Advanced Soil Mechanics
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Lecture – 15
Mathematical Formulation Axisymmetric

So, welcome back in the last lectures, we have seen the importance of 3 dimensional to 2 dimensional idealizations. We have seen the mathematical formulations related to 2D idealization where the important 2D idealizations we have seen are plane strain, plane stress and axisymmetric condition. Out of this the mathematical formulations related to plane strain and plane stress we have discussed. So, in todays lecture, we will see the mathematical formulation for axisymmetric condition.

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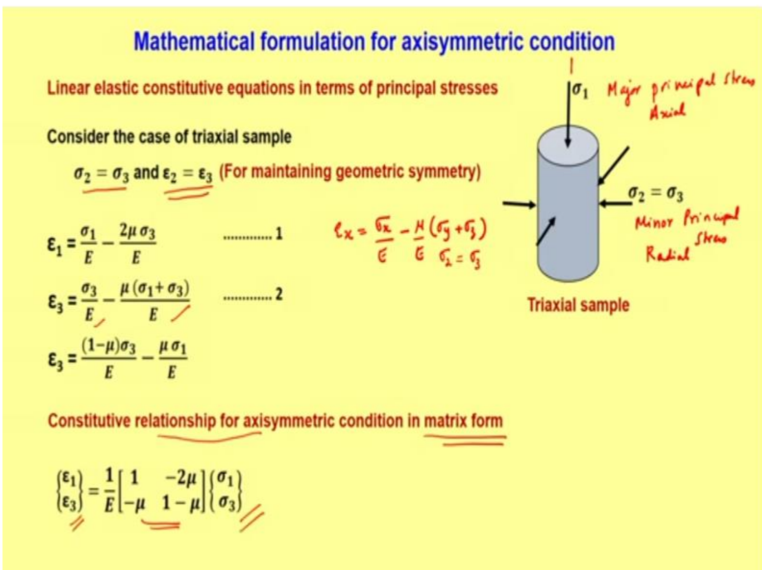

So, here we are dealing with linear elastic isotropic condition, in the previous lectures also we have done for the simple linear elastic models. So, here also we will see for the linear elastic formulation. Now, in this axisymmetric condition, we have already stated that, this is very important with respect to soil behavior, why because the triaxial samples what we discuss falls under axisymmetric formulation.

So, we will now, in the formulation for axisymmetric condition, we will specifically focus on triaxial sample, you can see that there is this cylindrical soil sample which is acted upon by actual stress and actual stress is normally considered as the major principle stress and the radial stress the order on stress or the radial stress is the minor principle stress. So, this is actual and this is a radial. So, this is a situation so, if you see that if σ_1 is exactly acting at the center is an axial load.

So, it has to act at the central axis. If this is considered as the reference, then this is symmetrical about the application of this load, it is also symmetrical with respect to its geometry. So, it is important that these are met. Additionally, in this particular formulation or in this particular triaxial sample, we know that $\sigma_2 = \sigma_3$ the radial stresses or all around stresses are same and to maintain the geometric symmetry.

So, we also need to have $\varepsilon_2 = \varepsilon_3$ that is for maintaining geometric symmetry. Now, we know the formulations which we have already seen before the formulation was let us say in general $\varepsilon_x = \sigma_x / E - \mu / E (\sigma_y + \sigma_z)$. So, this is the general equation, now, we are considering in terms of principle stresses. So, we have ε_1 is the major principle strain is equal to $\sigma_1 / E - 2 \mu \sigma_3 / E$.

Why? Because here $\sigma_2 = \sigma_3$ hence, it will become $- 2 \mu \sigma_3 / E$. So that is denoted as equation 1. Then we have $\varepsilon_3 = \sigma_3 / E - \mu (\sigma_1 + \sigma_3) / E$. So, there is no need of ε_2 because we $\varepsilon_2 = \varepsilon_3$. So, $\varepsilon_3 = \sigma_3 / E – \mu (\sigma_1 + \sigma_3) / E$ and that is denoted as equation 2 and $\varepsilon_3 = (1 – \mu) \sigma_3 / E - \mu \sigma_1 / E$.

