C + \kappa \ln p'_{yC} - \kappa \ln p'_B$$

.So, just like you have written for e_A, we can write

$$e_B = e_C + \kappa \ln \frac{p'_{yC}}{p'_B}$$

Now for e_A we have e_D, and for e_B we have e_C, and we also know that this point D and this point is related, how it is related? Because it is falling on the ICL line or NCL line, accordingly point C that is e_C can be written as N_o. N_o means it is corresponding to unit pressure

$$e_C = N_o - \lambda \ln p'_{yC}$$

λ is the slope of NCL. And e_D can be written as

$$e_D = N_o - \lambda \ln p'_y$$

If that is the case you can substitute for N_o, so

$$e_C = e_D + \lambda \ln p'_y - \lambda \ln p'_{yC}$$

$$e_C = e_D - \lambda(\ln p'_{yC} - \ln p'_y)$$

$$e_C = e_D - \lambda \ln \frac{p'_{yC}}{p'_y}$$

So, this is the expression for e_C in terms of e_D , so, you can substitute for e_C in e_A - e_B.

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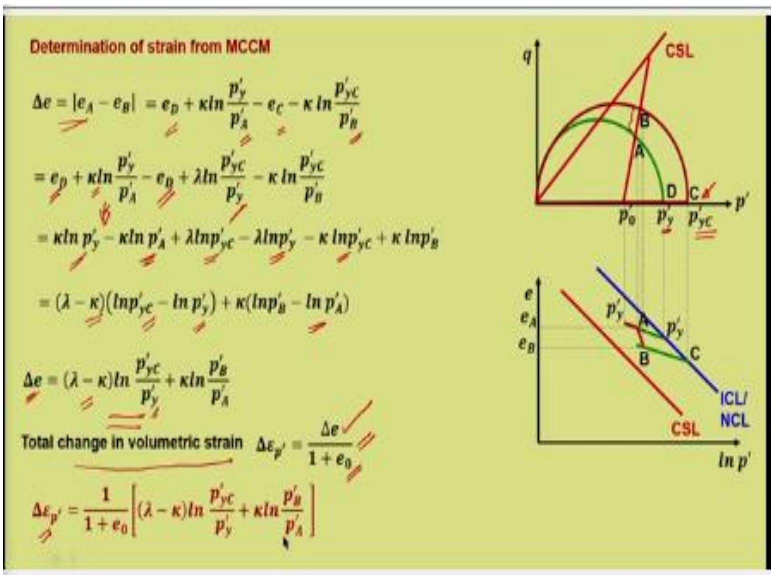

So, now our ultimate aim is to get the total volumetric strain Δe. So, we have now obtained the expression for e_A, we have obtained the expression for e_B, we have also obtained the expression for e_C in terms of e_D, so one will get eliminated. So, we know that

$$\Delta e = |e_A - e_B| = e_D + \kappa \ln \frac{p'_{yC}}{p'_y} - kln \frac{p'_{yC}}{p'_B}$$

By substituting for equations $e_A \; and \; e_B$. If that is the case

$$\Delta e = e_D + \kappa \ln \frac{p'_y}{p'_A} - e_D + ln \frac{\lambda p'_{yC}}{p'_y} - \kappa ln \frac{p'_{yC}}{p'_B}$$

Now if we expand we can write

$$\Delta e = \kappa ln p'_{yC} - \kappa \ln p'_A + \lambda ln p'_{yC} - \lambda ln p'_y - \kappa \ln p'_{yC} + \kappa \ln p'_B$$

Now we can rearrange the terms we get

$$\Delta e = (\lambda - \kappa)(\ln p'_{yC} - \ln p'_{y}) + \kappa(\ln p'_{B} - \ln p'_{A})$$

So, Δe can be written as

$$\Delta e = (\lambda - \kappa) \ln \frac{p'_{yC}}{p'_{Y}} + \kappa \ln \frac{p'_{B}}{p'_{A}}$$

So, what is actually these points, let us see, $\lambda - \kappa$ into ln, that is the expanded yield curve, we are trying to find out this incremental strain. So, total volumetric strain which it has undergone when it moves from when the ESP expands from A to B. So, for that what is the maximum yield stress corresponding to expanded yield curve, that is p'_{yC}.

