River Engineering
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Lecture – 06
Linear Momentum Equations

Good morning. Let us start a very interesting subject linear momentum equations which we use for river flow simulations. Today, I will derive linear momentum equations for 3-dimensional forms which is the Navier-Stokes equations as well as also we derive linear momentum equations for river flow. So, try to compare 2 equations and the complexity what is there. I will not go to derive by step by step.

I do encourage all of you to follow any of the fluid mechanics books for getting the step-by-step derivation of the Saint-Venanat equations or Navier-Stokes equations, but here I will highlight you the major assumptions, major concept what we follow to derive Navier-Stokes equations as well as Saint-Venanat equations for the river flow. Try to give you the examples and simulations files.

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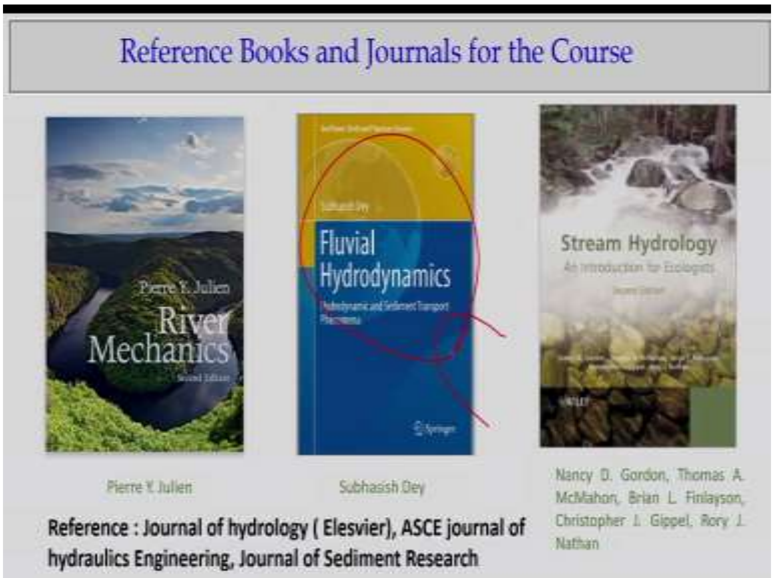

Before that, I just want talking about reference book that mostly we are following on the fluvial hydrodynamics books for these presentations, but you can look it the similar type of derivations are available in any others hydraulics book.

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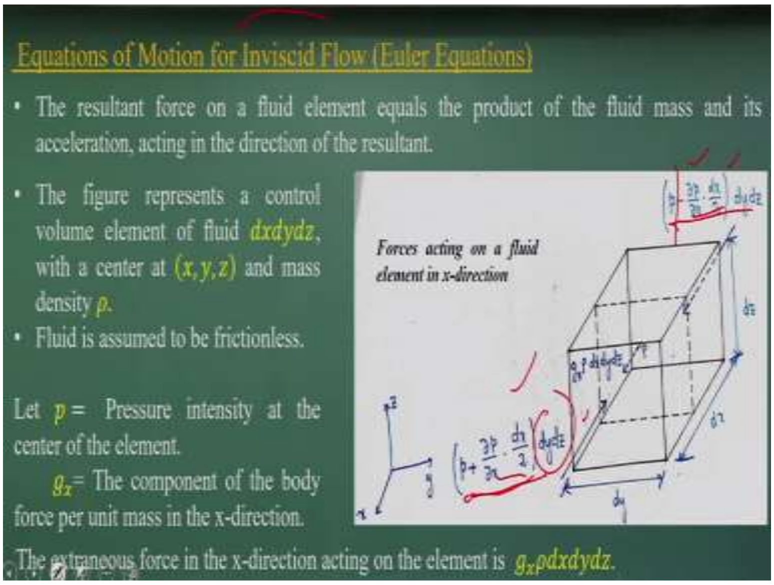

Let us go for the Euler equations that is what I need to summarize you, last class we discussed that when you consider an infinitely small control volumes which having the dimension of dx dy dz and if you consider the pressures and the velocity both the fields are continuous functions in these infinite small domains control volume what could be the mass conservation equations.

Again we are looking at what could be the linear momentum equations for inviscid flow that is what we have derived. For inviscid flow, there is no friction components, there is no viscous stress components. Because of that only you will have the pressure field variations in the fluid control volumes. So we have only pressure variations. There is no frictional part, no viscous effects, no shear stress components.

So only we have the pressure variations and considering these control volumes we try to find out what will be the pressure variations if I am considering these control volumes in a dx/2 distance backward and the forward what could be the pressure at the center point plane the pressure is p. That is what using a Taylor series expansion of first two terms we can define the pressure variations and multiplied with area we get the force, pressure multiplied by area is a force.

So, we know what is the force acting on this plane, similarly what is the force acting in the back plane of these infinitely small control volumes of dx dy dz using Newton’s second law of the motions where the sum of the force is acting is equal to the mass into accelerations which we know it. So, the same things we have applied for these control volumes as it has the pressure variabilities and the velocity variations.

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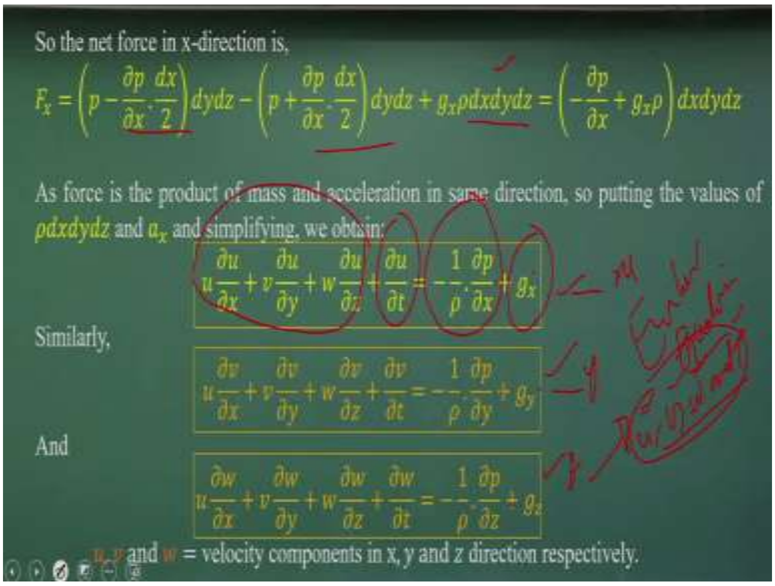

Considering that control volume we got these derivations. If you look at it is very simple derivation. The pressure force and equating with a gravity force component that what is equal to the mass into acceleration. That the accelerations component it has local acceleration and the convective accelerations. More detail about the local acceleration and convective acceleration you can follow any fluid mechanics books, in which fluid kinematics chapter you can follow.