That is if you rearrange this σ_3 and $- \mu \sigma_3 / E$ we get $(1 – \mu) \sigma_3 / E - \mu \sigma_1 / E$. So, based on ε_1 and ε_3, one can always arrange the constitutive relationship for axisymmetric condition in matrix form. This we have done for plane stress and plane strain as well; the same constitutive formulation has been done for axisymmetric condition as ε as a function of stress.

So,

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_3 \end{Bmatrix} = \frac{1}{\mathrm{E}} \left[\frac{1}{-\mu} + \frac{-2\mu}{1 - \mu} \right] \begin{Bmatrix} \sigma_1 \\ \sigma_3 \end{Bmatrix}$$

Now, one thing we have to note here is, if you consider this particular matrix, this is not a symmetric matrix, you have – 2μ and – μ here. So, this is something different from what we have seen earlier, where we had a symmetrical matrix. So, in this case it is not symmetric. Hence, the inverse may not be straightforward. I mean to say if I get the equation for σ_1 that will not apply for σ_3, so that we will have to see separately. So, this is the constitutive relationship in terms of ε_1 and ε_3.

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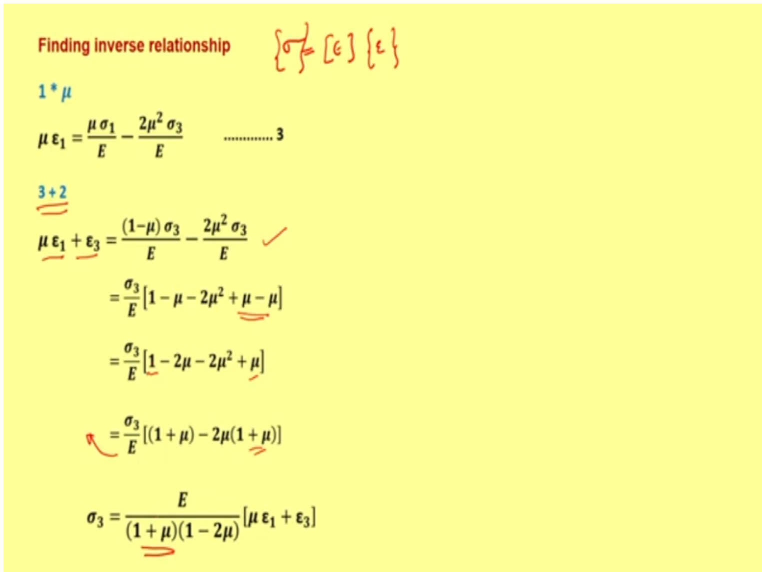

Now, we also need to know what is the inverse relationship that is what will be $\sigma = E\,\varepsilon$. So, we also need to know this particular inverse relationship, this we have seen in the earlier lecture also we need to do some mathematical rearrangement and simplification, so that we get the relationship in terms of σ. So, for that, we need to do some modification for equation 1. So, 1 is multiplied by μ which gives $\mu\,\varepsilon_1 = \sigma_1\,\mu / E - 2\,\mu^2\,\sigma_3 / E$ and that is denoted as equation 3.

Why we are doing this? We are doing this to eliminate one by one and we are left with only 1 stress and that will give you the required equation. So, we are just eliminating it. So, then this equation plus equation 2, equation 2 is first ε_3 that will give you $\mu\,\varepsilon_1 + \varepsilon_3 = \sigma_3\,(1-\mu) / E - 2\,\mu^2\,\sigma_3 / E$. I again strongly urge all of you to please write it down step by step and make sure that it is all right.