So, ln p'_{yC} that is a maximum, divided by the maximum yield stress of the initial yield curve, so that is the ratio. So, $\ln \frac{p'_{yC}}{p'_{Y}}$ into $\lambda - \kappa$ plus the elastic portion that is κ into $\ln \frac{p'_{B}}{p'_{A}}$, so that is the expanded point that is B upon A. So, that will give us the total change in volumetric change in that is represented by void ratio, so Δ e we have obtained.

So, we can very well define the total change in volumetric strain as $\Delta\epsilon_{p'}$, that is the total change in volumetric strain that is Δ. So, Δ is important incremental, please keep this in mind. So, $\Delta\epsilon_{p'}$ is equal to we know that it is change in void ratio upon initial void ratio. So, that is initial volume not initial void ratio initial volume, so change in volume is represented by

$$\Delta\epsilon'_{p} = \frac{\Delta e}{1 + e_o}$$

Now we have obtained the expression for Δe. So, we can write the total change in volumetric strain

$$\Delta e'_{p} = \frac{1}{1 + e_o}\left[(\lambda - \kappa) \ln \frac{p'_{yC}}{p'_{y}} + \kappa \ln \frac{p'_{B}}{p'_{A}}\right]$$

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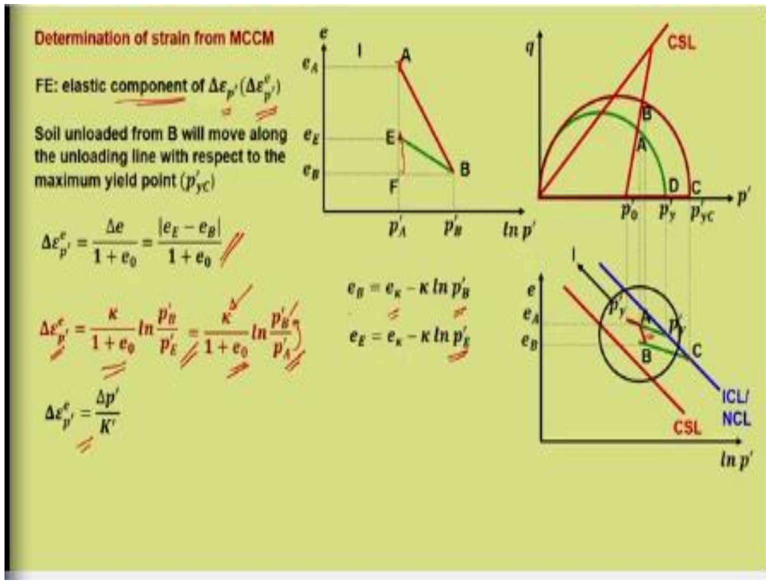

So, that is about the total change in volumetric strain, we will just highlight. Now what we want? We are not interested in total volumetric strain rather we want the components both the elastic volumetric strain and the plastic volumetric strain. So, we will see how to get that? Before that I will show the expanded view of this particular circle which is denoted as I. So, e ln p' we are just focusing on the part I.

So, AB which is this is represented by this line, and there is unloading that is happening. So, if you consider this e_A e_B is the void ratio and the points are e_F, now what is this unloading? Like if you consider with reference to point A, we can say that this is the void ratio that it got regained, when the load is released what happens is this much of the strain or this much of the void ratio it regained, the soil regain, so that is nothing but the elastic component.

So, p'_A, p'_B the pressure FE this particular distance, that is FE is the elastic component of $\Delta\epsilon_{p\prime}$ and that is represented by $\Delta\epsilon^{e}_{p\prime}$. Now soil unloaded from B will move along the unloading line with respect to the maximum yield point p'_{yC} and that is what I have already explained. Now you are unloading it from B means that corresponds to the unloading-reloading line at corresponding to point C.