You can try to understand it how interesting to know it what is local acceleration? What is the convective acceleration? That is what is equal to the force due to the pressure in balance that is what per unit mass that is the reasons we have this part, then you have a gravity force component. Since it is a three-dimensional, three-coordinate directions, we can write this the same newtons second law.

So, we can have 3 linear momentum equations, only it is difference between these is that first one for the x direction and then this we have y direction and you have a z direction and if you look at this u, v, w, those things are changing and the pressure gradient in, this is the x direction, this is y direction, this is z direction, acceleration component at x direction, y direction, z direction.

A simple force i.e. mass into acceleration that is what is we have applied it, force is equal to mass into acceleration at a smaller infinite small control volume where we consider the pressures field have the continuous functions, the velocity has a continuous functions in terms of the x, y, z and the t, the space coordinates and the time coordinates. So, considering that we can develop these equations and this equation is known as Euler equation.

But try to look at these equations these are non-linear partial differential equations. This is the non-linear partial differential equation in terms of u, v, w and the phi. So, 4 unknowns we have, so we need to need another equation which is the continuity equation, it is available to us in a 3-dimensional form. So with these 3 equations plus the continuity equations, the set of the equations you can solve it, today's world it is possible using the numerical solutions.

If we solve it, we can get u, v, w and the p variations of any fluid flow problems. That is with an assumption of inviscid flow, the flow regions where the viscosity does not dominate much or the significancy of the viscous stresses are very, very less which can be neglected for a fluid domain, we can apply this Euler equation solvers with having these 3 linear momentum equations and the continuity equations.

That is what we do it the solutions of the Euler equations, but it is a non-linear partial differential equation with having 4 dependent variables u, v, w and p and 4 independent variable x, y, z that is the space coordinates and the t is the time dimension. That is what we do it and nowadays it is easy to solve these Euler equations with a continuity equation to get the pressure field as well as the velocity field.

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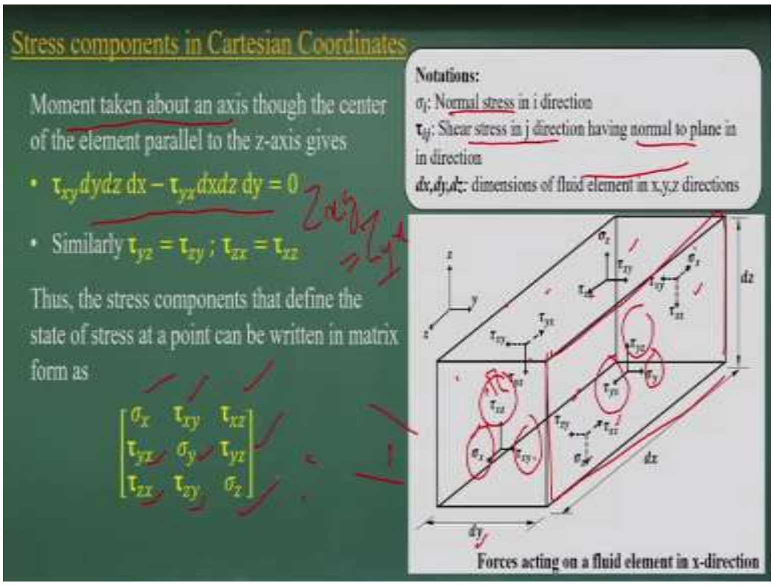

Let us go for the same things, we will extend it to next level. So let us go for the Navier-Stokes equation derivations. As I said it earlier, I am not going for step by step derivations which you can get in any fluid mechanics books. Basic my idea is that what the assumptions we have are and how we are considering it and how we derive the Navier-Stokes equations with these assumptions for the viscous flow.

So, we are now moving from the Euler equations to Navier-Stokes equation, Euler equation does not consider the viscosity part whereas Navier-Stokes equations we consider the viscosity parts. That means in a fluid domain you consider a very, very tiny size of control volumes which is infinitely small control volumes in which you consider it not only the pressure and the velocity, the field variation, continuous functions, also we introduce a stress component.

We consider the control volumes and over this control volume surface we define the stress components. That is what we follow it, the shear stress component in z direction and the normal stress. Now, on the surface if you look at this x y z plane, so I have the plane surface and in this plane surface normal components given σ_x means it is a plane the perpendicular to the x direction that are normal stress what I am getting it here.

Normal stress in the x direction over this plane that is what it is this plane having dy and the dimensions in z is dz, dimensions is the dy dz in this plane which is perpendicular to x that the normal stress I defined as σ_x. How do I define the shear stress components? Shear stress components are defined like τ_{xy}. What does it indicates here? τ_{xy} indicates that it is on the surface which is the perpendicular to x plane.

The perpendicular to the x plane, the first coordinate is perpendicular to the x plane, on that surface it is that. The second subscript indicates for us in which direction it acts, in this case it acts in the y direction. So this is τ_{xy}. It is acting on the surface which is perpendicular to x axis and that axis part what we are looking at that which direction they were.

