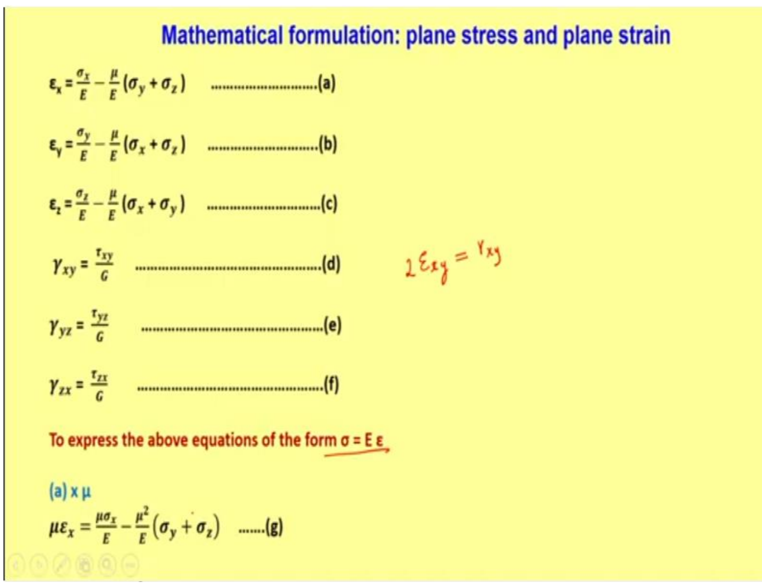
Advanced Soil Mechanics
Prof. Sreedeep Sekharan
Department of Civil Engineering
Indian Institute of Technology – Guwahati

Lecture – 14
Mathematical Formulation Plane Stress Plane Strain

Welcome back, in this lecture, we will see the mathematical formulation for plane strain and plane stress. In the last lecture, we discussed how a given 3D problem can be idealized to a 2D cases. And we understood that it is plane strain, plane stress and axisymmetric condition. So, for a simple linear elastic isotropic case, we will see how the mathematical formulation of plane strain and plane stress would look like.

(Refer Slide Time: 00:58)



So, we will start with the general linear elastic equations where εx is related to σx, σy and σz as given. So, $\varepsilon_x = \sigma_x / E - \mu / E (\sigma_y + \sigma_z)$ this is where we start with a typical linear elastic problem. Similarly, we have ε_y, ε_z the engineering shear strain γ_{xy}, γ_{yz} and γ_{zx}, one can also represent γ_{xy} in terms of the pure shear strain or tensorial shear strain.

Which is given by ε_{xy} the relationship is it is $2\varepsilon_{xy} = \gamma_{xy}$. So, $\gamma_{xy} = 2\varepsilon_{xy}$. So, one can also replace this by this in that case, you will have this particular side it will change to $\gamma_{xy} = 2\varepsilon_{xy}$. So, then it will be $\varepsilon_{xy} = \tau_{xy} / 2G$. So that is the difference. So, we will start with this now, for plane strain or plane stress now, in the case of planes train case, the given condition is the strain is equal to 0.

So, we need to have the inverse of this relationship as well that is stress in terms of strain. So, first let us now this is strain in terms of stresses. So, we will try to find the inverse of this that is stress as a function of strain. So that is what we will do first, now, we need to do some mathematical rearrangement and then find out what is the expression for σ. So, we will start this we need to obtain the equations of the form $\sigma = E\varepsilon$.

So, for that, I strongly suggest all of you to follow these steps and work it out for yourself, because in this it may look a bit abstract. So, when you when we discuss the whole formulation, then it will be easy for you but I strongly suggest all of you work it out on your own. So, first what I will like to do is I will try to eliminate one of these stresses. So, one of these stresses gets eliminated for that what are the different steps to be followed. The first step is multiply equation a by μ. So, you will have $\mu\varepsilon_x = \mu\sigma_x / E - \mu^2 / E (\sigma_y + \sigma_z)$.