There is a maximum yield stress is p'_{yC} . So, $\Delta\epsilon^{e}_{p'}$ can be written as

$$\Delta\epsilon_p^e = \frac{\Delta e}{1 + e_o} = \frac{|e_E - e_B|}{1 + e_o}$$

So, this will give the elastic component, because this much is the elastic volume change or regaining. So, we know that $e_B = e_\kappa - \kappa \, ln \, p'_B$, the way we have done it before and $e \, E = e_\kappa - \kappa \, ln \, p'_E$. So, if you rearrange this and you substitute for $e_E - e_B$ what we are going to get is this

$$\Delta\epsilon_{p'}^e = \frac{\kappa}{1 + e_o} \ln \frac{p'_B}{p'_E} =$$

That is $\epsilon_{p'}^e$ is the required elastic volumetric strain increment. So, volumetric strain increment the elastic component is $\epsilon_{p'}^e$ We can also write it

$$\Delta\epsilon_{p'}^e = \frac{\kappa}{1 + e_o} \ln \frac{p'_B}{p'_A}$$

because p'_E and p'_A both are same. So, you will get now this is the reference point when the ESP moves from A to B what is the elastic component of volumetric strain? That is $\Delta\epsilon_{p'}^e$ = κ is the critical state parameter that represents the unloading-reloading line divided by 1 + e_o, 1 + e_o is nothing but the initial volume, ln which is the final point p'_B which is the initial point p'_A on the yield curve, I mean to say from yield curve to yield curve. We also know

$$\Delta\epsilon_{p'}^e = \frac{\Delta p'}{K'}$$

because this is an elastic parameter, it is a linear response, p' which is the stress upon the bulk modulus K'.

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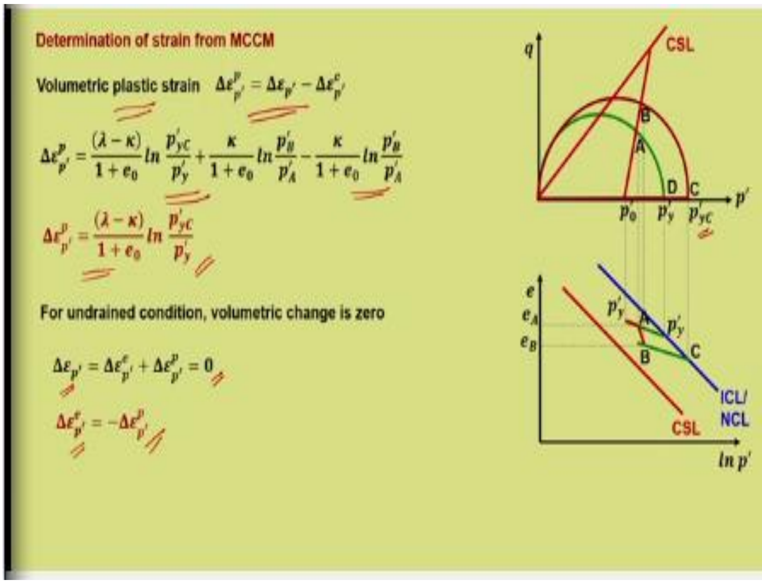

Now we have set the volumetric elastic strain, we need to also determine volumetric plastic strain. We have determined the total volumetric strain; we have determined the elastic volumetric strain. So, obviously the plastic volumetric strain is total minus the elastic component, and that is what is written here,

$$\Delta\epsilon^p_{p'} = \Delta\epsilon_{p'} - \Delta\epsilon^e_{p'}$$

Symbol Δ is important as I told in the beginning incremental stress. So, if you substitute this in the equation,

$$\Delta\epsilon^p_{p'} = \frac{(\lambda-\kappa)}{1+e_o}\ln\frac{p'_{yC}}{p'_y} + \frac{\kappa}{1+e_o}\ln\frac{p'_B}{p'_A} - \frac{\kappa}{1+e_o}\ln\frac{p'_B}{p'_A}$$

There is a total volumetric strain minus the elastic component of volumetric strain will give the plastic strain and that if you expand the terms will get cancelled off and finally we will be left with

$$\Delta\epsilon^p_{p'} = \frac{\lambda-\kappa}{1+e_o}\ln\frac{p'_{yC}}{p'y}$$

So, the plastic component can be represented by the critical state parameters. Now for undrained condition we know that the total volumetric strain will be 0.