It can work in z because it is a perpendicular to x plane that can work in the z direction or the y directions. If it is acting in the y direction, we define it as τ_{xy}. If it is acting in the z direction, we define it as τ_{xz}. Please draw a control volume and designate it all these shear stress components that is what is always difficult for students, but please try to do it that draw a small infinite small control volume dx dy dz.

Then scale all these normal stress and the shear stress components like here τ_{xy}, this is τ_{xz}. Same way if you are coming to this plane if you look at that I am coming to this plane which is perpendicular to the y direction. This plane perpendicular is in the y direction, if it is that I have a σ_y, I have a τ_{yx} because this component acting along the x direction, I have τ_{yz}.

Same way I can notate it in each plane we can define the stress components. That is what I am encouraging as a student you please draw the control volumes infinitely small control volumes with sides dx dy dz and considering these notations please sketch all these stress components acting on these all the surfaces. Please visualize that. That means you have to consider control volumes with having 6 faces and each face we want to derive it.

We have to define what is a stress component acting on it, stress into the area is a force. So that is what we are looking at we have to derive as a stress component and we are designating with 9 stress components of σ_x, τ_{xy}, τ_{xz} and σ_y, τ_{yx}, τ_{yz} and σ_z, τ_{zx}, τz_y so that is what is called tensors or stress matrix. So, any small infinitely small control volumes if you consider dx dy dz and that what we can define as a stress component on the all these 6 faces

All the 6 surfaces or faces we can define this shear stress component. This is valid for the solid mechanics, this is valid for the fluid mechanics. So, any small solid object if you take it you can define as a stress field like this or stress tensor like this with having 9 components, but in some of the cases we can take a moment about the axis through the center of the element and I can write this part which indicates is that just I am equating the moment.

If I going through the axis which is the center of the elements I can get force multiplied by any distance, force into distance if I equate this part I am getting $\tau_{xy} = \tau_{yx}$. The same concept I can use it for other faces to find out that $\tau_{yz} = \tau_{zy}$ and $\tau_{zx} = \tau_{xz}$ .So, that means now instead of 9 components of matrix of shear stress, we can consider it only 6 because other 3 are equal to that.

We have now only unknowns of 6 stress components that is what is a simplification, which is what we do it for this infinitely small control volume.

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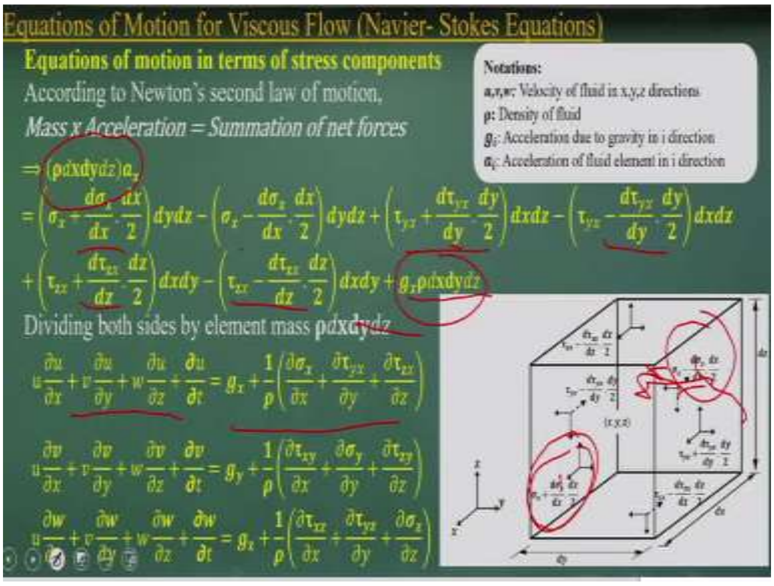

Now if you understand this concept, then it is very easy to derive this Navier-Stokes equation, so because we have considered infinitely small control volumes over that we have defined the stress field. Now we have to define it the stress fields what if we have defined in the 6 components, they also have a continuous function within that. So again, we can follow the Taylor series approximations for different faces what could be the value if at the centroid we know it.

So, that is the reasons what we do it like for example if you look at these figures that I am looking the σ_x variations at $\frac{dx}{2}$ distance on this surface

Then $\sigma_x + \frac{d\sigma_x}{dx}.\frac{dx}{2}$ and $\sigma_x - \frac{d\sigma_x}{dx}.\frac{dx}{2}$ is the approximation from Taylor series

Exactly the same way as we did for u and v components, we are doing same approximations as a σ_x as a function variability in the space and the time that can be defined as what could be the value if I know the centroid value, what will be the value at $\frac{dx}{2}$ positive side or $\frac{dx}{2}$ in the negative side. Just that means I can know it what are the force components are acting on this x directions.

Some are the normal stress component as well as the shear stress components. This is a normal stress multiplied by area, is a force because of normal stress what is the force is acting it, this is the shear stress along this x direction, how it is acting and also the shear stress along this x direction because of the z field how it is coming and then you have the force component because of gravity on these infinitely small control volumes.

That way in ρdVa_x, which is mass multiplied by acceleration, dV is volume of the control volume which is equal to dx dy dz. Let me summarize what we have done it now for the same small control volumes we have trying to write or trying to apply the newtons second laws for that control volumes where the stress distributions we are considering and from that stress this one we are trying to find out what is the force acting along the x direction.

That is what we are doing, that is what will be the mass into accelerations in the x direction that is what we have done it, simple things, the same Newton's second law of motions, nothing else, but we have applied for infinitely small control volumes with stress distributions which is representing a force acting on this surface and also the gravity force as you know it. So, if you look at that if I simplify these equations, I will get this part.

So that means again we are coming back to the equations which will have this local accelerations and convective acceleration terms, then we have the terms in terms of σ_x, τ_{xy}, τ_{xz} along the x direction. Same way, looking these equations you can write what could be the equations for the y direction and the z direction, it is a simple thing. Just look at this equation, you can write it.

