Unsaturated Soil Mechanics
Dr. T.V. Bharat
Department of Civil Engineering.
Indian Institute of Technology, Guwahati

Week – 07
Lecture – 21
Analytical Methods for Transient Flow-I

Hello everyone in the previous lecture we discussed Transient flows through horizontal and vertical columns or the transient flows without influence of elevation head and with the influence of elevation head we have considered and we got analytical solutions.

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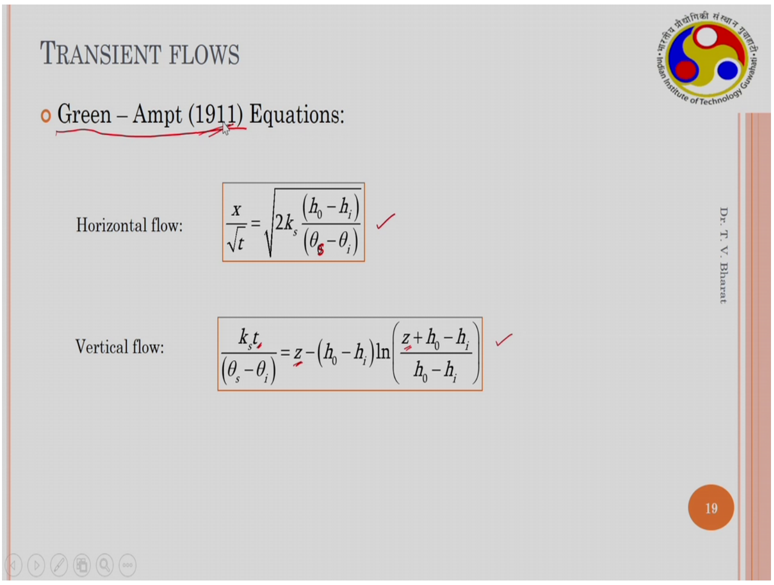

These equations are called Green and Ampt equations. This is given by the two gentlemen Green and Ampt in 1911 very old literature.

This particular equations this is for the horizontal flow the x by square root of time is equals to square root of 2 k s times h naught minus h i by theta naught minus theta s. Here theta naught or theta s. So, here x is the spatial distance and t is a time and k s is the saturated hydraulic conductivity of the soil h naught is positive head you maintain it could be 0 or more than 0. And h i is initial head and theta s is saturated volumetric water content and theta i is initial volumetric water content.

Similarly for the vertical flows we have this expression instead of x you will have the independent variables z here this expression is implicit and you can obtain for a given time what is waiting front location in the vertical direction can be obtained using this particular equation. Here the z is assume to be positive upward. So, when there is a infiltration against the gravity that is taking place then we have used positive value for z and this is a expression for this. Again once again I remind you that this expressions are divided in 1911.

So, when the computation was nearly a very costly and during that time the analytical expressions for the transient flows were derived. This expressions can be used for the coarse grained soils, but not applicable for fine grained soils because shorb wetting front generally you do not find for most of the soils, but that approximation is can be born for coarse grained soils; for some coarse grained soils.

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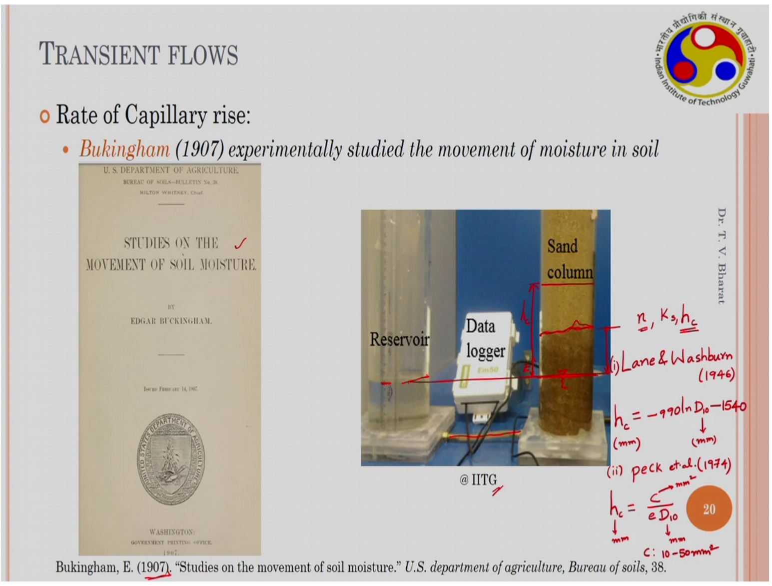

Interestingly capillary rise in soils has been studied slight little earlier by Buckingham in his classical work on studies on the movement of soil moisture which is published in 1907 very old work which is a classical work in soil science literature. In this particular work here done establish the rate of capillary rise he found the capillary rise data with time for several soils.