$$\Delta\epsilon_{p'} = \Delta\epsilon^e_{p'} + \Delta\epsilon^p_{p'} = 0$$

So, this component $\Delta\epsilon_{p'}$= 0, in that case we get the expression

$$\Delta\epsilon^e_{p'} = -\Delta\epsilon^p_{p'}$$

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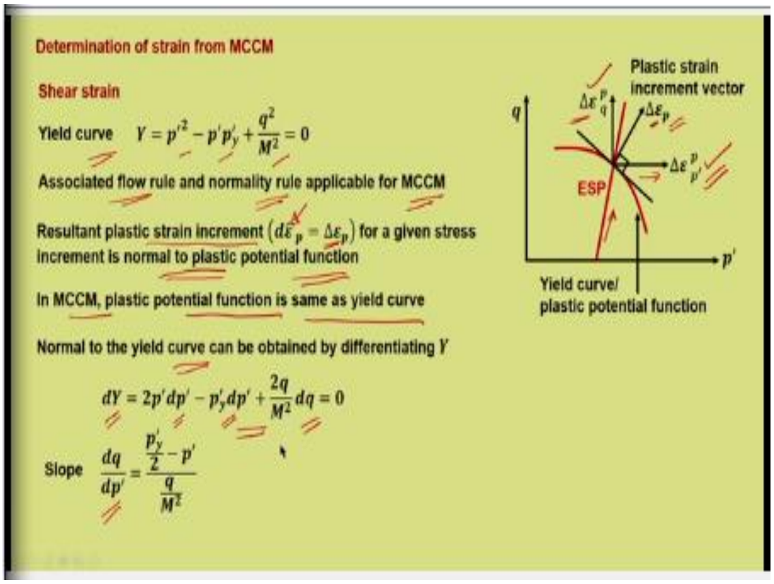

So, we have now completed the volumetric strain part, now we have left with shear strain. So, we need to determine the shear strain. Now for determining shear strain, we know that about the plastic potential and how it expands. So, yield curve Y is represented as

$$Y = p'^2 - p'p'_y + \frac{q^2}{M^2} = 0$$

Because this is MCCM and the yield curve corresponds to ellipse, this is the equation of the ellipse. We have already seen that it follows associated flow rule and normality rule which is applicable for MCCM, we have discussed this in detail please refer back if you want to revise it. Now the resultant plastic strain increment which is represented as $\Delta \widehat{\epsilon_p} = \Delta\epsilon_p$, we are just taking for sake of convenience that is the incremental form. For a given stress increment, we know that because of normality rule it is normal to the plastic potential function.

And we also know because of associated flow rule in MCCM plastic potential function is same as yield curve. So, we do not have 2 equations, the equation for yield curve itself is good enough for defining plastic potential function. So, here in q p' this is what we have already seen, this is the ESP, this is the yield curve the same as the plastic potential function, this is the direction which is normal to the curve at this particular point of contact is $\Delta\,\epsilon_P$ which is plastic strain increment vector.

We have used this particular symbol initially which is considered same as $\Delta\, \epsilon_P$. Now here you have the component of $\Delta\epsilon_{p'}^{p}$, which we have already determined in the previous slide. We are left with $\Delta\epsilon_q^p$ which we not know which is the plastic component of deviatoric strain, and this we have already determined. So, our next task is to find out what is $\Delta\epsilon_q^p$.

Now normal to the yield curve can be obtained by differentiating Y. Now basic calculus says that we can get the normal to the yield curve by differentiating the equation of yield curve. So, we get tangent and from tangent we can get the normal, so