Again, if you can see with this particular operation, you can see that this is now in terms of σ_3. So, σ_3/E if we take it outside we have $(1 - \mu - 2\,\mu^2)$ and with some adjustment that is plus μ minus μ, you will get a proper equation. So, $\sigma_3/E\,*(1 - 2\mu - 2\,\mu^2 + \mu)$. Further, if you again rearrange this we will get $\sigma_3/E\,*[(1+\mu) - 2\,\mu\,(1+\mu)]$. So, again $(1+\mu)$ can be taken outside this will give σ_3 as E divided by, so if you take (1 +μ) outside here it will be σ_3 (1 +μ) and you are left with 1 - 2μ. So, σ_3 will be equal to E will go on the other side E will go here. So that will read E /(1 +μ) (1 - 2μ) * [μ ε_1 + ε_3] which is from here.

That is the equation for σ_3 in the earlier lectures we have seen that once we have got it for 1 particular stress, you can just reproduce it for other stresses as well there in that particular case, you could see that the matrix was symmetrical where ε in terms of σ was symmetrical but in this case, it is not symmetrical.

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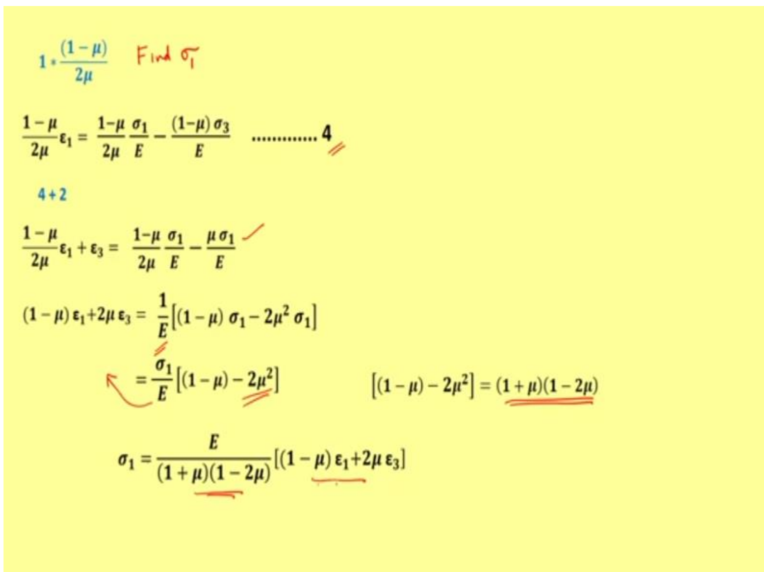

Hence, we cannot reproduce for σ_1 we have to again find it separately. So, we will see how if you multiply equation 1* (1 – μ)/ 2 μ. Why we are doing this? We need to find σ_1 expression. So, multiply (1 – μ)/ 2 μ. This will give (1 – μ)/ 2 μ * ε_1= (1 – μ)/ 2 μ * σ_1/ E - (1 – μ) * σ_3 / E. So, call it as equation 4. Now, if we again do this operation 4 + 2, we will be left with this (1 – μ)/ 2 μ * ε_1 + ε_3 = (1 – μ)/ 2 μ * σ_1/ E – μ * σ_1/ E.

So, σ_3 got eliminated in this operation. So, this we are left with now σ_1 now, we need to rearrange it, if you multiply by 2 μ, we will be left with (1 – μ)* ε_1 + 2 μ ε_3 = 1/E [(1 – μ) * σ_1 – 2μ^2 * σ_1]. So, σ_1 /E [(1 – μ) – 2μ^2]. So, this again can be further simplified which will give [(1 – μ) - 2 μ^2] = (1 + μ) *(1- 2 μ) substituted back here.

We can write the expression for $\sigma_1 = [E/ (1 + \mu) *(1- 2 \mu)] *[(1 – \mu)* \varepsilon_1 + 2 \mu \varepsilon_3]$.