If I try to write the equation motions on along the y axis instead of u I can use the v, instead of g_x I will use g_y and similar way corresponding stress components I can write. Once I write the y axis things, I can write for z axis because all are the similar, only we have changing the scalar velocity field and g_x, g_y, g_z components and the stress components. So the basic concept is just to apply newtons second law into these smaller control volumes.

Where we have defined the stress variabilities, stress continuum and on that basis we have derived these components. So, try to understand this and I am encouraging you to write the equations so that you can feel that what it is coming in.

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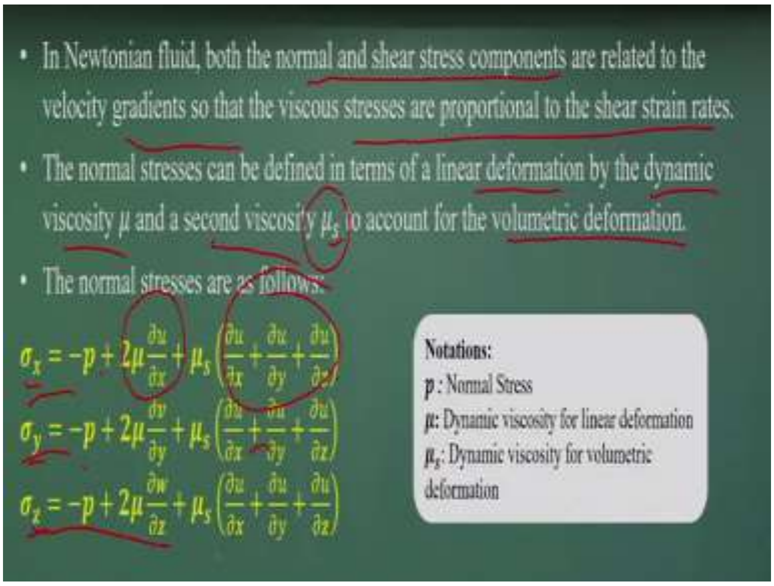

Now if you look at that still we have written the equations of motions in terms of accelerations components, in terms of the stress component, we have not written in terms of velocity gradients. So, to reduce the number of dependent variables we have to write the stress in terms of velocity gradient. As we know from Newtons laws of viscosities that the shear stress is having a linear proportionality to the velocity gradient or the shear strain rate for Newtonian fluid, the fluid which obeys that laws.

So if you look at that part that means again I can derive the stress components that I have in terms of velocity gradient that is what is Newton's laws of the viscosity. So, if you look at that part that is what is saying that this normal and the shear stress components we can relate it to the velocity gradient. Viscous stress are proportional to the shear strain rate which is there in Newtons laws of viscosities.

If you look at that now what is that we have the problem, the problem is not a 1-dimensional problem, this is a 3-dimensional control volume we have considered it. It will have the 2 deformations, one is a linear deformations another will be volumetric deformations. So we need to consider the stress due to the linear deformations and the stress due to the volumetric deformation.

That is what we consider the linear deformations by the dynamic viscosity and the second viscosity we just introduce it, it is not a dynamic viscosity, it is another viscosity which consider the stress formations because of volumetric deformations. Thus, any object when you have the stress field, it will go through linear deformations as well as volumetric deformation. If you consider that things, I am not going more details.

The normal stress in x, y and z directions can be written in terms of pressure, in terms of velocity gradient, in terms of volumetric deformations into the μ_s, μ_s stands for secondary viscosity. Please go through advanced fluid mechanics book to try to understand why do we have the two factors, why do you have a plus and minus, but try to write, try to have a visualization. Now I can derive the normal stress in the x direction is a function of the p, which is the pressure.

The velocity gradient representing the linear deformations and I have the volumetric deformations that is what I can write. Same way I can write for σ_x, σ_y and σ_z. So normal stress components we have written in terms of p, in terms of velocity gradient which have a proportionality constants of μ is a dynamic viscosity and μ_s is the second viscosity which takes care of volumetric deformations, the stress generations due to the volumetric deformations.

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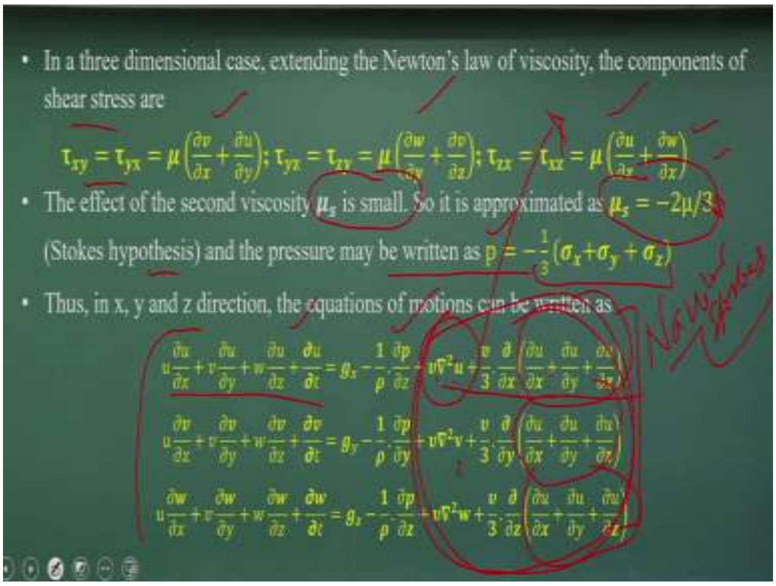

If you are considering that and go for the next level looking that what could be happened to the shear stress component that is what any fluid kinematic chapters if you go through we can easily write it the shear stress component in terms of the velocity gradient with a μ stands for a dynamic viscosity, so we can write it. Then the Stokes hypothesis says that the second viscosity effect is a small, that is not big because your volumetric deformations will not have that much significance.

