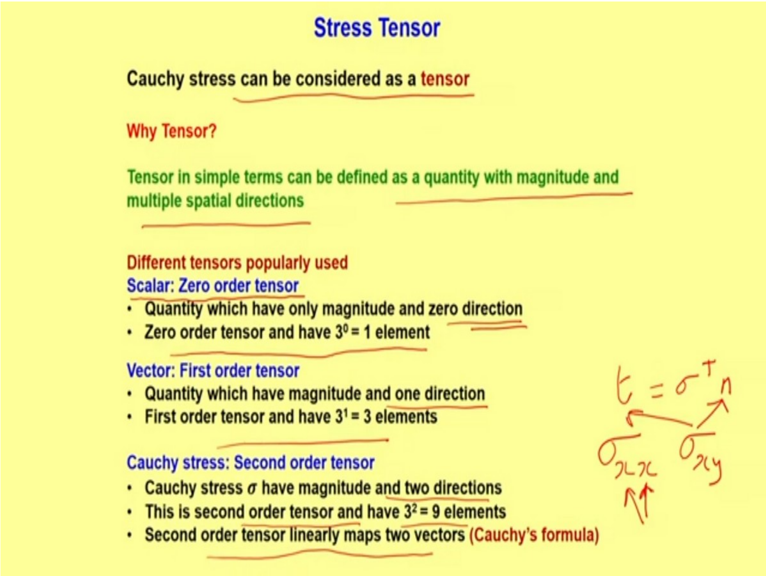
Advanced Soil Mechanics
Prof. Sreedeep Sekharan
Department of Civil Engineering
Indian Institute of Technology, Guwahati, Assam

Lecture - 03
Stress Acting at a Point - Stress Tensor

So, welcome back, this is the next lecture on stress tensor. So in the last lecture we have categorically seen what is cauchy's stress, σ. And we have seen that definition of σ indicates what is the internal force that gets developed within on a plane or within a body at a point due to some action of external forces. Now what actually is σ. In fact σ is a stress tensor.

(Refer Slide Time: 01:10)



So, cauchy's stress it can be considered as a tensor. So, now we are going to define a new term what is known as tensor. If you want to study or if you want to do modeling in continuum mechanics, as the complexity of the problem increases, it is always convenient to define what is known as tensor. And we have already stated stress is a tensor quantity. Now what is a tensor? We know what are scalars, we know what are vectors. So tensor is also a similar kind of quantity. So why tensor, because it is very convenient to express stress as a tensor.

In short stress itself is a tensor. In simple terms, we can say that tensor can be defined as a quantity with magnitude and multiple spatial directions. So, possibly you will think like what is the difference between a tensor and a vector. Vector also has a magnitude and a direction, but we will see that vector has magnitude, but it will have only one direction whereas in the case of tensor multiple directions are there.

So that is the essential difference and tensor is a more general term. And the subsets of tensor are scalar, vector and any other tensor of higher order. So different tensors which are popularly used are that is what I told the first one is scalar the simplest tensor is scalar and it is called zero order tensor. A quantity which have only magnitude and zero direction, scalar we all of us know that it does not have any direction it has only magnitude.

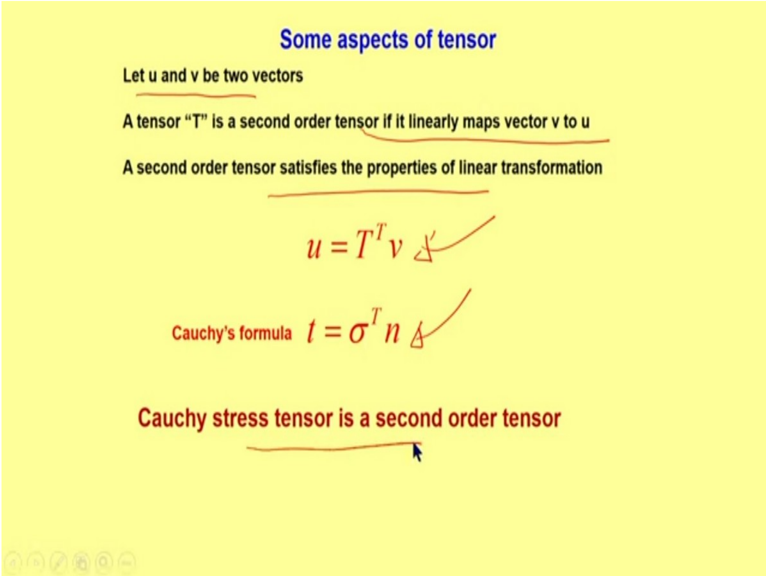
So we call it as zero direction. And zero order tensor which is a scalar have 3^0, where 0 represents the number of direction. So $3^0 = 1$ element and that is true, it is merely a number which shows the magnitude. Scalar is a 0 order tensor. The second one is vector which is the first order tensor. Vector has it is a quantity which have magnitude and one direction you can see that vector has only one direction.