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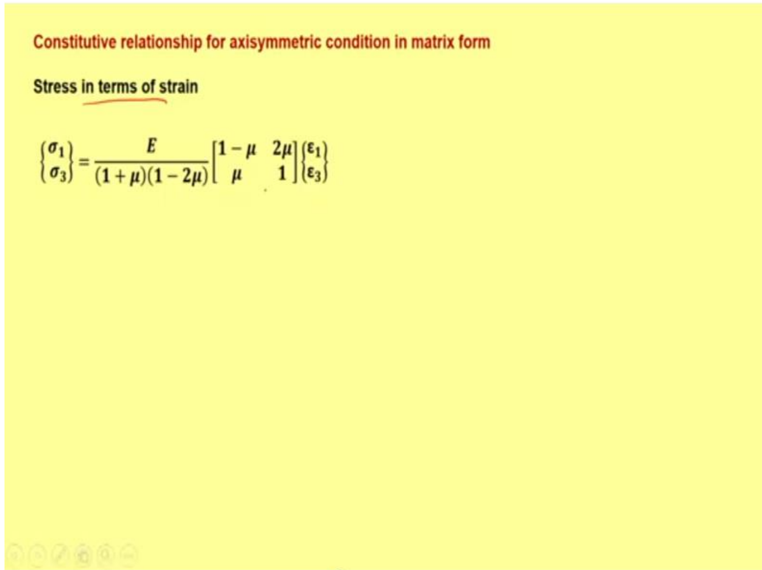

So, one can again rearrange the constitutive relationship for axisymmetric condition in matrix form that is stress in terms of strain as

$$\begin{Bmatrix}\sigma_1\\ \sigma_3\end{Bmatrix} = \frac{\mathrm{E}}{(1\ +\ \mu)(1-\ 2\,\mu)}\left[\frac{1-\ \mu}{\mu} + \frac{2\mu}{1}\right]\begin{Bmatrix}\varepsilon_1\\ \varepsilon_3\end{Bmatrix}$$

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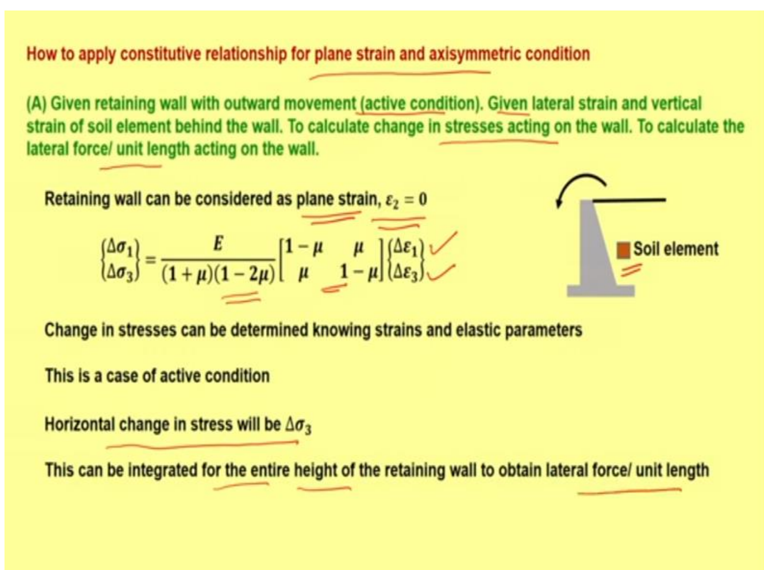

Now, let us see what is the application of these constitutive relationship we will see this with a simple example. Whereas, the same constitutive relationship can be used to solve complex problems as well. But for understanding it better, we will see a very simple example which may not need constitutive relationship as such to solve the problem but for us to understand how this can be applied. So, how to apply constitutive relationship for plane strain and axisymmetric condition.

I am purposely avoiding plane stress because we are in advanced soil mechanics this particular course and specifically in geomechanics, we will have a lot of plane strain and axisymmetric cases. So, the first example is that there is a retaining wall with outward movement, outward movement means, we know that it is an active condition also lateral strain and vertical strain of a soil element behind the wall are given, we need to calculate change in stresses acting on the wall to calculate the lateral force per unit length acting on the wall.