The experiment involves somewhat like this; this figure is not from this work this is setup from our lab at IITG. And you will have a soil column it could be sand column in this case and you have a reservoir, water reservoir these two are connected through this pipes.

So, you have a water table maintain at particular level. So, this is the water table that is maintained which can be read on the scale. So, when the water table level in the reservoir is this at equilibrium the water table level in the soil also is expected to be here somewhere here.

So, this is the water table and z is measured upward positive from here onwards and here you can see the capillary rise that is taking place and capillary fringe you can be seen the fringe the way the water moves in sand which is called a capillary fringe you can be seen here.

So, the rate at which the movement of water takes place within the soil mass was obtained for different soils where are the porosity values for different and all compassion densities could be different therefore, the porosity varies or the saturated hydraulic conductivity various then the at equilibrium what is a maximum a capillary height that was achieved, that was found which is indicated with h c capillary height.

So, for these three different values for example, the capillary height values for different soils were try to correlated this is the capillary height is correlated with different soil pore size distribution parameters by several researchers afterwards also for example, several works in later on were conducted by lane and wash burn. This is in 1946 he established relationship between h c and detain of soil h c measured in millimetre and here minus 990 log d 10 minus 1540. This is one im-prickle equation he derived by conducting experiments on several soils and here detain is also substituted in mm.

So, h c was predicted using such empirical equations. So, this is one such empirical equation, another empirical equation given by peck et al this is in 1974. So, here the h c is equals to c by e D 10. Here D 10 is substitute in mm and c s units of mm square and h c has units of mm, e is a void ratio and c generally it varies between 10 to 50 mm square.

So, such empirical correlations are available for predicting the capillary rise or capillary height for different coarse grained soils generally, because for fine grained soils it takes enormous time to reach equilibrium and moreover the height will be very significantly high.

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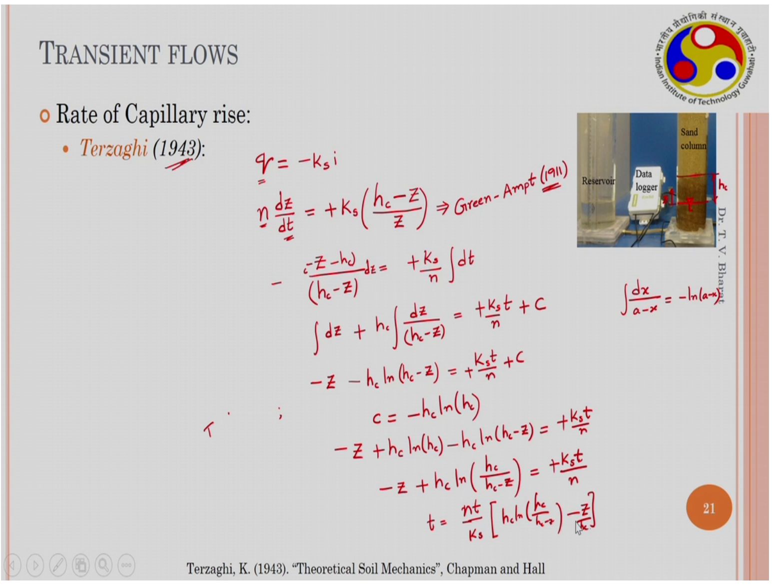

Little afterwards in 1943 Terzaghi's in famous book theoretical soil mechanics provides one solution for rate of capillary rise. The rate at which the capillary rise occurs in soils a simple analytical solution is derived which has some interesting aspect which I will discuss after the derivation.

So, which is a again assumes that the Darcy velocity value it is. So, he uses q equals to minus k s i take also minus k s i and q we substitute the flux or the Darcy velocity as earlier we substitute as n times d z by d t. So, this is velocity and this is porosity of the soil. So, this is equals to minus k s he assumes that the saturated hydraulic conductivity still valid for this particular soil because he also assumes that there is a sharp wetting front exist within the soil when the capillary rise takes place which assumption is similar to green and ampt equations.

So here he assumes the gradient that governs for the flow is h c the capillary height minus z which is a distance for example, this is a water table within the soil mass and z varies upward and which is, now if this is the capillary height. So, this is the capillary height h c minus z which is the head that controls and distance is z.