We can approximate it by $\frac{2}{3}\mu$ that is we can approximate it and similarly with the pressure we can quantify as the average normal stresses and negative indicates it acts the opposite directions of the stress field. So, if you look it that way you have

$$P = 1/3\ (\sigma_x + \sigma_y + \sigma_z)$$

Just trying to locate that if my infinite smaller control volume is tending to the 0 that is what will be the average, normal stress is equal to the pressures and the directions to take care of we have introduced the minus. So, now if you look at that if I put it all the stress components into the basic linear equations of motions, I will get same concept, that means I will get local accelerations, convective accelerations, the pressure terms are given.

Laplace terms are coming it here plus I have a component because of volumetric deformation. That is substituting all the variables and which as I said it earlier I am not going to do a step by step derivations, please look at the step by step variations in any of the books, but let us understand this equations which is the Navier-Stokes equation. Navier-Stokes equations for any fluid flow we can solve it if you look it, but what is there in this equation?

If you look at that it is exactly whatever we have this if I consider dynamic viscosity, the kinematic viscosity is 0 because the non-viscous flow systems it comes out to be Euler equations that is supposed to be. But that means any Euler equations we have accelerations component, we have the component of gravity force component, we have the force per unit area, pressure on balance but we have introduced now the viscous effect that is what you try to understand.

This is what we have considered the viscous effect as the Laplace equations of u and this part, it is a volumetric part. So if you try to understand this, this is the part what we have introduced in the Navier-Stokes equations, additional components to the Euler equations as we are considering the fluid has viscosities and that frictional things if I consider as a linear deformations, as a volumetric deformations, we can make this additional components.

Any of the computational fluid dynamics that they in very detail discuss about these equations, but I just want to summarize it what we are trying to get, we are again getting a non-linear partial differential equation with a more nonlinear component like Laplace second order components are there. So, if you look at this way, the more complicated equations we are getting.

It as nonlinear partial differential equations to solve the fluid flow problems considering the viscous effects and here also we have this 4 dependent variables, what are they? They are velocity u, v, w, velocity scalar components and the pressures but we have the 4 equation, 3 linear momentum equations and 1 continuity equation. So, if considering that we can solve it and today it is possible to solve it.

There lot of solvers are there for the Navier-Stokes equations commercially or many sources are there nowadays you can get a Navier-Stokes equation solvers that is not a big issue to solve these equations, but it looks a very complicated equations in terms of non-linear partial differential equations with 4 dependent variable u, v, w and p, 4 independent variable the space coordinates like x, y, z and the time. So, how to solve these things I am not going to tell most things now.

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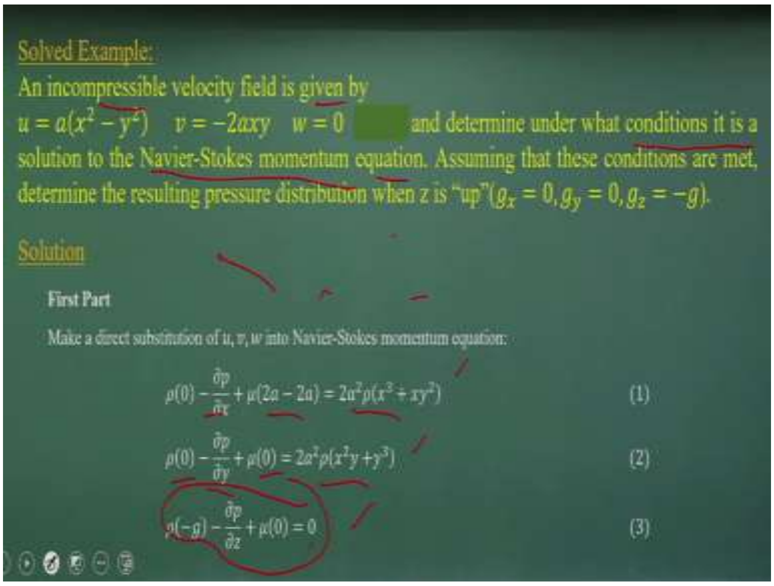

Let us come back to just solve these equations taking example problems which is we are considered from FM White Fluid Mechanics text book just to demonstrate it that we can use this equation to get some analytical solutions for a fluid flow problems. Let us consider the flow is incompressible, density does not vary with time, and velocity field is given.

And if you look at this velocity field you can interpret many things like the velocity field does not have the time components, it has only x and y, there is no z component and the w part is equal to 0, so this have to consider it before solving the problems because when you try to get analytical solutions you try to understand what type of velocity field is given to us and the density field. Density says that is incompressible that means density is not varying.

So, if you are consider that part we have to determine under what conditions, that means we are looking the solutions of the Navier-Stokes equations. If this is the velocity field what is the pressure field we are looking at or does this velocity field satisfy the Navier-Stokes equations that is what we are looking at. So, what we do it like we just try to apply this Navier-Stokes equation and substitutes all the terms here.

All these terms we just compute it in Navier-Stokes equations of the first x and y and z directions. substitutes all these things, this is what we do it first partial derivative, second partial derivative with respect to x, with respect to the y, then we substitute these values and we try to find out that what will be these 3 equations.

If you look at these 3 equations very good you can find out these equations are very simple because we can integrate it. It does not have any component, so we can integrate this, integrations with this we can get it next part.

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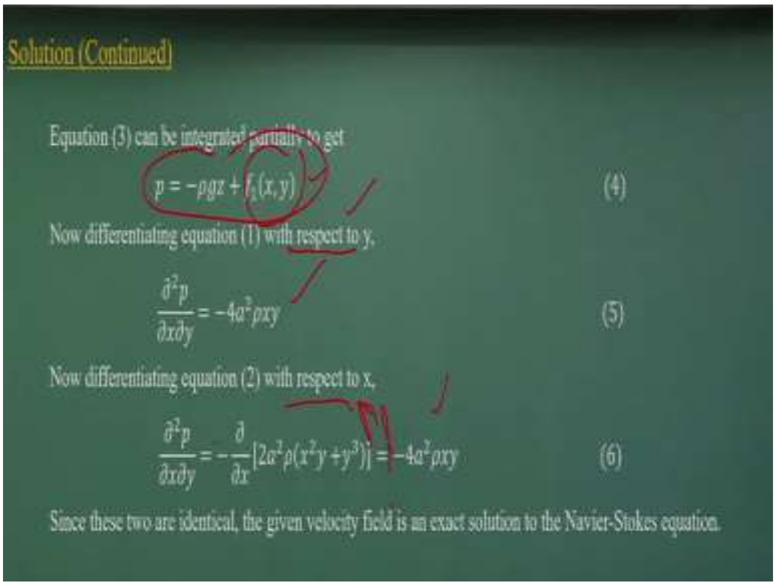

So, that if we integrate it you can see this pressure is varying like this, but there is a constant which may be functions of x and y, we do not know it. Second point is coming okay we have integrated one equation to get it what could be probable solutions of this with an unknown function $f_1(x)$, is it correct one? So what do we do it again we use this equations to check it other two equation do they satisfy ?