Now, we are supposed to calculate what are the changes in the stresses, due to this movement. Now, due to this movement outwards which is an active condition the strains are given. I am not I purposely avoided the values here, because the same will be attempted in assignments, you just need to understand how or what is the logic by which we can use the constitutive relationship. So, retaining wall, what is the speciality of a retaining wall? We have already seen that the behavior of a retaining wall can be idealized.

Where 2d behavior and that too in terms of plane strain, because the length in one direction is long as compared to the cross section and the strain in one direction. So, we are dealing in principle stresses and principle strain. So, the ε_2 can be considered equal to 0. So, this is a retaining wall we are talking about a given soil element whose lateral and vertical strain is given and this retaining wall is moving outward.

So, we know since it is a plane strain problem, we know that we can apply this particular constitutive relationship. So, here I have added $\Delta\ \sigma_1$ just to show that it is change in stress. So,

$$\begin{Bmatrix} \Delta\sigma_1 \\ \Delta\sigma_3 \end{Bmatrix} = \frac{E}{(1\ +\ \mu)(1 -\ 2\ \mu)} \begin{bmatrix} \frac{1-\ \mu}{\mu} + \frac{\mu}{1-\mu} \end{bmatrix} \begin{Bmatrix} \Delta\varepsilon_1 \\ \Delta\varepsilon_3 \end{Bmatrix}$$

here you note that this is a symmetric matrix which is different from the asymmetric matrix that we obtained in axisymmetric case, now, what is given $\frac{\Delta\varepsilon_1}{\Delta\varepsilon_3}$.

Now, once we know $\frac{\Delta\sigma_1}{\Delta\sigma_3}$, we can always calculate change in stresses by knowing the elastic parameters also. One more thing we need to understand here is in this problem we are assuming everything to be linear elastic. Now, how far it is true, how far it is close to reality that is something different but here we are assuming the behavior to be linear elastic. So, this is an active condition.

We have already discussed that horizontal change in stress will be $\Delta\ \sigma_3$ in active condition; we have minor principle stress acting horizontally. So, what we are looking at is $\Delta\ \sigma_3$ now, once you substitute the values of $\Delta\ \varepsilon_1$, $\Delta\ \varepsilon_3$, μ and E we can determine $\Delta\ \sigma_1$ and $\Delta\ \sigma_3$. Now $\Delta\ \sigma_1$ is in vertical $\Delta\ \sigma_3$ is in horizontal because it is a moment outwards.

Now, once we know $\Delta\ \sigma_3$ we need to find out what is the lateral force per unit length. So, you are just determining the force. Now, what is the force this can be done by integrating $\Delta\ \sigma_3$ for the entire height of the retaining wall this will give us lateral force per unit length same problem with its numericals we will see in the assignment.

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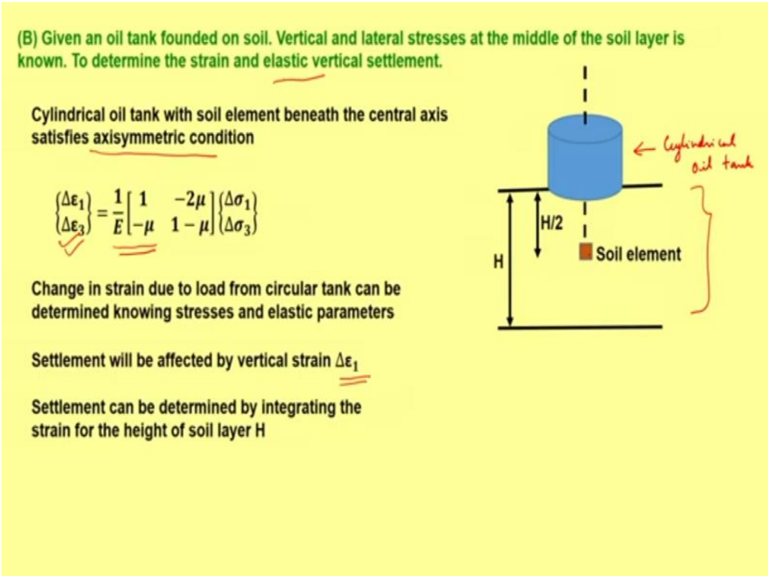