(Refer Slide Time: 04:00)

(g) + (b)

$$\mu\varepsilon_x + \varepsilon_y = \frac{\mu\sigma_x}{E} - \frac{\mu^2}{E}(\sigma_y + \sigma_z) + \frac{\sigma_y}{E} - \frac{\mu}{E}(\sigma_x + \sigma_z)$$

$$\mu\varepsilon_x + \varepsilon_y = \frac{\sigma_y}{E} - \frac{\mu\sigma_z}{E} - \frac{\mu^2}{E}(\sigma_y + \sigma_z)$$

$$\mu\varepsilon_x + \varepsilon_y = (1-\mu^2)\frac{\sigma_y}{E} - \frac{\mu\sigma_z}{E}(1+\mu) \quad \text{.........(h)}$$

(g) + (c)

$$\mu\varepsilon_x + \varepsilon_z = \frac{\sigma_z}{E} - \frac{\mu\sigma_y}{E} - \frac{\mu^2}{E}(\sigma_y + \sigma_z)$$

$$\mu\varepsilon_x + \varepsilon_z = (1-\mu^2)\frac{\sigma_z}{E} - \frac{\mu\sigma_y}{E}(1+\mu) \quad \text{.........(i)}$$

$(h) \times \frac{(1-\mu)}{\mu}$

$$(1-\mu)\varepsilon_x + \frac{(1-\mu)}{\mu}\varepsilon_y = (1-\mu^2) \times \frac{(1-\mu)}{\mu} \times \frac{\sigma_y}{E} + \frac{\sigma_z}{E}(1-\mu^2) \quad \text{.........(j)}$$

Then do g + b. So, this equation is g, why should you do g + b? In the process of doing g + b. So, this is g + b we will eliminate σ_x. So that is the whole idea in whatever manner you can eliminate you can do that but this is one particular step g + b will give $\mu\varepsilon_x + \varepsilon_y = \mu\sigma_x / E - \mu^2 / E (\sigma_y + \sigma_z) + \sigma_y / E - \mu / E (\sigma_x + \sigma_z)$. Now, if you simplify this, the left hand side remaining same you will get $\sigma_y / E - \mu\sigma_z / E - \mu^2 / E (\sigma_y + \sigma_z)$.

So, in the process of this, we have got rid of σ_x. So, $\mu\varepsilon_x + \varepsilon_y = (1 - \mu^2)2\ \sigma_y / E$, if you combine these 2, this one and this one, you will get this - $\mu\sigma_z / E\ (1 + \mu)$. So, again if you rearrange this and this, you will get this expression. So, now, you are left with 2 stresses, we need to eliminate again 1. So call this as h, then similarly, you do g + c.

So, g + c once you do you will get $\mu\varepsilon_x + \varepsilon_z = \sigma_z / E - \mu\sigma_y / E - \mu^2 / E\ (\sigma_y + \sigma_z)$ and rearranging, you will get $(1 - \mu^2)\ \sigma_z / E - \mu\sigma_y / E\ (1 + \mu)$, you can see a lot of similarity between h and i. Now, it is very easy for us to again simplify this multiply h * $(1 - \mu) / \mu$, you multiply this particular equation by $(1 - \mu) / \mu$. So, we will be left with $(1 - \mu)$, μ and μ get cancelled off, So, it will be $(1 + \mu)\ \varepsilon_x + (1 - \mu) / \mu\ \varepsilon_y = (1 - \mu^2) * (1 - \mu) / \mu * \sigma_z / E + \sigma_z / E\ (1 - \mu^2)$. So, μ again gets cancelled off from this particular expression. So, call this as j.

(Refer Slide Time: 06:43)

(i) + (j)

$$\varepsilon_x + \frac{(1-\mu)}{\mu}\varepsilon_y + \varepsilon_z = \frac{(1+\mu)}{\mu} \times \frac{\sigma_y}{E}\left[(1-\mu)^2 - \mu^2\right]$$

$$\frac{\mu(\varepsilon_x + \varepsilon_z) + (1-\mu)\varepsilon_y}{\mu} = \frac{(1+\mu)}{\mu} \times \frac{\sigma_y}{E}[1-2\mu]$$

$$\sigma_y = \frac{E}{(1+\mu)(1-2\mu)} \times \left[(1-\mu)\varepsilon_y + \mu(\varepsilon_x + \varepsilon_z)\right] \text{..........A}$$