That is what we did it, differentiate with respect to y respect to x, then you just compare it and we found it these are also the same values. That means this velocity field what is given satisfies the Navier-Stokes equations. Now we are looking at what could be the pressure field.

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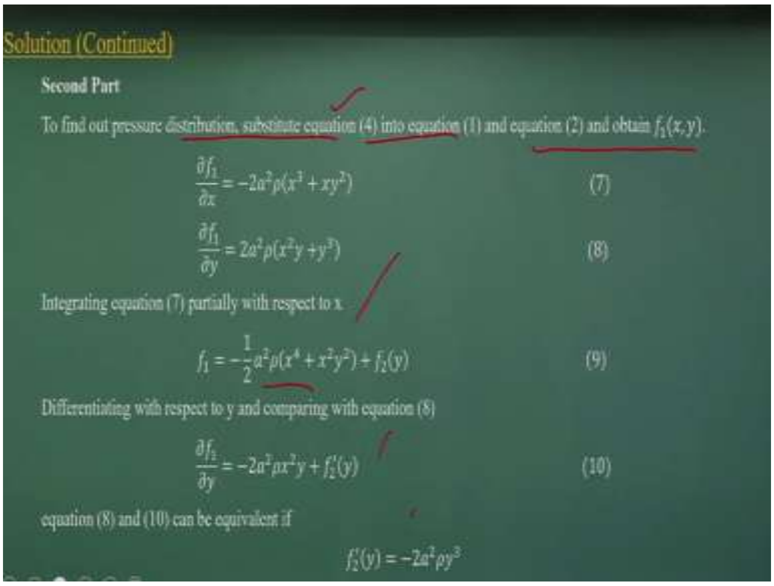

Again, these are all I can say that is just differences in integrations to just find out that what will be the integration constant values. That is the reasons again we substitute equation 4 into equation 1, 2 into the 1 and obtain what will be the $f_1(x)$. So, once you know this $f_1(x)$, then we integrate it and do the differentiations still we get the $f_2(y)$ value.

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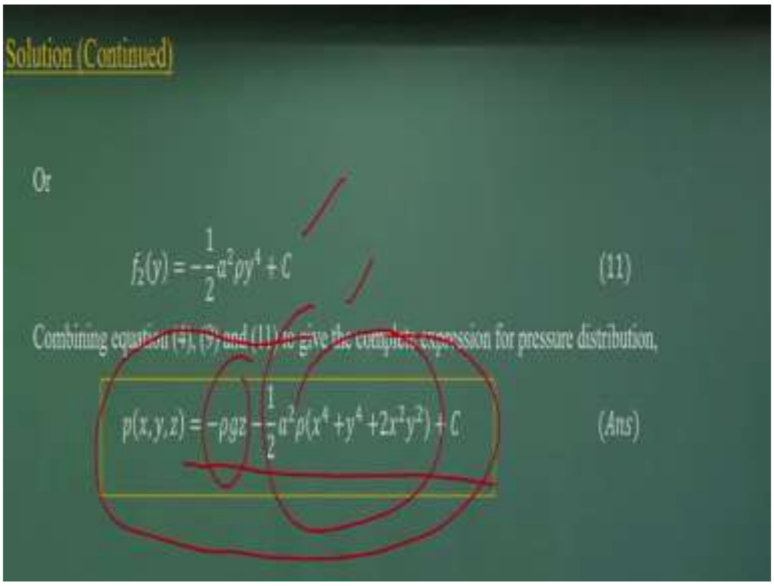

Now we will substitute it which gives us the pressure field. This part we got it. This is the part we got it after integration and differentiation of the previous equations and that is what is giving us the pressure field. We satisfy the velocity field, pressure field for this the flow domains where the Navier-Stokes equation holds good.

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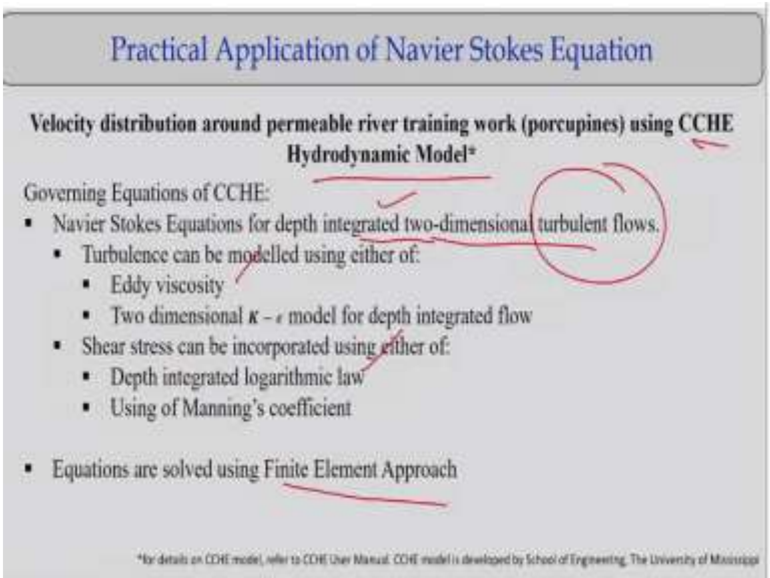

Before completing these things, I just want to show you that a similar way we have applied the Navier-Stokes equations for the fluid flow problems in channels, we know it many of the times we put a small obstruction like a porcupine structures, the river training works and try to know it does it work?, how much it is working in terms of velocity reductions. That is what we try to do with 3-dimensional models, CCHE 3D, 3-dimensions models which has the Navier-Stokes equations with depth integrated.