So, this is assumed to be the gradient for the flow. So, when z equals to 0 there is this gradient becomes infinity that means, it wants to achieve this value and when z is equals to h c that means, it reaches the capillary height then gradient become 0.

So, this is a hydraulic gradient he used, this expression is very one similar to the Green and Ampt equations which were given in 1911 and this is 1943.

So, we can understand that rate of capillary rise equations are derived based on existing green and Ampt model for the transient flows through soils during that time. So, he further he simplifies this expression. So, if we simplified. So, this you get this expression zdz by h c minus z is equal to minus k s pi n integral dt.

So, after rearranging the terms all z terms are taken into one side and time terms are taken into other side and we integrate on both sides then. So, this can be simplified further into if you put minus here and add h c to it I will put d z somewhere here and minus h c then this expression becomes minus first two terms if you used this is minus d z plus h c times d z by h c minus z is equal to minus k s by n this is time t plus constant of integration.

So, this is what results and this is minus z plus the integration of dx by a minus x is minus log a minus x. So, therefore, so, this becomes minus h c log h c minus z there is equals to k s t by n plus c. If we invoke the initial condition that is at time t equal to 0 z is equal to 0 there is no flow there is no rise. So, then if we invoke that the c is equals to minus h c log h c minus z sorry h c.

So, this is the value for c if we substitute we get minus z. Then if you bring this value this side plus h c minus h sorry plus h c log h c minus h c log h c minus z minus k s t by n. So, this can be written as minus z minus of h c. If you take common or plus h c if you take common. So, this simply becomes log h c by h c minus z is equal to minus k s t by n.

So, therefore, the t is equals to n by k s. If I take common h c also then log see if I put k s plus k s here then everything should be fine I think then this is simply. So, this is plus k s then everything is fine then. So, t is equals to minus (Refer Time: 14:33) k n t by k s k s times h c log h c by h c minus z minus z by h c.

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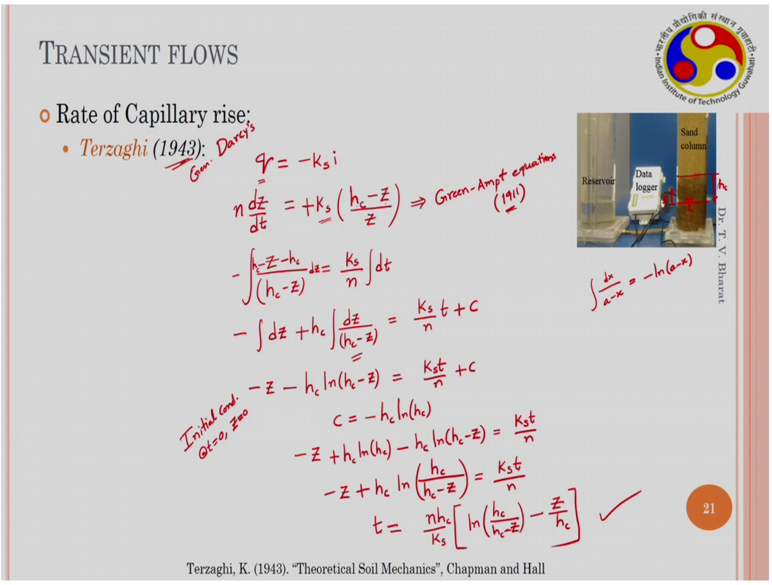

So, it assumes the validity of Darcy's law general Darcy's law that is we consider total head here. So, if we consider that this can be written as minus k s the hydraulic gradient he considers is h c that is capillary height minus z divided by z. This form is a very much similar to the green and Ampt equations which are derived or given in 1911.

If you notice the time interval where the Tezarghis expression is given in 1943 for the capillary rise rate and by then already the Green and Ampt equations are available for transient flows through unsaturated soil. So, therefore, it is very much clear that Terzaghi has considered the grant green and Ampt equations for the derivation of capillary rise rate in unsaturated soils.

So, here this expression q is equals to minus k s time this one if is it written and here the q is similar to the green and Ampt their theta s minus theta i it is written. So, here initially the soil is assumed to be completely dry then theta is equals to 0. Then that is theta s which is nothing, but n times d z by dt there is a velocity term.

So, then this is flux is equals to minus k s again he assumes that the saturated hydraulic conductivity provides because he assumes that there is a wetting front; sharp wetting front exist during the capillary rise which is similar to what shown here. If this is a water table location and above this the z the elevation is shown with z and if this is the capillary height h c then the gradient is considered to be h c minus z that is a head that governs the flow divided by the z at any given location for example, at z equal to 0. So, it has it becomes infinity that means.