$$dY = 2p'dp' - p_y'dp' + \frac{2q}{M^2}dq = 0$$

So, if you rearrange it the slope

$$Slope\ \frac{dq}{dp'} = \frac{\frac{p_y'}{2} - p'}{\frac{q}{M^2}}$$

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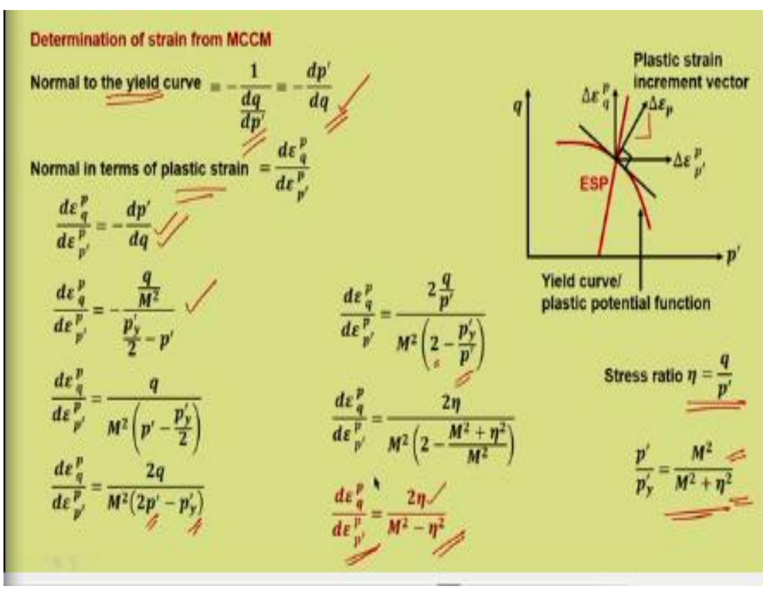

Now normal to the yield curve that is if you multiply the slope of normal and tangent it is going to be -1 and that is what we are written here.

$$Normal\ to\ the\ yield\ curve = -\frac{1}{\frac{dq}{dp'}} = -\frac{dp'}{dq}$$

So, the reciprocal of minus of $\frac{1}{\frac{dq}{dp'}}$ is the normal to the yield curve and that is represented as $-\frac{dp'}{dq}$. So, normal in terms of plastic strain we know from this particular figure it is Normal in terms of plastic strain = $\frac{d\epsilon_q^p}{d\epsilon_{p'}^p}$

So, this slopes can be equated, so we can always write

$$\frac{d\epsilon_q^p}{d\epsilon_{p'}^p} = -\frac{dp'}{dq}$$

Now $-\frac{dp'}{dq}$ can be written as

$$\frac{d\epsilon_q^p}{d\epsilon_{p'}^p} = -\frac{\frac{q}{M^2}}{\frac{p'_y}{2} - p'}$$

Which comes from the previous slide. We have already obtained the equation for dq upon dp'.

So, we are taking the minus sign inside we have

$$\frac{d\epsilon_q^p}{d\epsilon_{p'}^p} = \frac{2\frac{q}{M^2}}{M^2\left(2 - \frac{p'_y}{p'}\right)}$$

Which can be further simplified as

$$\frac{d\epsilon_q^p}{d\epsilon_{p'}^p} = -\frac{2q}{M^2(2p' - p'_y)}$$

$$\frac{d\epsilon_q^p}{d\epsilon_{p'}^p} = -\frac{\frac{q}{M^2}}{M^2\left(p' - \frac{p'_y}{2}\right)}$$

q/p' is nothing but stress ratio η which we have seen in our previous discussions. And we have also derived the equation for p'/p'_y which is equal $\frac{M^2}{M^2+\eta^2}$, please refer to MCCM discussion for this. So, these 2 relationships are known which can be substituted in this. So, if you substitute

$$\frac{d\epsilon_q^p}{d\epsilon_{p'}^p} = \frac{2\eta}{M^2\left(2 - \frac{(M^2+\eta^2)}{M^2}\right)}$$

So, you finally get this expression

$$\frac{d\epsilon_q^p}{d\epsilon_{p'}^p} = \frac{2\eta}{M^2 - \eta^2}$$

So knowing the critical state parameter and the stress ratio we can find out $d\epsilon_q^p$.

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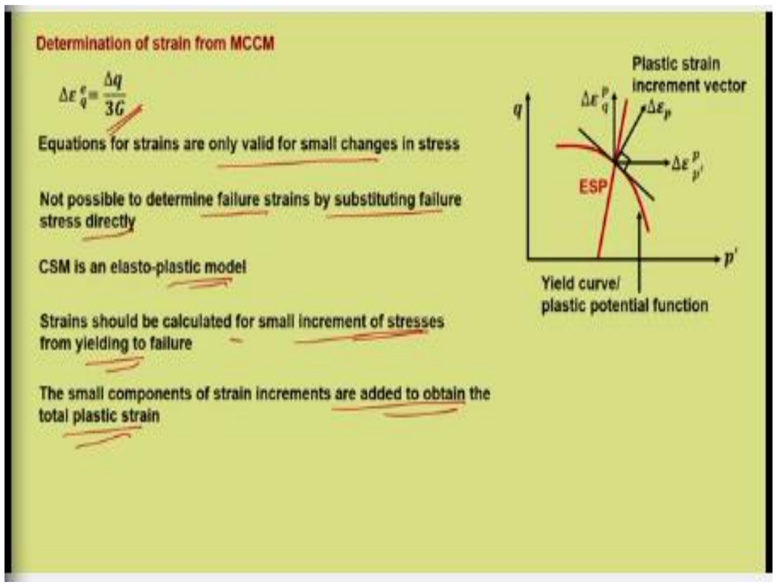