Similarly,

$$\sigma_x = \frac{E}{(1+\mu)(1-2\mu)} \times \left[(1-\mu)\varepsilon_x + \mu(\varepsilon_y + \varepsilon_z)\right] \text{..........B}$$

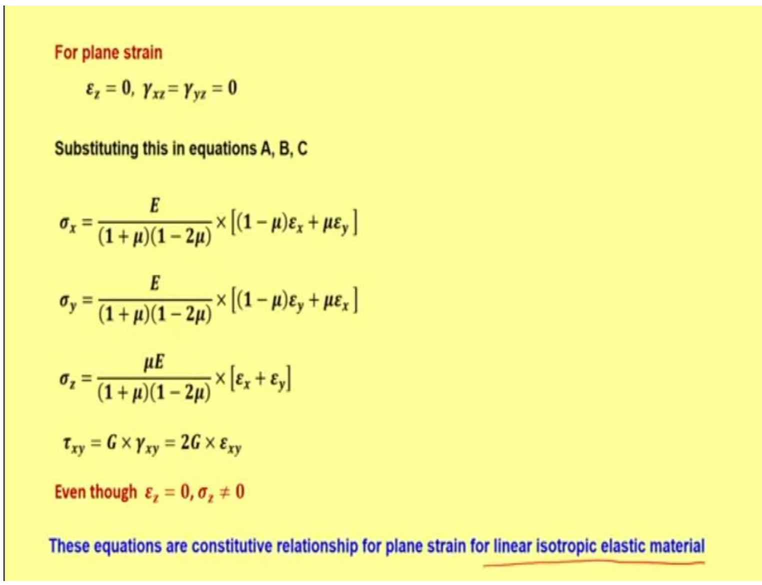
$$\sigma_z = \frac{E}{(1+\mu)(1-2\mu)} \times \left[(1-\mu)\varepsilon_z + \mu(\varepsilon_x + \varepsilon_y)\right] \text{..........C}$$

Now, add i + j because you will have this is j, this is i, one of the stress we will get again eliminated that is why we have multiplied it by $(1 - \mu) / \mu$. So, when you do i + j we will get $\varepsilon_x + (1 - \mu) / \mu\ \varepsilon_y + \varepsilon_z = (1 + \mu) / \mu * \sigma_y / E\ [(1 - \mu)^2 - \mu^2]$ simplifying, we will get $\mu\ (\varepsilon_x + \varepsilon_z)$. So, you can combine these 2 because this multiplied by μ. $\mu\ (\varepsilon_x + \varepsilon_z) + (1 - \mu)\ \varepsilon_y$. The same expression by μ, here again you can do the simplification $(1 + \mu) * \sigma_y / E\ [(1 - 2\mu)]$. So, this μ and μ goes away finally, we will be left with the expression for σ_y. So, in the process σ_z also got eliminated. So, you can see that $\sigma_y = E/\ (1 + \mu)\ (1 - 2\mu) * [(1 - \mu) / \varepsilon_y + \mu(\varepsilon_x + \varepsilon_z)]$. Now, once you got the expression for σ_y it is more or less the same for σ_x and σ_z.

So, this is equation A similarly, you can write σ_x and σ_z the expression which is A, B and C. So, this is the required form for applying the condition for plain strain, you can also do by after applying the condition for plain strain you can simplify. So, we have done it before, so that it becomes very easy to just apply the condition for plain strain. So, you have linear elastic equations in terms of epsilon that is strain in terms of stresses and now, we have inverted this and we have got now stress in terms of strain.

So, these are the equations, you can note that these remains same E/ (1 + μ) (1 – 2μ) and here it is [(1 – μ) / ε_y + μ(ε_x +ε_z)].