Accordingly the number of elements will be 3^1 which is equal to 3 elements. So if you have x y z axis you have vector in 3 different directions. So that is possible. So that is what it means it has 3 elements. So it has one direction every vector it is associated with only one direction. And it has 3^1 which is 3 elements and specifically cauchy's stress is known as a second order tensor.

Why cauchy's stress σ have magnitude and 2 directions. Now what are these 2 directions? Now we will see it specifically how these 2 directions come into picture when you define a stress component and it is very easy also if you remember cauchy's stress, we represent it as σ_{xx} or σ_{xy}. So there are 2 symbols associated and that is why it is always associated with 2 directions it is associated with on which plane it acts, that means the normal to that particular plane.

It is also dependent on which direction that particular traction acts. So we will discuss that a bit later, only to specify here is cauchy's stress is a second order tensor. It has magnitude and it has 2 directions. So this is second order tensor and it has 3 square equal to 9 elements which we have already seen in the cauchy's stress tensor. There are 9 elements and the second order tensor it linearly maps 2 vectors that also we have seen. We have seen that $t = \sigma^T n$. So it linearly transforms to vectors that is cauchy's formula.

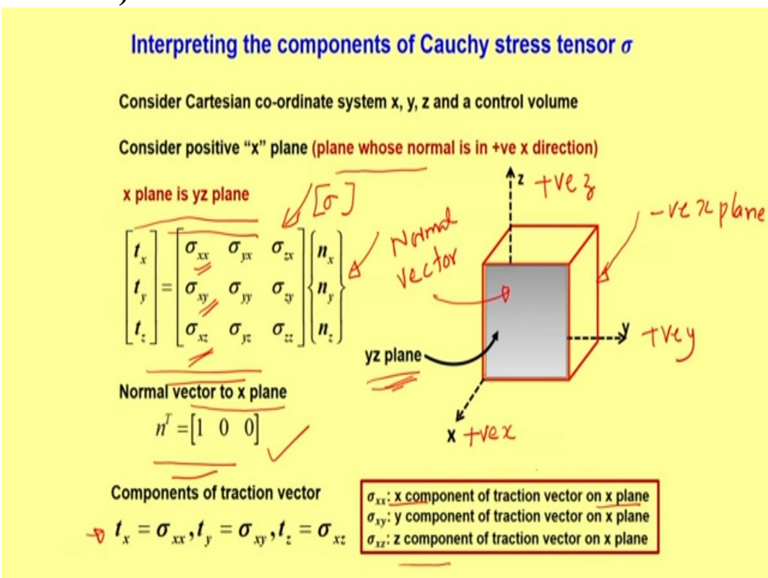
(Refer Slide Time: 06:12)



Now, some aspects of tensor to be very specific, this may not be useful, but then this is important to understand the tensor. Let us say there are 2 vectors u and v a tensor T is a second order tensor if it linearly maps vector v to u as can be shown here $u = T^T v$ and the second order tensor satisfies the properties of linear transformation. So this is what has been written, t maps v to u or there is a linear mapping of v to u.

If you compare this with cauchy's formula, it is more or less the same thing that is how we define cauchy's stress tensor is a second order tensor. Having said that, now the next job is to interpret the components of Cauchy's stress tensor. We know that there are nine elements. Now what are these nine elements. What does it represent?

(Refer Slide Time: 07:21)



So, for that we need to define the Cartesian coordinate. So you have a Cartesian coordinate x y and z. And to make it simple a control volume is also shown, control volume is a very common terminology which is used in continuum mechanics or any other form of mechanics. In fact, this control volume is not required, but to make things simple and for one to understand it has been shown.