Another case is given an oil tank founded on soil. Vertical and lateral stresses at the middle of the soil layer are known to determine the strain and elastic vertical settlement. So, it is specified here it is elastic vertical settlement. So, our problem becomes linear elastic. Now, cylindrical oil tank with soil element beneath the central axis it is satisfies the axisymmetric condition, we will see how, now consider this to be the cylindrical oil tank, now, we are considering a soil element which is exactly beneath the central axis of this cylindrical oil tank.

If you are considering this then it obviously satisfy the requirements of axisymmetric condition once we know it is axisymmetric we can always use the corresponding equation or the constitutive relationship and

$$\begin{Bmatrix} \Delta\varepsilon_1 \\ \Delta\varepsilon_3 \end{Bmatrix} = \frac{1}{\mathrm{E}} \begin{bmatrix} \dfrac{1}{-\mu} + \dfrac{-2\mu}{1-\ \mu} \end{bmatrix} \begin{Bmatrix} \Delta\sigma_1 \\ \Delta\sigma_3 \end{Bmatrix}$$

. So, this is the equation which we will be using you can see that this matrix is asymmetric.

Now, change in strain due to load from circular tank can be determined if we know the stresses and the elastic parameters. Now, what is given vertical and lateral stresses at the middle of the soil this is given already? So, once we know this we can always calculate what is the strain. Now, settlement is mostly governed by the vertical strain $\Delta\ \varepsilon_{1.}$ So, once we substitute this $\Delta\ \varepsilon_1$ is obtained using $\Delta\ \varepsilon_{1.}$ we can calculate the settlement how by integrating for the entire height. So, settlement can be determined by integrating the strain $\Delta\ \varepsilon_1$ for the height of the soil layer H.

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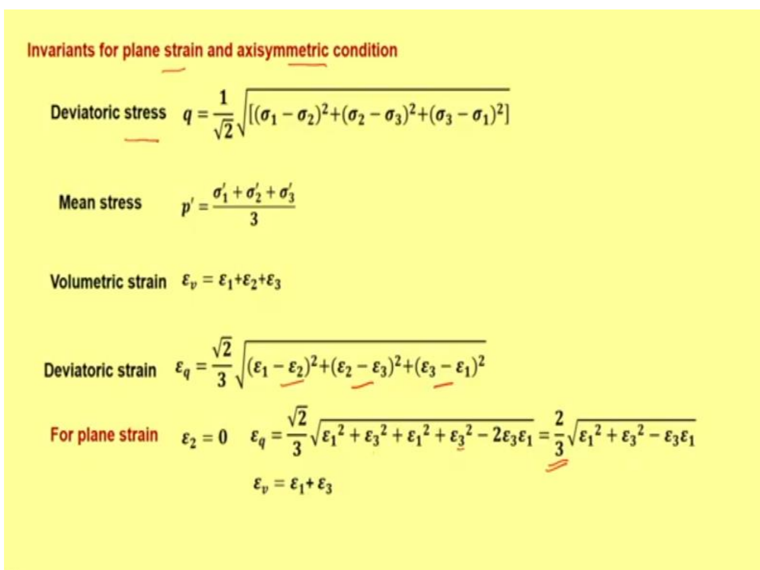

So that is what we have to do once you get this problem for assignment, then you will understand it better. Now, we will discuss a bit on invariants for plane strain and axisymmetric condition, what are invariants? We have already discussed which is not affected by the coordinate axis. So, those invariants in terms of plane strain and axisymmetric condition we will see. So, we have already discussed, what is the general expression for deviatoric stress which is q?