(Refer Slide Time: 09:19)

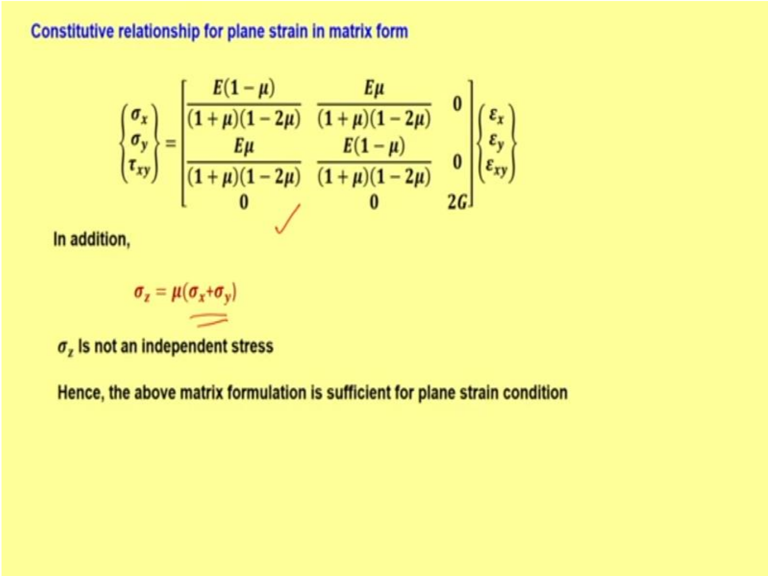


So, now, it is easy, we know for plane strain conditioned strain in one direction is 0. So, ε_z = 0, γ_{xz}, γ_{yz} = 0. If we put this in the expression, we will get that is we are substituting in equations A, B, C we will get σ_x = E/ (1 + μ) (1 – 2μ) * [(1 – μ) ε_x + με_x]. So, σ_z the expression is this is slightly different from the earlier one because here the contribution of σ_z goes away.

So, you are left with μ E/ (1 + μ) (1 – 2μ) * [ε_x + ε_y] and τ_{xy} = G γ_{xy}, earlier we have written γ_{xy} = τ_{xy} / G or it is 2G ε_{xy} where you are replacing γ_{xy} / 2, ε_{xy}. Now, one important point is like even though σ_z = 0, σ_z is not equal to 0 for plane strain condition. So, these equations are known as the constitutive relationship for plane strain for linear isotropic elastic material.

So, constitutive relationship, we know that is the relationship between stress and strain. And it represents the material characteristics as well. Here, this is specific to plane strain condition, we have derived or we have formulated equation the constitutive relationship for plane strain corresponding to a simple linear elastic isotropic material.

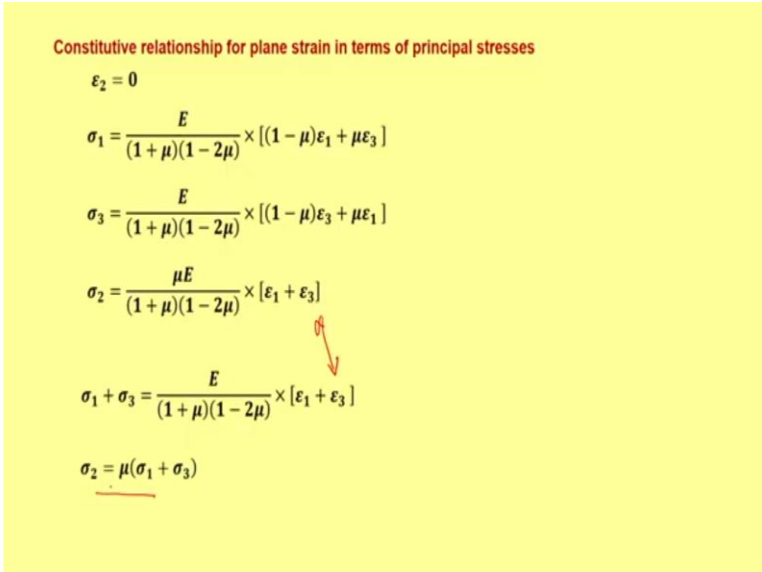
(Refer Slide Time: 11:12)



So, we are just rearranging that the whatever expression that we got for plane strain in matrix form. So, you have σ_x, σ_y, τ_{xy} these are the 3 stresses which are present whatever we have got in the previous slide, we are just rearranging it and you have E (1 – μ)/ (1 + μ) (1 – 2μ), E μ/ (1 + μ) (1 – 2μ). So, these are symmetric the other components are 0 and there is 2G corresponding to τ_{xy}. In addition, we also know $\sigma_z = \mu(\sigma_x + \sigma_y)$.