So we have a Cartesian coordinate x y z. So you can consider positive x plane, now x is an axis what is meant by x plane. It means the plane on which x direction is the normal to that plane that is what is written here the plane whose normal is in positive x direction. So it is called positive x plane.

So what will be negative x plane. The negative x plane will be here, because the outward normal to this plane is in the negative x direction. So this is negative x plane. So you need to understand this very carefully. Consider positive x planes, so we are talking about this particular plane. That is positive x plane why because normal to this particular plane x this is the yz plane, this is y, this is z. So this plane is yz plane.

Now for this yz plane the normal is in the direction of x. So that is what it means. So x plane means, positive x plane means, yz plane which is shown here so, positive x, negative x plane both are there. Now we will come back to cauchy's formula and cauchy's stress σ. So now, normal vector to x plane this positive x direction. Please understand the normal vector to x plane. So this is the x plane.

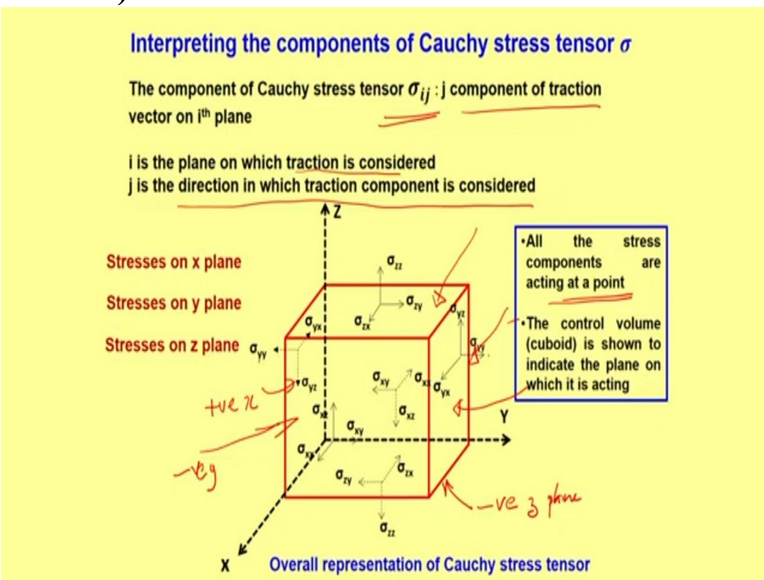
The normal is x. So you can easily write what is the normal vector. So this is the normal vector. So for x direction it is [1 0 0]. So n^T is given in this manner and similarly for y it will be [0 1 0] and for z, [0 0 1]. So normal vector to x plane is defined that is n transpose is given. Now what are the components of traction vector t_x, t_y and t_z. You already have this to be, that is t_x, t_y, t_z is equal to σ and n.

So if you substitute the value of n that is for positive x plane [1 0 0] here, so it will be [1 0 0], and do the matrix operation, you will see that $t_x = \sigma_{xx}$, ty will be equal to σ_{xy}. So this is σ_{xx}, σ_{xy}. So t_y will be equal to σ_{xy} and t_z is equal to σ_{xz}. So what does it mean, it means that the components which are present in cauchy's stress tensor these are components of traction vector in a given direction.

So if you see, you can see that σ_{xx} is the x component of traction vector on x plane. So there are 2 references which are coming and that is why we said that there are 2 directions. It is the x component that is the traction vector is in the x direction and it is acting on x plane. So there are 2 things which are coming. Similarly, you have σ_{xy}. σ_{xy} is the y component of traction vector acting on x plane.

Similarly, you have z component of traction vector acting on x plane. Similarly, other components of cauchy's stress tensor can be identified based on cauchy's formula. So that is what is the meaning of each of the terms which are present in the cauchy's stress, it is nothing but the components of traction vector acting in specific direction.

(Refer Slide Time: 13:04)



So, the component of cauchy's stress tensor in general is σ_{ij}, it is the j component of traction vector it can be x, it can be y, it can be z. So it is a j component of traction vector acting on i^{th} plane. So first index i, it shows which plane it is associated with. Which plane means, which is the normal and j is the direction of that component, direction of the component of traction vector.