And we have also known that this particular deviatoric stress basically comes from the definition of von Mises failure criterion. So, here

$$q = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

Now, what is the implication once you consider it to be plane strain and axisymmetric because we know the conditions. So, we just need to apply that into this equation and get the corresponding expression.

The next invariant is mean stress

$$p' = \frac{\sigma'_1 + \sigma'_2 + \sigma'_3}{3}$$

Volumetric strain $\varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ and deviatoric strain

$$\varepsilon_q = \frac{\sqrt{2}}{3} \sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2}$$

So, we need not repeat it again because this we have already discussed. Now, for plane strain, we know $\varepsilon_2 = 0$, there are no other shear strain. So, we do not have to discuss we are discussing in terms of principle stresses and strains.

So, $\varepsilon_2 = 0$ that will give this equation, we need to substitute $\varepsilon_2 = 0$ that will give the expression

$$\varepsilon_q = \frac{\sqrt{2}}{3} \sqrt{(\varepsilon_1)^2 + (\varepsilon_3)^2 + (\varepsilon_1)^2 + (\varepsilon_3)^2 - 2\,\varepsilon_3\varepsilon_1}$$

$$\varepsilon_q = \frac{2}{3} \sqrt{(\varepsilon_1)^2 + (\varepsilon_3)^2 + -\,\varepsilon_3\varepsilon_1}$$

So, we have got the equation for deviatoric strain corresponding to plane strain condition. And here in case of volumetric strain it will be simple $\varepsilon_v = \varepsilon_1 + \varepsilon_3$ because $\varepsilon_2 = 0$.

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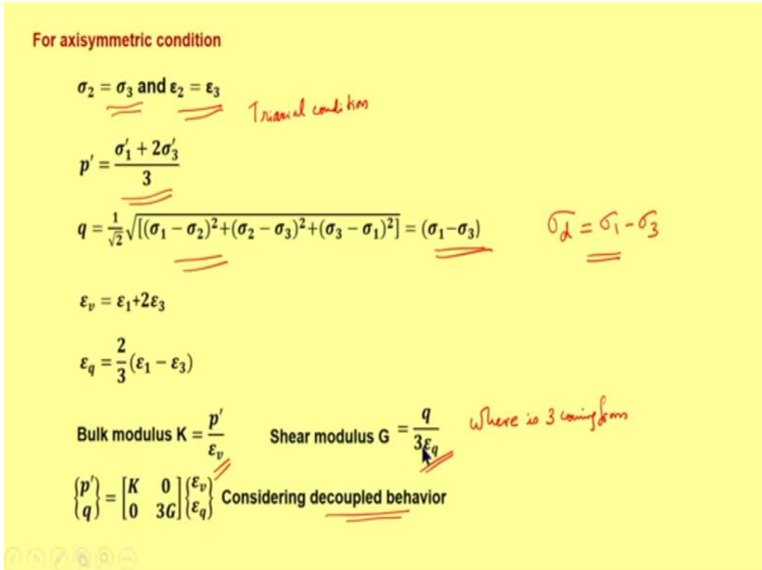

Now, for axisymmetric condition we know the conditions are $\sigma_2 = \sigma_3$ and $\varepsilon_2 = \varepsilon_3$, we just need to substitute this into the previous equations to get

$$p' = \frac{\sigma'_1 + 2\sigma'_3}{3}$$

Please note this expression does the mean stress this is very important and we will be using this quite extensively throughout this course. So, please make a note of this mean stress and this is specifically valid for triaxial condition.