So, if you substitute this in the expression you will get this particular value as well. So, σ_z is not an independent stress. So, if you just insist on this particular matrix that is sufficient for solving the plane strain problem, because σ_z is again a function of σ_x and σ_y it is not an independent stress. Hence, the above matrix formulation is sufficient for plane strain condition.

(Refer Slide Time: 12:23)

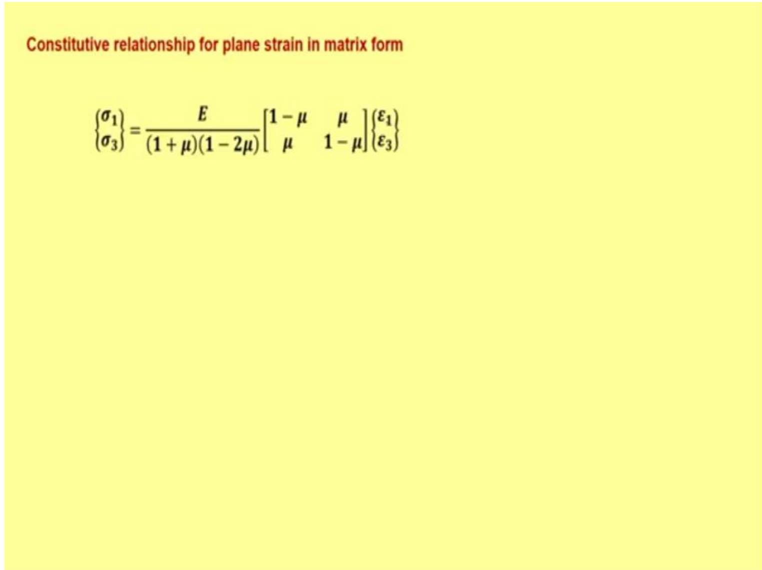


Now, constitutive relationship for plane strain in terms of principle stresses. So, some more will go away. So, you have σ_1= E/ (1 + μ) (1 – 2μ) *[(1 + μ) ε_1 + μ ε_3] what is the difference in earlier formulation and planes principle stresses, the shear stress component goes away that is the only difference replace σ_x, σ_y, σ_z by σ_1, σ_2, σ_3.

So, here σ_3= E/ (1 + μ) (1 – 2μ) *[(1 + μ) ε_3 + μ ε_1] and σ_2= μ E/ (1 + μ) (1 – 2μ) *[ε_1 + ε_3]. And σ_1 + σ_3 if you take σ_1 + σ_3 you will be left with E/ (1 + μ) (1 – 2μ) *[ε_1 + ε_3]. Now, you please compare this equation with σ_2 it is same. So that is how you can very well replace σ_2 equal to only this μ is additional here remaining E/ (1 + μ) (1 – 2μ) *[ε_1 + ε_3] is σ_1 + σ_3.

So, one can always write σ_2= μ (σ_1 + σ_3). Earlier we have returned the same kind of expression. So, it is similar to σ_z= μ (σ_x + σ_y). So, here in terms of principle stresses one can write σ_2= μ (σ_1 + σ_3).

(Refer Slide Time: 14:06)

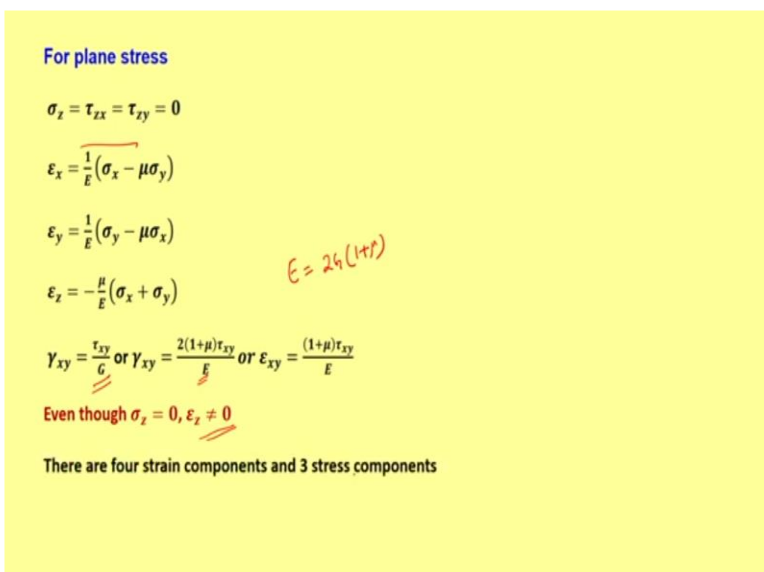


So, constitutive relationship for plane strain in matrix form one can write this is exactly in terms of 2D, see the effect of σ_2 is not considered here, you can write