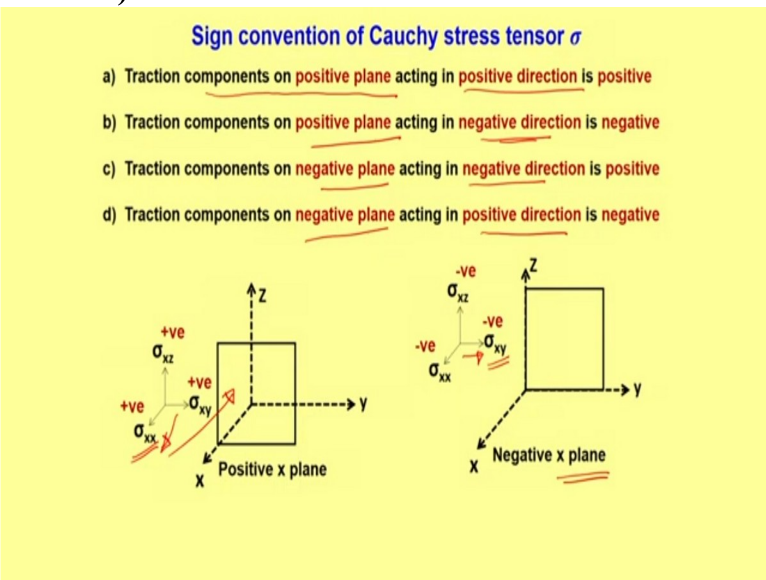
So i is the plane on which traction is considered, j is the direction in which the traction component is considered. So we can see the overall representation of cauchy's stress tensor. So first is stresses acting on x plane. Now which is the x plane, this is the x plane. So there are 2 x plane, this is negative x and this is positive x. So what are the stresses which are acting, we have σ_{xx} in the direction of x. So, all of them are acting on x plane.

Then we have σ_{xy} and σ_{xz}. Similarly, on the other side other plane that is negative x plane we have σ_{xy}, σ_{xz} and σ_{xx}. It is identical but it is on the other side. Then we have stresses on y plane. Now what is meant by y plane? A plane with y direction as the normal. So you are talking about this and this. So you have positive y and this is negative y.

Similarly, so in this you have σ_{yy}, which is the direction, in the direction of y and you have σ_{yx}, σ_{yz}, similarly σ_{yx} and σ_{yz}. Then we have stresses acting on z plane, what are the stresses acting on z plane and what are the z plane, this is positive z plane and this is negative z plane and this stress is acting as σ_{zz}, σ_{zx}, σ_{zy}. Similarly, here also you have σ_{zz}, σ_{zx} and σ_{zy}. So these are the representation of the components of Cauchy's stress on a given control volume.

So, all these stress components are acting at a point. Now we need to keep in mind since I have shown a control volume in the figure, and that is only for understanding how the stresses are oriented. Otherwise it does not serve any purpose. We need to still understand that whatever stress components are there in Cauchy's stress tensor, it is acting at a point and the control volume, the cuboid is shown only to indicate the plane on which it is acting. So that notion we should not forget. So it is a stress acting at a point. Now having said that, we need to now define some sort of sign convention of Cauchy's stress tensor.

(Refer Slide Time: 16:51)



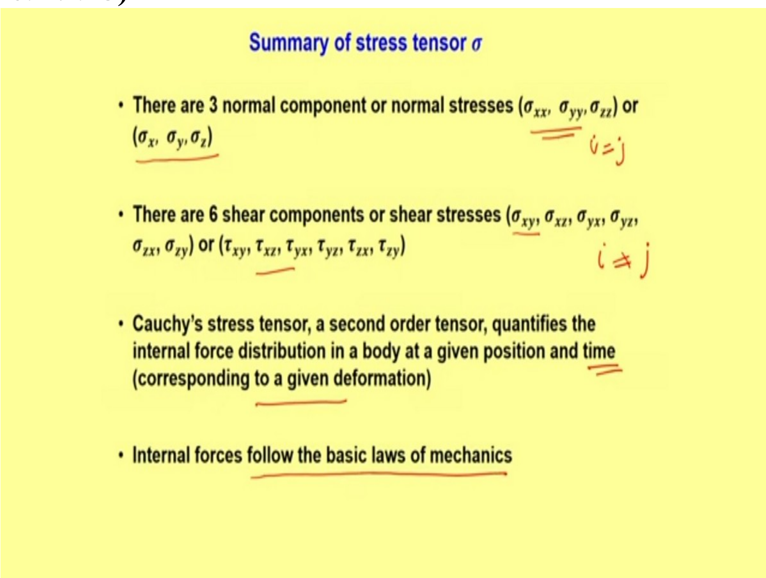
So the given sign conventions are the traction components on positive plane. So now we have already marked what is a positive plane. So the traction component on positive plane acting in the positive direction means the direction of x y z which is in the positive direction, so is positive. So you have positive plane and the traction component is acting in the positive direction, so it is positive. Similarly, if you have a positive plane and the traction component is acting in a negative direction, so it is negative.