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Where $\sigma_2 = \sigma_3$ and $\varepsilon_2 = \varepsilon_3$. So, p' we have obtained, q will also change, $q = \frac{1}{\sqrt{2}}$ into this = $\sigma_1 - \sigma_3$, you will be very familiar with this particular form of deviatoric stress. Now, when you consider triaxial testing, we know that there is a deviatoric stress which causes shear failure and there we have taken $\sigma_d = \sigma_1 - \sigma_3$ where it comes from, it comes from here from this general expression.

When it satisfies the conditions of access symmetry, then you get $\sigma_d = \sigma_1 - \sigma_3$ which is popular and we know this, because we have studied triaxial test. So, it comes from the So, the genesis is q in this particular form and $\varepsilon_v = \varepsilon_1 + 2\varepsilon_3$ and $\varepsilon_q = 2 / 3\ (\varepsilon_1 - \varepsilon_3)$ your mere substitution gives these expressions. We also know bulk modulus K = p' / ε_v and shear modulus G = q / 3 * ε_q.

Now, please make a note here if you consider bulk modulus it is mean stress upon mean strain or volumetric stress upon volumetric strain gives bulk modulus but in the case of shear modulus G it is deviatoric strain upon 3 ε_q where is 3 coming from again this will be dealt as an assignment problem. But you may just look at it from where it will come. So, once we know bulk modulus and shear modulus in this expression, we can also write the constitutive relationship in this form where

$$\begin{Bmatrix} p' \\ q \end{Bmatrix} = \begin{bmatrix} K & 0 \\ 0 & 3G \end{bmatrix} \begin{Bmatrix} \varepsilon_v \\ \varepsilon_q \end{Bmatrix}$$

We all already we have discussed about the decoupled behavior by which the soil is studied which means to say the volumetric component and the shear components are separated or the deformation components are separated. So, here this means stress deals with the volume q deals with the deformation or the shear behavior and that K, 0, 0, 3G, 3G comes from here, because 3 goes here. So, 3G is q into ε_q ε_v ε_q. So, this is how it is done. The task for you is to understand where this 3 comes from.

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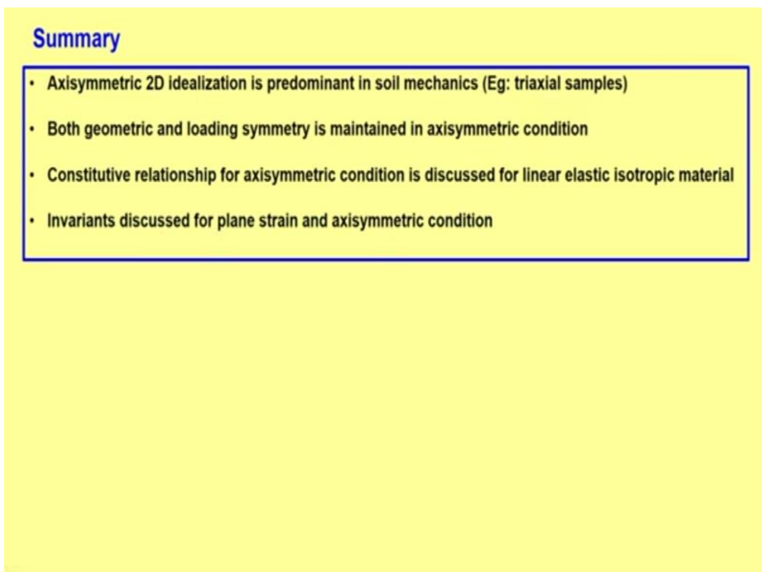

So, to summarize this particular lecture axisymmetric 2D idealization is predominant in soil mechanics. This is very important we need to understand more problems relevant to axisymmetric condition. Both geometric and loading symmetry is maintained in axisymmetric condition. Constitutive relationship for axisymmetric condition is discussed for linear elastic isotropic material. And the invariants are discussed for plane strain and axisymmetric condition.

So, with this, we are almost nearing to the end of module 1 where we have discussed some of the basics of continuum mechanics which may help you to learn further. So that is all from this lecture, we will see in the next lecture. Thank you.