$$\begin{Bmatrix}\sigma_1\\ \sigma_3\end{Bmatrix} = \frac{E}{(1\ +\ \mu)\ (1 - 2\mu)}\left[\frac{1\ -\ \mu}{\mu} + \frac{\mu}{1\ -\ \mu}\right]\begin{Bmatrix}\varepsilon_1\\ \varepsilon_3\end{Bmatrix}$$

here we have considered only 2 dimensional stresses. So, what we have done is the mathematical formulation for plane strain.

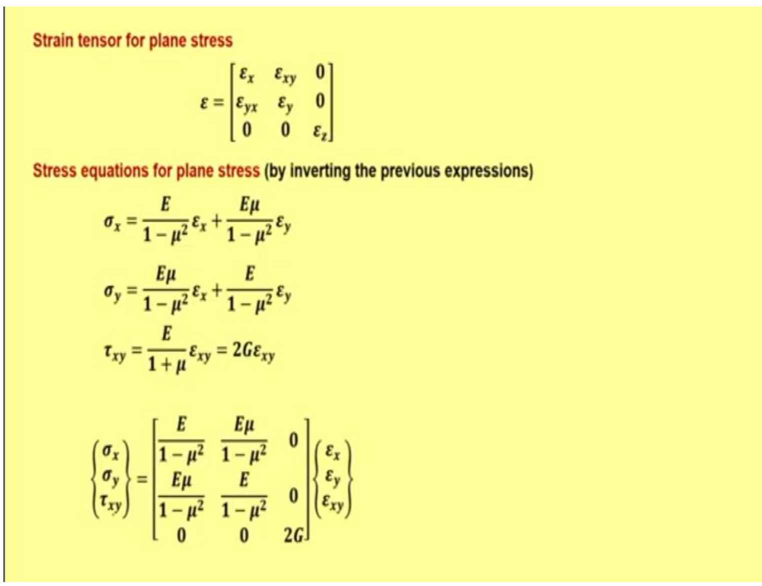
(Refer Slide Time: 14:38)



Now, let us see for plane stress, it is the condition of one Stress. Stress in one direction is 0. So, we will have to use the very first equation of strain in terms of stresses and substitute this particular condition. So, we will see how it will look like. So, you will have ε_x = 1 / E *(σ_x - μ σ_y), ε_y, ε_z and γ_{xy} which is equal to τ_{xy} / G or γ_x = 2 (1 + μ) τ_{xy} / E.

Because you have replaced G by this expression we have the expression E = G (1 + μ). So, by substituting that you have obtained in terms of E. So, here is a mere substitution of σ_z, τ_{zx}, τ_{zy} = 0. So, we are left with these equations. So, here also we have to note that even those σ_z = 0, ε_z not equal to 0, we know this, because we have the contribution of σ_x and σ_y in the other lateral direction that strictly It will also influence the string. So, hence ε_z is not equal to 0. So, there are 4 strain components and 3 stress components altogether.