For negative plane, if the plane is negative and the traction component is acting in negative direction, so it is positive. And the final case is negative plane traction component direction is positive direction, it is negative. So this is one sign convention, you can see that there are numerous sign conventions which are available and one may use at his convenience, but if you follow one sign convention, you need to follow it for throughout.

So this is one convenient way of defining sign convention, there are assigned conventions which are available based on movement also, sometimes it may be difficult to understand. So this is very easy and very easy to define as well, one example is given here. So this is the positive x plane, and the stresses acting are σ_{xx}, σ_{xy} and σ_{xz}. If you consider σ_{xx}, this is acting on a positive x plane.

And σ_{xx} is in acting in the positive x direction. So that is why it is positive, similarly $\sigma_{x}y$ and σ_{xz}. Now consider the case of negative x plane, if you consider σ_{xy}, this is a negative plane, negative x plane whereas, this is acting in positive y direction. So negative plane positive y direction, so it is negative. Similarly, all the stress components sign can be assigned. So this is the sign convention of Cauchy's stress tensor. So what are the summary that we have understood till now.

(Refer Slide Time: 19:25)



There are 3 normal component or normal stresses, σ $_{xx}$, σ $_{yy}$ and σ $_{zz}$. You can see that in this figure, you have σ $_{xx}$, σ $_{yy}$ and σ $_{zz}$, these are acting in the same direction as that of normals. So there are 3 normal components or normal stresses σ $_{xx}$, σ $_{yy}$, σ $_{zz}$ or it is merely stated as σ_x, σ $_{y}$, σ $_{z}$ which is a common terminology, which we normally use in mechanics.

There are 6 shear components or shear stresses where ($i \neq j$)

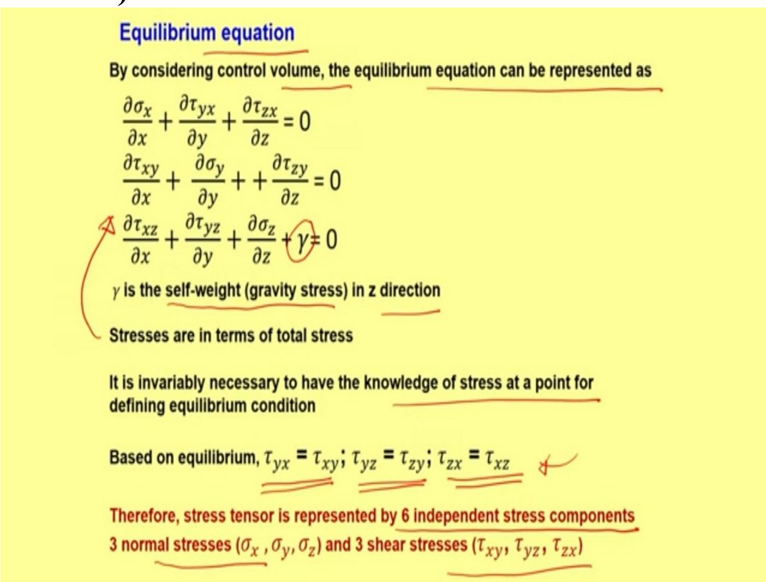
$$(\sigma_{xy}, \sigma_{xz}, \sigma_{yx}, \sigma_{yz,}\sigma_{zx}, \sigma_{zy})$$
$$(\tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yz,}\tau_{zx}, \tau_{zy})$$

For (i=j),

$$(\sigma_{xx}, \sigma_{yy}, \sigma_{zz})$$

So these are shear components of traction or shear stresses, it is written either in σ form or in τ form. Cauchy stress tensor a second order tensor, quantifies the internal force distribution in a body at a given position and time corresponding to a given deformation. Why time is important, because we are considering the condition corresponding to a given deformation. And internal forces, which that gets developed it follows the basic laws of mechanics.