So, the gradient is with full capacity it has infinite gradient or large gradients. So, when z is equals to h c that means, when the flow reaches the estimated capillary height. So, because the capillary height is should be known in this particular equation if it reaches the capillary height then the gradient is 0 and the flow stops.

So, that is expression for when we simplify this here we need to observe one thing here as we are writing the flow is taking place from higher head to the lower head because this is the higher head and it is taking towards the lower head this should be positive value and then you will get positive quantities and it is solvable.

Now, when you simplify this particular expression by rearranging the terms on both sides you get z d z by h c minus z and integrate on both sides. So, you get and if I take it on the other side k s by n and integrate the d t term. So, then if we integrate this we can simplify this a priori. So, you can taking negative here and a put negative here and then you can d z I will write here then I can add c h c and deduct h c.

So, then I can simplify this as d z minus d z. If I write the first two terms then this is 1. So, therefore, minus d z and plus h c integral d z by h c minus z which is equals to k s by n t plus an integration constant. So, this expression when you solve here this is minus z plus h c.

So, the integration for this is 1 by a minus x is minus log a minus x. So, therefore, the integration for this is you get negative here minus h c log h c minus z is equal to k s t by n plus c. If we invoke the initial conditions, the initial condition is at time t equal to 0 z is equals to 0 there is no flow, the capillary rise is 0.

So, therefore, c becomes minus h c log h c. So, if I substitute this constant here you get minus z and take the c on other side plus h c log h c minus h c log h c minus z is equal to k s t by n and if I simplified this minus z plus h c log h c by h c minus z is equal to k s t by n. So, this can be further written as t is equals to n by k s. If I take the h c common this is log h c by h c minus z minus z by h c. So, this is the expression given by Terzaghi in 1943 in theoretical solve mechanics book and this expression is an implicit expression describing the rate of rise.

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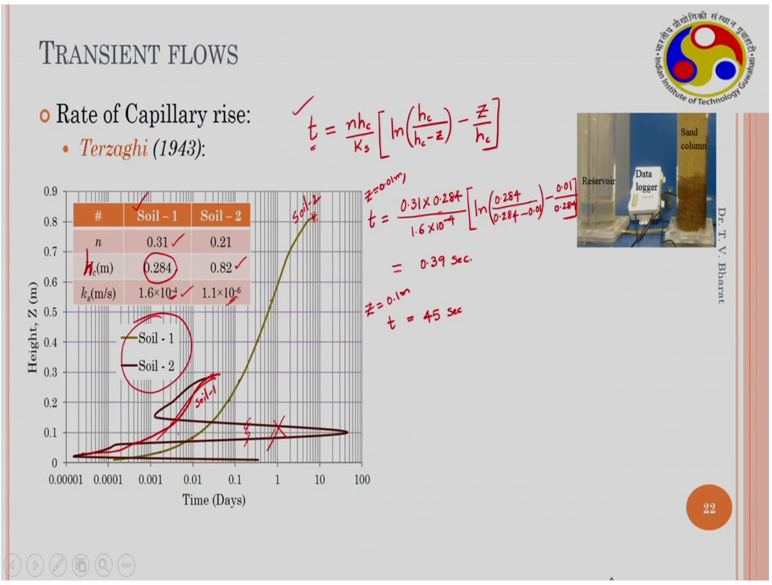

So, there are some problem with this figure here. So, ignore this soil to figure it should be somewhat like this.

So, there is some issue and then you can ignore this soil to data you can only assume the soil one data now let us see. So, the expression for time t is equals to n h c by k s times log h c by h c minus z minus z by h c. Here for a given soil see you take soil one the porosity of the soil is known k s of the soil is known H c should be determined or H c should be known the capillary rise at equilibrium should be known for a given soil then if such information is known one can predict the rate at which the capillary rise takes place through partially saturated soils can be determined.

So, for example, if this data is given this particular data is given we substitute for time equals to at any given says space. Here it is an implicit form therefore, what is a time taken for reaching the capillary rise to given a elevation can be formed. So, for example, the time required to reach 0.1 metre 0.01 metre can be obtained.

So, which is n equals to 0.31 times H c is 0.284 this is substituted in metres k s again substituted in metres metre per second this is 1.6 into 10 to the power minus 4 the data is taken from here times log H c is 0.284 this should have been smaller letter small h c 0.284 divided by 0.284 minus z I said 0.01metre minus 0.01 metre divided by H c 0.284.