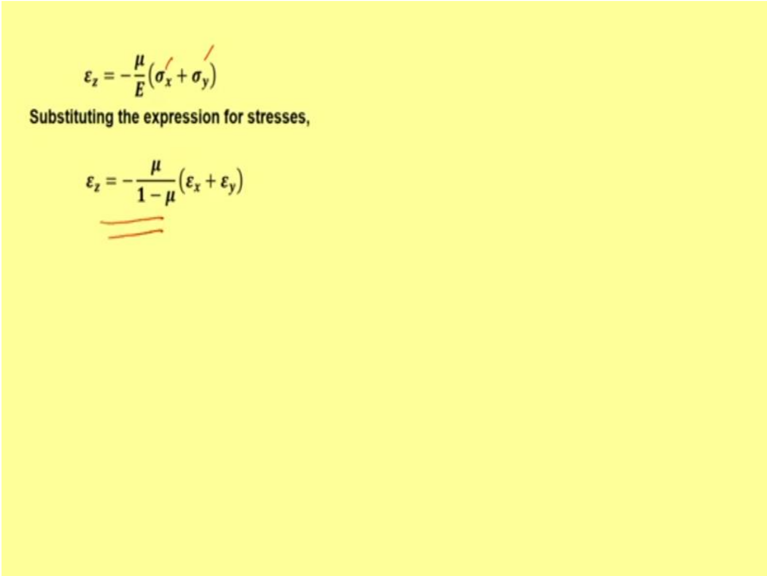
(Refer Slide Time: 16:16)



So, strain tensor for plane stress is denoted in this particular manner, ε_x ε_{xy} and we have ε_z as well. Stress equations for plane stress, by inverting the previous expression, we can get σ_x= E/ (1 – μ^2)* ε_x + E μ/ (1 – μ^2)* ε_y, we are doing the inversion, it is a simple rearrangement. So, one can always try this, you will get σ_x= E/ (1 – μ^2)* ε_x + E μ/ (1 – μ^2)* ε_y and you have σ_y and τ_{xy}.

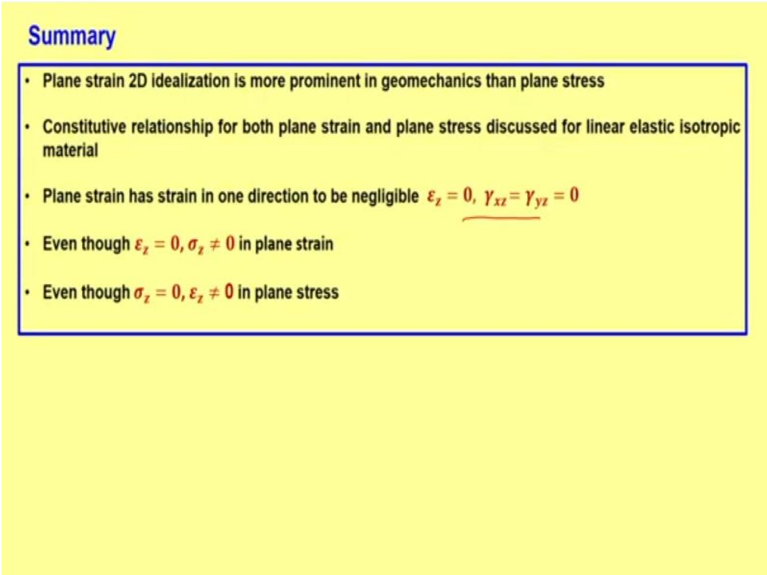
So, this is stress in terms of strain. So, to write the stress tensor, you will have = E/ (1 – μ^2), E μ/ (1 – μ^2) to 0. So, E μ/ (1 – μ^2). So, this is the expression for this is the matrix representation of stress for plane stress condition.

(Refer Slide Time: 17:23)



And $\varepsilon_z = -\mu / E * (\sigma_x + \sigma_y)$ substituting the expression for stresses that a σ_x and σ_y we have seen in the previous slide, if you substitute it, one can get $\varepsilon_z = -\mu / (1 - \mu) * (\varepsilon_x + \varepsilon_y)$.

(Refer Slide Time: 17:46)



So, in this lecture, we have done the mathematical formulation for plane stress and plane strain condition. So, plane strain 2D idealization is more prominent in geo mechanics than plane stress. Constitutive relationship for both plane strain and plane stress discussed for a simple case of linear elastic isotropic material. Plane strain has strain in one direction to be negligible. That is this particular expression, ε_z, γ_{xz}, $\gamma_{yz} = 0$.

And also even though $\varepsilon_z = 0$, σ_z is not equal to 0 in the plane strain condition. Similarly, we have even though $\sigma_z = 0$, ε_z not equal to 0 in the plane stress condition. So, this is all about the mathematical formulation of plane strain and plane stress. In the next lecture, we will see the mathematical formulation for again for the simple case with respect to axisymmetric condition. Thank you.