The turbulent flow which will come later on. We will have a Eddy viscosity, k-€ model and the depth integrated logarithmic law and all and it is a solver of the finite element approach that solutions what gives is just going to give you confidence that we have put the structures here, we try to measure the velocities using the numerical models.

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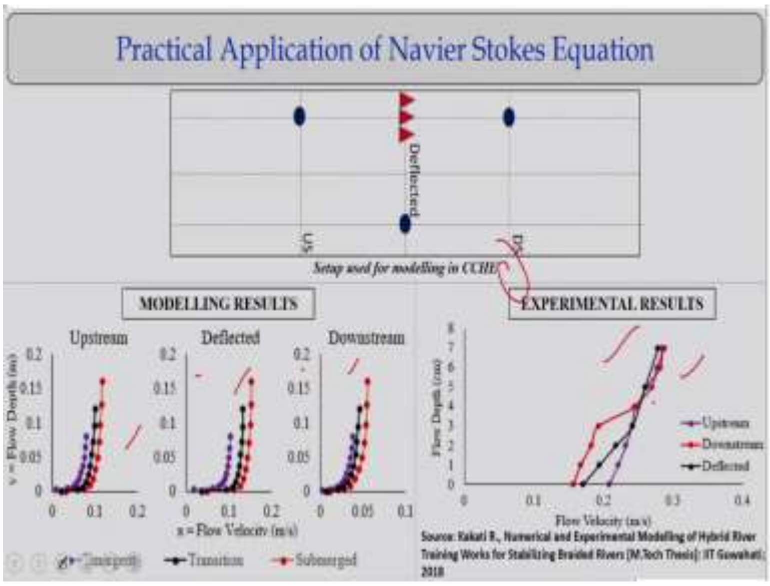

And if you look at this velocity distributions obtained from flume experiments. You can see that there is a velocity reduction because of these obstructions that is what is clearly visible from upstream and the downstream velocity distributions. There are significant velocity reductions are happening in experimental data as well as the numerical modeling of Navier-Stokes equations.

So, now it is possible, there are the models available which solves the Navier-Stokes equations with some approximations like depth integrated concept with turbulent structures, we can get the solutions like what we have done it for a river training work having the porcupine structures, a small obstruction we can see that how much velocity reductions are happening.

Same things you can look at how much velocity reductions happening at the experimental levels. So, those things we can conduct and it is possible nowadays we can do it as we consider as Navier-Stokes equation solvers.

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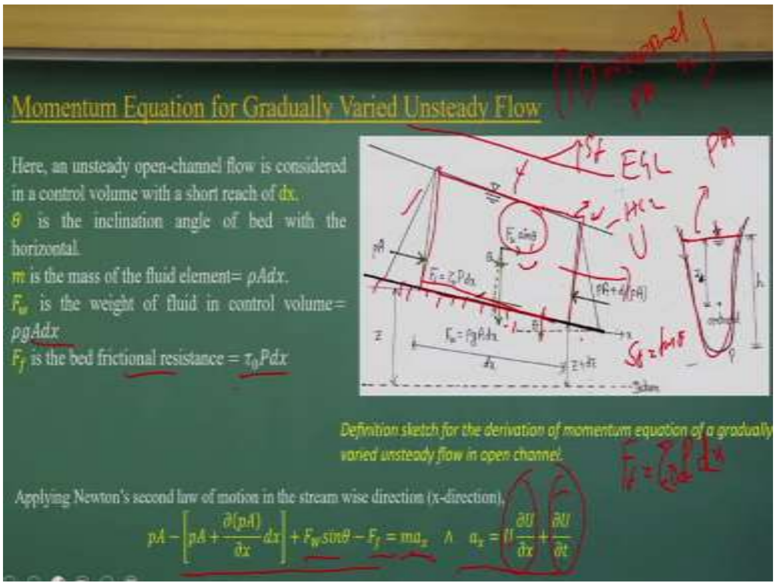

Let us come back to derive the momentum equations for river flow, unsteady gradually varied flow. The variations in the gradually as well as unsteady flow. If you look at a river as we have considered here is the river, this is the bed, this is the free surface and there will be the energy gradient line. This is a hydraulic gradient line and if I consider these channels having making angle Θ.

The channel is making or the river is making angle Θ and this is the free surface, this is the energy gradient line, the slope of energy gradient line we define as S_f and we also define the slope of bed is S_0 which is equal to tanΘ. But this cross-section geometry of the rivers can have a complex like this as we are looking for area integrated concept, it can have a like this, the free surface can be here and Z_c is the centroid point where the pressure acts it.

So, in this field p is having the variations, the pressures acting on this field is having the variations, A also varies. So p multiplied by A also have the variabilities, pressure into area is the force, force also we are considering is a variability from along the x direction. Along the x direction, the force into area varies. If this is my control volume, I can easily interpret it what are the forces acting on this control surface.

One is the force components hydrostatic or hydrodynamic part, here is hydrodynamic force components are there acting on these two surfaces, the force acting on the free surface we can negligible it but there will be a force, the frictional force acting on this. The frictional force acting on the bed surface we can quantify as τ_o P dx if you look at this τ_o stands for the bed shear stress.

The shear force acting on the bed times of the perimeter of the flow, weighted perimeter of the flow and to the dx coordinates. That is what the frictional force acting on this. Frictional force we have designated as a bed shear stress acting on this bed that what we are defining it. Then we are defining the gravity force components. As it is an inclined phase you can find out it has a component in x directions, along this direction.