So, this comes out to be a very small value 0.39 seconds. So, to reach 0.01 metres it hardly takes 0.39 seconds for this particular soil having these properties. Similarly when z is equals to, here z is equals to 0.01 metre if z is equals to 0.01 metre, but time required is 45 seconds.

So, in this manner we are using different data points at different heights or elevation the time we have found and drawn in this manner. So, this is a data somehow there is error in plotting. So, dimensions. So, this is somehow distorted, but the data should have been like this.

So, as the hydraulic conductivity decreased nearly 2 4. So, this is for soil one right. So, this is for soil two this is for soil one. So, the colour also there is some error in this while copying from a spreadsheet there is a error in the error or the entire figure got distorted. As the capillary height is 0.82 for soil 2. So, this is a data for this and 0.284. So, the data somewhat like this it should come here 0.284.

So, this is soil 1 and this is soil 2 for the soil 2 data the capillary height is 0.82 and the rate is very small because it is taking nearly 8 days to reach equilibrium because the hydraulic conductivity is 10 power minus 6 mitre per second. Here the equilibrium time is very less hardly 0.03 days are nothing, but few seconds because the hydraulic conductivity is 10power minus 4metre per second.

So, with this one can used Tezarghis expression, its very simplest expression to find out the capitalize rate. However, researchers have found that the Terzaghis expression over estimates capitalize rate. This is because especially the limitations in the expression, one of the major limitations is that he assumes that the saturated hydraulic conductivity for the soils he uses saturated hydraulic conductivity for k.

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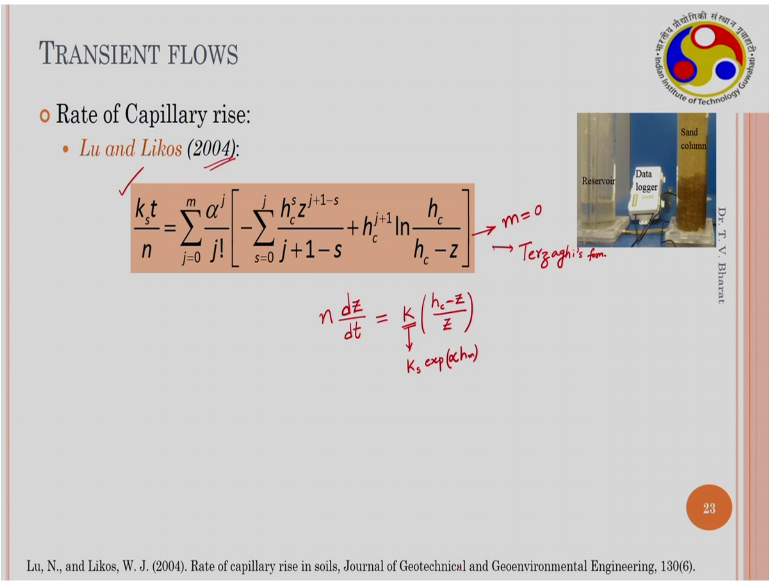

So, which is the major limitation and in 2004 after in 1943 and Lu and Likos in 2004. They have given an expression by modifying the hydraulic conductivity function. So, they uses the same expression here k times h c minus z by z.

So, everything else is same instead of k is equal to k s they use gardeners expression they use expression for k as k s time exponential of alpha h m. So, this is what the expression they use and they got a close form analytical solution which is a k s times t by n which is similar to what we got is equals to sigma j equals to 1 to m alpha power j divide by j factorial times minus sigma s is equal to 0 to j, h c power s into z power j plus 1 minus s divide by j plus 1 minus s plus h c power j plus 1log h c by h c minus z.

This expression will be same as Terzaghis expression if the series form is not considered if j is equal to 0 to m equal to 0 if you substitute m equal to 0 this expression will converge to Terzaghis form. So, the Lu and Likos in his in the paper rate of capillary rise in soils published in the ac journal. They showed that this particular expression is very good match for coarse grained soils for predicting rate of capillary rise. When compared to Terzaghis expression Terzaghi expression over estimates significantly.