Now the elastic part of the deviatoric strain, we already know that is $\Delta\ \epsilon_q^e\ = \frac{\Delta q}{3G}$. Now equations for strains are only valid for small changes in stress, I mean to say incremental form. So, the moment you are talking about yielded point from first yielding to the next small increment of stress that is represented by another yield curve. So, between the initial yield curve and the reference yield curve which we are talking about there is a small stress increment and we are talking about that small strain increments.

So, all the equations that we have discussed till now is referring to that small incremental strain. Not possible to determine failure strains by substituting failure stress directly. So, if you want to determine what is the failure strain, we should not simply substitute the failure stress and get the result. We need to get the small increments; you keep on adding it and then get the final strain. So, CSM is an elastoplastic model, so it is not elastic model, so you cannot simply substitute point stress and get the strain.

So, strains should be calculated for small increment of stresses from yielding to failure. So, you will have different components of strains from the first yielding to the failure. So, the total components of strain increments are added to obtain the total plastic strain. So, these small, components up to failure is added together to get the total plastic strain.

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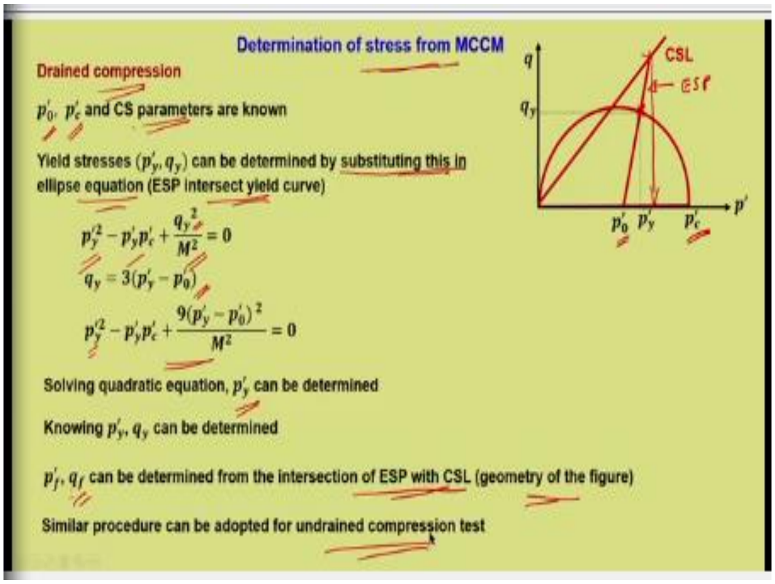

So, drained compression, if you want to determine the stress for MCCM that is corresponding to drained compression. So, what we have done is, we have determined all the strain components, knowing the MCCM model and the critical state parameters, now what we are referring to is. We are just summing up to say how do you determine the stresses which we have already discussed before. But in prediction when we discussed about this we did not specifically mention.

So, when I say determination of stresses in MCCM what are the stresses which we are concerned about? We want yield stresses that is when the ESP meets the yield curve that is the initial yield curve, so that yield point is important. So, q_y p'_y corresponding to the first yielding, then we are interested in the failure stresses, that is q_f and p'_f. Along with that the strain and stiffness comes into picture but what we are more interested in this particular lecture or this particular course is the stress components.

So, we have yield as well as failure stress components in terms of q and p', so how to determine this? And we also need to find out whether it is for drained compression or for undrained compression? So, that is what we will just quickly glance through how to obtain the stress components in this particular slide. Now another aspect is the pore water pressure, the excess powder pressure corresponding to yielding and the excess pore water pressure corresponding to failure, how do we do that?