(Refer Slide Time: 21:04)



Now one particular aspect why stress at a point that information is needed is to, define the equilibrium equation. So it is a application of why you need to have the knowledge of stress at a point. Now stress at a point is very important to define the equilibrium equation as we have seen in the beginning, you have seen that there are certain requirements which need to be satisfied like equilibrium condition, the compatibility condition and so on.

Now for defining the equilibrium condition, we need to specify equilibrium equation. I will not go into the derivation of this equilibrium equation it is a very basic and which is mostly seen by most of you. So by considering a given control volume, the equilibrium equation can be represented as follows.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0$$

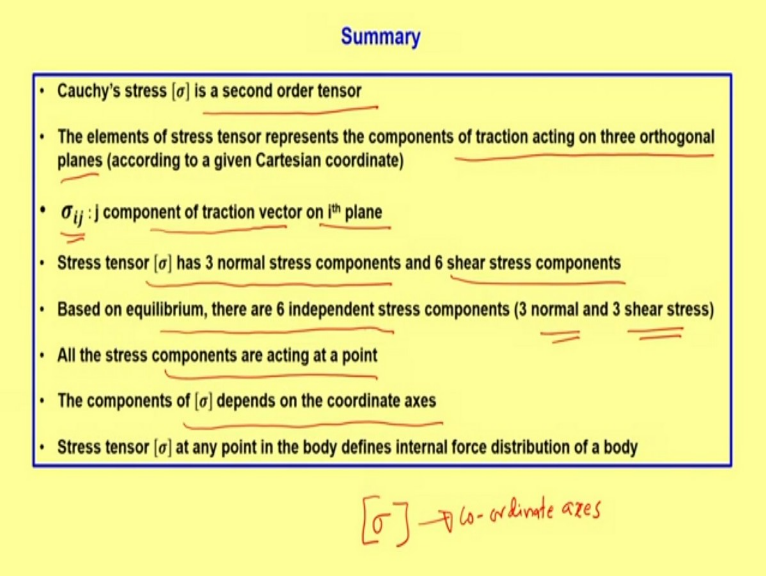
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \gamma = 0$$

And you can see that the components of equilibrium equations are the stress tensor components, the only new term is γ.

Where γ is a self-weight or the gravity stress which acts in the z direction or in the vertical direction and to be very specific stresses are in terms of total stresses in this particular equation, it is invariably necessary to have the knowledge of stress at a point for defining equilibrium condition. Now based on equilibrium, we can say that $\tau_{yx} = \tau_{xy}$, $\tau_{yz} = \tau_{zy}$ and $\tau_{zx} = \tau_{xz}$.

Therefore, the stress tensor is represented by 6 independent stress components, there are nine components in the cauchy's stress tensor just because of this condition, we have 6 independent stress components, and they are 3 normal stresses, σ_x, σ_y, σ_z and 3 shear stresses τ_{xy}, τ_{yz} and τ_{zx}, where $\tau_{xy} = \tau_{yx}$. So this is what it is. So that is how it boils down to 6 independent stress components.

(Refer Slide Time: 23:24)



So the final summary of what we learned in this particular lecture is cauchy's stress, σ is a second order tensor. The element of stress tensor represents components of traction acting on 3 orthogonal planes according to a given Cartesian coordinate. σ_{ij} means j component of traction vector acting on i^{th} plane. Stress tensor σ has 3 normal stress components and 6 shear stress components. But based on equilibrium, there are 6 independent stresses were 3 normal and 3 shear stresses.

All the stress components are acting at a point which is a very relevant point and which is very important. The components of σ depends on coordinate axis, please note here as such σ it is not dependent on coordinate axis, but the components of σ, I mean to say σ_x, σ_{xy} those are the components or the traction vector components, they are dependent on the coordinate axis. So there is a distinction which needs to be very clear, one should not get confused with σ as a whole and the components of σ.

σ as a whole it is not dependent on any axis, but the components of σ keeps changing, but the overall σ representation of internal force remains the same depending on the reference axis, the components magnitude value keeps changing. Stress tensor σ at any point in the body defines the internal force distribution of a body. So this is all about this particular lecture, we will see in the next lecture.