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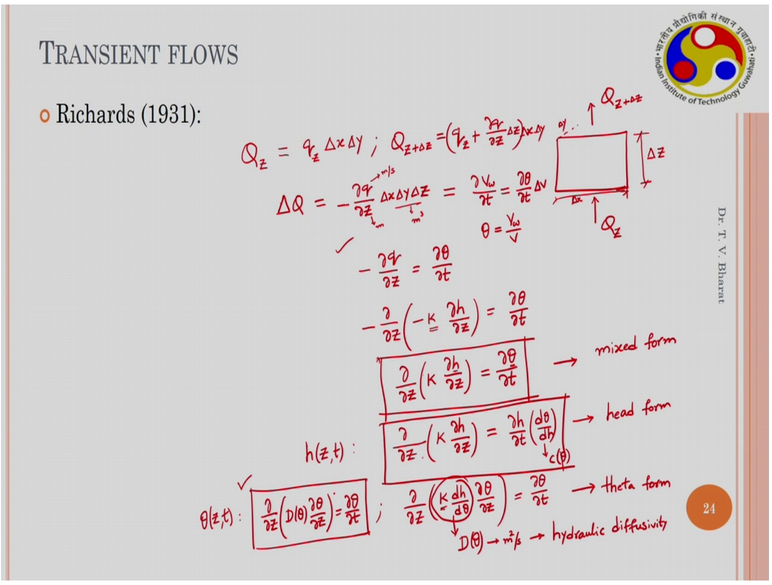

So, far we have seen several preliminary expressions for transient flows through unsaturated soils, but then 1931 researcher called a Richards had given an expression for transient flow through unsaturated soils, this expression is even now it is used.

This is based on Darcy's generalized Darcy's law and combining with mass conservation expression which I discussed earlier also which involves considering unit volume, considering a representative volume when the flow takes place through this representative volume.

So, Q z in the upward direction and Q z plus delta z because this quantity is delta z. And this is a delta x and other direction is a delta y. So, the flux which is this is volumetric flux. So, the volumetric flux which is going into the element is volumetric flow rate Q z of the water which is entering into the element, elemental volume is there is Darcy's velocity Q z times cross sectional area delta x delta y.

Similarly, the amount which is coming out is Q z plus there is some change in the flux dou Q by dou z times delta z times delta x and delta y. So, this change exits there is what is the difference between these two the difference is dou q minus dou q by dou z times delta x delta y delta z.

So, this difference exists if this is a Transient flow. So, in the Transient flows this is equal to what. So, this is Q has units of metre per second. So, this is equal to this quantity is equal to the change in volume of water with time because there is a change in the volume of water, when water is water content increases in this particular element then the volumetric flow rate also changes. So, therefore, in Transient flows this is equal to this if you look at the dimensions this has units of metre per second and this has metre and this is metre Q.

So, this has units of metre cube per second and here volume of water is metre cube and t equal to time. So, dimensionally this form is correct this equation is correct, now this can be written as dou theta by dou t times volume element because theta is defined as volume of water by total volume.

So, this can be written as this. So, therefore, the expression which you get using mass conversation is dou q by dou z is equal to dou theta by dou t using mass conversation; if you use generalize Darcy's law to express q then this becomes dou by dou z of minus k s gradient. So, there is d or dou h by dou z this is a head the hydraulic potential varies in the vertical direction that is why the flow is taking place only in vertical direction. So, that is why the gradient is dou h by dou z is equal to dou theta by dou t.

So, this expression dou by dou z of sorry, this should not be k s this is k. So, this is unsaturated hydraulic conductivity times k times dou h by dou z is equal to dou theta by dou t this expression is Richards expression Richards equation this is very famous equation which is even now used for solving Transient flow problems.

So, this original expression is in mixed form it is called mixed form because the dependent variables on one side the right hand side is theta volumetric water content and the dependent variable on the left hand side is h suction head or looks are present as dependent variables this form is called mixed form. This can be written in the other form as a dou by dou z of k dou h by dou z.

By using chain rule you can write this as dou h by dou t times d theta by d h. So, this quantity is a slope of your soil water (Refer Time: 34:49) of c theta it is called its quantity is called specific moisture capacity. So, this is head form because when you have a smooth soil water characteristic curve function smooth function of soil water characteristic curve the slope can be determined and then if you solve you will get h of z comma t using this expression so, this is called head form.

Similarly you can write this expression as dou by dou z of k this can be modified again using chain rule. This can be written as d h by d theta times dou theta by dou z is equal to dou theta by dou t. So, this particular quantity is called diffusivity, hydraulic diffusivity d theta which may be different from diffusion coefficient which is used in and diffusion this has units of metre square per second.

Because it has units of metre per second and h has units of metre this as units of metre square per second this is called hydraulic diffusivity. So, this expression finally, is dou by dou z of d theta times or I will write it here this expression is dou by dou z of d of theta dou theta by dou z is equal to dou theta by dou t this form if you solve you get theta of z t. So, this is called theta form.