Along the x direction as a gravity component of F_w sinϴ and F_w cos ϴ components. So that resolving we have done it, that is what is the bed frictional resistance is τ_o times of wetted perimeter and length. F_w is the weight of the fluid in this control volume, so we have considered it and ϴ is inclusive. Now again we have to apply the same Newtons second law okay. Find out the net force difference in the x directions that is a pA and the force acting on this part.

Then the force component due to the gravity, frictional force is equal to mass into acceleration. If you look at that is the same concept we are assuming it, only we have changed the control volume, we have changed the force components. We have a different assumption for that like in a free surface there is no force component. We have considered this hydrodynamic force is a pA which is variability we have considered it.

And we find out the frictional resistance force and all that is what is some of the forces we did it is equal to mass into acceleration that is the component. And the accelerations we can define as only this one directions area average U value that is what we can define as local velocity and the convective velocity in the x direction. So, we have a one-dimensional equation in x direction.

Let us try to conceptualize that we have considered the river and we are just looking at the longitudinal directions what is happening, we are not considering much lateral direction what is happening or the depth variations what is. We are just looking at along the longitudinal directions how these force components are there that is what is the area average velocity we have, we are looking at the local accelerations, we are looking the convective acceleration terms.

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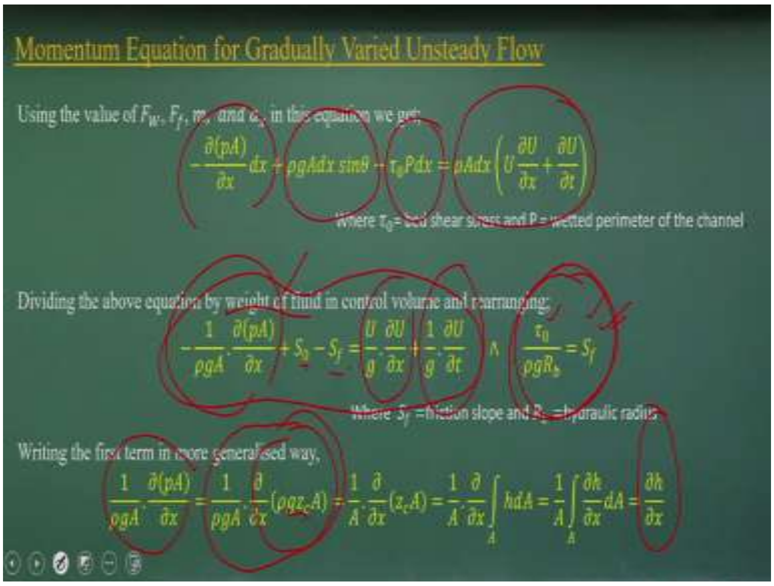

Now we have to simplify these things the equations components. So if you look at this part, we can get these components as well as will get the components the gravity force components and we have the bed shear force components, then we have acceleration components what we have. Just rearrange in these equations, divide by the weight of the fluid, I will get the equations on this part.

So in this equation if you look it that is quite interesting to know what is this value that is why it is quite interesting for us. Otherwise we know the S_0 is a bed slope, S_f is a friction slope, we have a local acceleration component, we have a convective acceleration components and we can define the bed shear stress in terms of friction slopes that is any of the open channel book you can find out.

With a small control volume we can get the relationship between the bed shear stress and the friction slope. If I try to find out what is these values with approximations of hydrostatic pressure distributions with approximation of hydrostatic differences, this component I will get it finally is equal to $\frac{\partial h}{\partial x}$. So that is why it is quite interesting approximations where you have done it, in this case we have considered is a hydrostatic pressure distribution.

Since is a hydrostatic pressure distribution, we can compute what could be the pressure at the centroid of this area and that one if you simplify it finally we are getting this component is nothing else, the gradient of the flow depth, h stands for here the flow depth, gradient of the flow depth that is what will give it as this component. This is the bed slope component, friction slope component, local accelerations and convective accelerations component.

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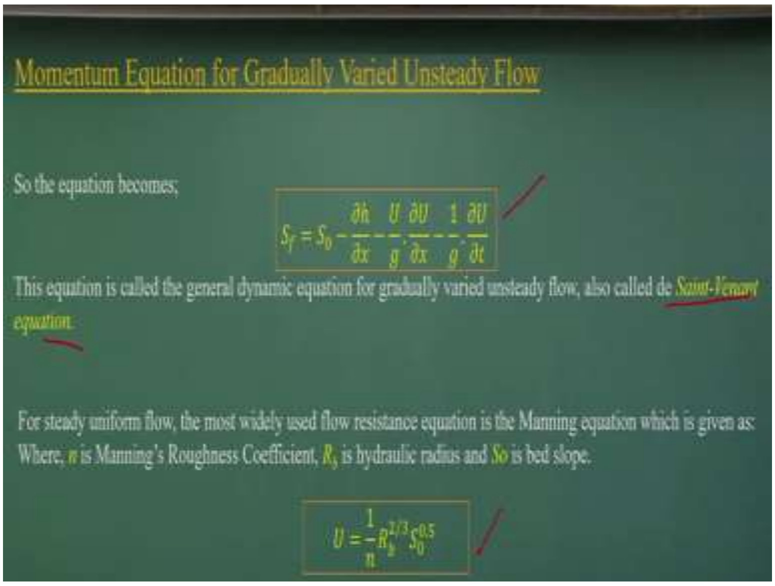

Now if I substitute all these things, I am getting these equations which is known as Saint-Venant equations. It is known as the Saint-Venant equations and this is you know about the Manning equations. More detail about the Saint-Venant equations we are going to discuss in the next class.

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Just to conclude this part I want to say you that we definitely do a lot of things to try to understand the rivers, but try to understand this quote that a good river is nature's life work in song. So just try to understand what do we mean by the rivers, we try to frame the rivers in mathematical forms but still our knowledge is limited for the natural rivers. With this let me conclude this lecture.