We know if it is undrained test, we know that there is a total stress path, so we know that it is at 3. Draw that; get the difference between total stress and the effective stress. So, determining pore water pressure is also a straightforward task. So, for drained compression, let us see p'_o that is the initial state of the soil p'_C is the maximum isotropic compression point and the critical state parameters are known.

So, this is the yield curve which corresponds to the maximum yield stress p'_C, p'_o is the initial point at which it has been unloaded let us say it is an LOC point, this is the ESP. Now this is the point where it yields which is given as $p'_y$and q I. So, let us give an example how to determine q_y and p'_y? We know from this figure ESP meets the ellipse at this particular point that is q_y and p'_y, we know the equation of ellipse.

So, yield stresses p'_y, q I can be determined by substituting this in the ellipse equation because it intersects, ESP intersects the yield curve. So, now this is the equation that is

$$p'^{\,2}_y - p'_y p'_c + \frac{q_y^2}{M^2} = 0$$

From this geometry we can also write

$$q_y = 3\left(p'_y - p'_o\right)$$

Substituting for q_y in this particular equation, we will get

$$p'_y - p'_Y p'_C + \frac{9\left(p'_y - p'_o\right)^2}{M^2} = 0$$

Now this can be simplified into a quadratic equation in terms of p'^2_y . So, solving the quadratic equation we will get the value of p'_y and once we get p'_y we can get q_y, so knowing p'_y q_y can be determined. Similarly p'_f and q_f can be determined from the intersection of ESP with CSL, this particular point.

Now this is nothing but the geometry, so from the geometry of the figure we can determine q_f and p'_f. Rather we have discussed how to determine q_f and p'_f in the previous lecture, the same geometry. Similar procedure can be adopted for undrained compression test. So, let us try to summarize what we have done essentially in this lecture is to demonstrate how we can determine the strains from the MCCM model.

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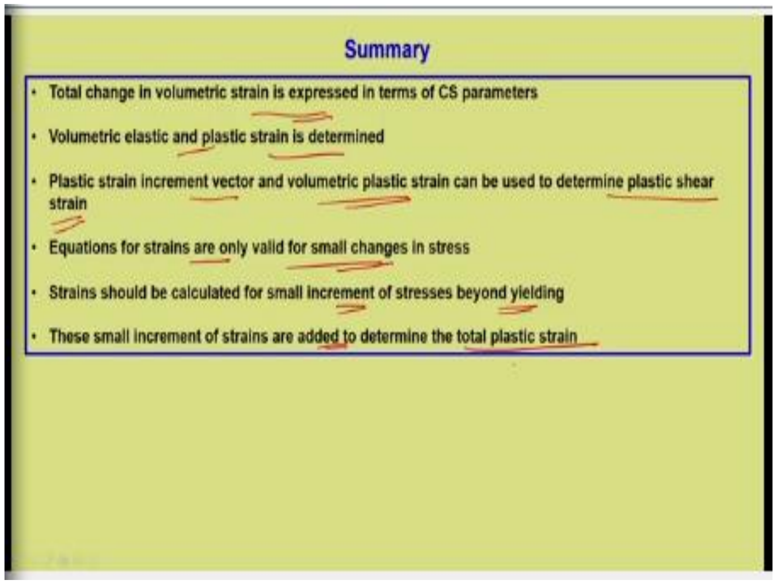

The total change in volumetric strain is expressed in terms of critical state parameters. Volumetric elastic and plastic strain is determined. Plastic strain increment vector and volumetric plastic strain can be used to determine plastic shear strain, so this also we have discussed. Equations for strain are only valid for small changes in stress specifically when it yields and when there is expansion of yield curve.

Strains should be calculated for small increment of stresses beyond yielding. These small increment of strains are added together to determine the total plastic strain. So, now we have completed most of the aspects of the critical state module. Now we are left with the final aspect of critical state where we are going to sum up all these information and understand what is known as the boundary surface?

The state boundary surface where it limits all the possible soil states? So, the next lecture will be on state boundary surface and with that we intend to finish this particular module 4. So, that is all for now, thank you.