Many researchers have studied all this by using several numerical techniques to solve these expressions and get the solutions, they found that the gives good solution because it does not approximate any mass conservation principles, but head form was found to be not conserving the mass and which gives erroneous results. And this particular form we have utilized earlier for when we discussed multi step out flow technique this is again simplified further into this particular form.

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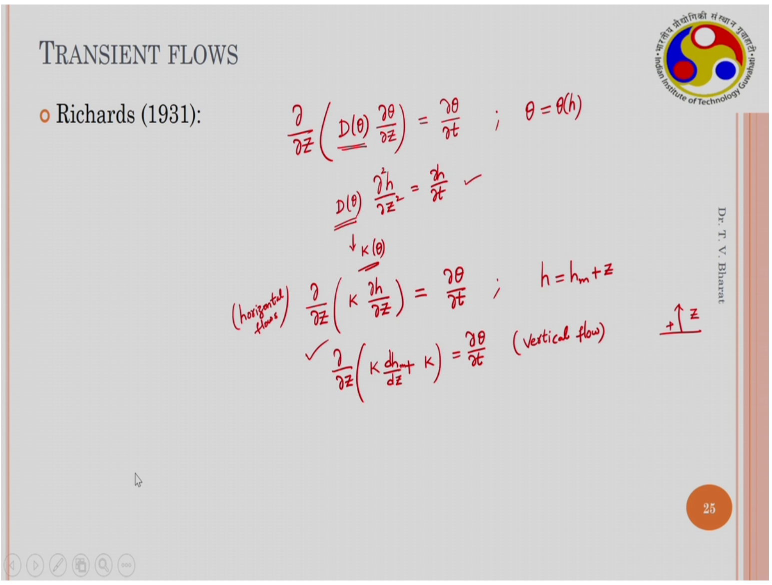

Dou by dou z of D of theta dou sorry dou theta by dou z is equal to dou theta by dou t. This particular form when we simplified, we assumed that the variation of D with respect to z is ignorable then are we assume that the variation of theta with respect to h is linear. Then we got an expression for dou square h by dou z square equal to dou h by dou t. And then Richards provided analytical solution for this particular expression and we estimated d theta and from knowing D theta and solver characteristic we estimated k theta earlier.

So, this Richards expression is a very popular expression popular equation which is used for solving many problems. In this expression dou by dou z of k dou h by dou z is equal to dou theta by dou t is a mixed form we wrote, if the flow is taken place in vertical direction this will be transformed into k here h consist of both metric suction head and plus elevation head.

If we considered upward positive then if there is a infiltration taking place in upward direction or evaporation taking place from initially saturated soil then this expression becomes d h by d z plus k is equal to d dou theta by dou t. If you considered this is negative then you get minus k.

Because when we substitute h is equal to h m plus z you get this expression. So, this is expression for vertical flows. Vertical flows or the influence of gravity is considered, this expression is valid for horizontal flows or the influence of gravity is ignored.

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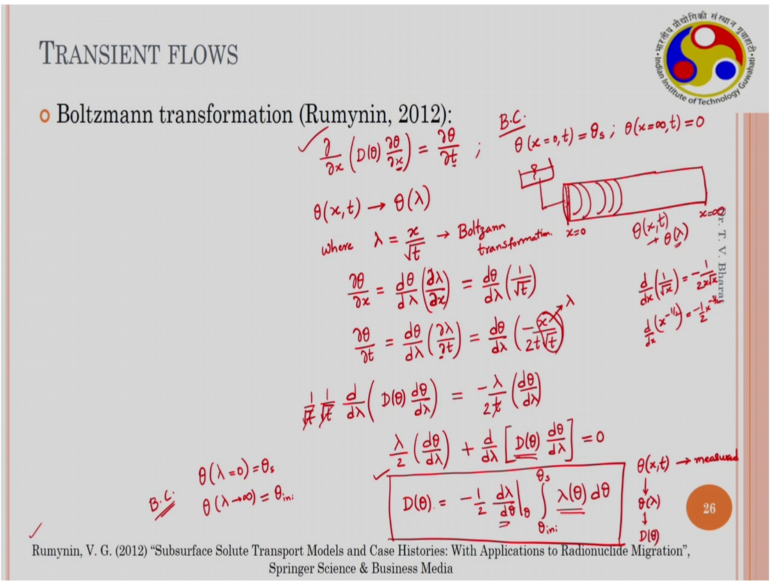

For the expression which I just discussed that is theta form that is dou by dou z of d theta of dou theta by dou z is equal to dou theta by dou t for this particular theta form researchers derived analytical solution earlier so, which is called Boltzmann transformation.

So, this particular derivation can be found in Lu Likos text book or many other resources like Rumynin which is published in 2012 on subsurface solute transport models and case histories with applications to radionuclide migration and many other researchers in many other literature also this particular formulation is provided. And all us most of the soil science literature we will have the derivation for this particular form of expression form of equation using Boltzmann transformation.

So, because this is a partial differential equation the analytical solution for this particular non-linear form of equation is not possible difficult.

So, therefore, if these two dependent variables can be transformed into z and t for example, this is a horizontal flow then it will be better if you write x that means it is a horizontal flow. So, the influence of elevation are gravity is not there. So, for this particular form you have two independent variables like x and t.

If these independent variables can be transformed into one independent variable then this can be this will become simply ordinary differential equation then solutions may be possible. So, this concept is utilized based on Boltzmann transformation. So, here the simple boundary conditions are considered, the boundary conditions are theta x is equal to 0 and time any given time this is equal to theta s and theta at x is equal to infinity at any given time this equal to 0.

So, this means that you have a infinite column that means, very long column long soil column which is connected to you may have some porosity or a something here porous plate and which is connected to a reservoir.

So, when the flow takes place through the soil mass you discard the experiment or you stop the experiment before the flow reaches the other boundary this x is equal to infinity boundary and this x is equal to 0, at time t is equal to any given time. So, theta is equal to theta s all the times.

So, at this point this is a saturated water content this is saturated water content at any given time and the flow does not reach the other boundary. So, now, the independent variables theta of x t is a independent variables the independent variables x t if they can be transformed theta of x t can be transformed to one independent variable that is lambda send a where lambda is equal to x y square root of t this is called Boltzmann transformation. This is called Boltzmann transformation now how to convert this. So, the dou theta by dou x terms.

So, this can be written as dou theta by dou lambda times d lambda d x using chain rule, but as the theta is not dependent on only one single independent variable this will be d instead of dou. So, this is d theta by d lambda and this is solved dou this is dou dou lambda by dou x.

So, if lambda is equal to x plus square root of t this is d theta by d lambda times d lambda by dx is simply 1 by square root of t. Similarly dou theta by dou lambda is equal to d dou t sorry, dou theta by dou t dou theta by dou t is equal to d theta by d lambda times dou lambda by dou t.

So, this is equal to d theta by d lambda times this one is nothing, but minus x by t root t because the derivation of D by dx of 1 by square root of x is minus 1 by 2 x root x and because this is a x power minus half and the derivative is simply minus half and x power minus 3 by 2. This can be written as 1 by 2 x minus 1by 2 x square root of x.

So, similarly here x is t. So, therefore, d theta by d lambda into minus x minus there and you are derivating partial derivative only derivative for t is considered. So, therefore minus x by t square root of t.

So, if you substitute these into the main expressions then dou by dou x gives d by d lambda of 1 by square root of time by t divide lambda d theta and dou theta dou theta by dou x is again d theta by d lambda and 1 by square root of time can be taken out and this expression which is equal to.

So, dou theta by dou t is minus x x by square root of t is again lambda from the Boltzmann transformation. So, this is simply minus lambda by t d theta by d lambda if you get two here. So, as this t gets cancelled on the either side this expression simply becomes lambda by 2 d theta by d lambda plus d by d lambda of d theta d theta by d lambda is equal to 0.

So, this is ordinary deferential equations this can be solved by integrating the boundary conditions will be transform to theta of x is equal to 0 means lambda is equal to 0 this is equal to theta s. And similarly when x is equal to reaches infinity lambda also reaches infinity which approaches to infinity which is equals to theta initial; initial value could be 0 or anything.

So, then if you integrated then you get value for if you integrate twice then you get d theta. So, then d theta can be written as minus half d lambda by d theta at theta theta initial to theta s lambda of theta times d theta. So, this is the solution for the theta form of Richards equations by considered the Boltzmann transformation.

So, essentially when you conduct such an experiment when you obtain the variation of theta with x and time if you get then this can be transformed to this can be written as theta of lambda if that information is known.

So, how lambda varies with theta or theta varies with lambda that is known. So, because a lambda is simply express square root of time. So, for different values of x and different values of time if you get the data of theta then this can be compiled and theta of lambda or lambda of theta can be written, once the data is substituted here you can get d of theta there is a diffusivity.

How diffusivity changes with volumetric water content can be obtained using this particular expression. So, from the measurements of theta of x t is measured from the data from this particular data which is converted to theta of lambda or lambda of theta. So, then from this using this expression d of theta can be obtained the diffusivity changes with volumetric water content.

Thank you.