Unsaturated Soil Mechanics
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Week – 06
Lecture - 16
HCF Modelling

Hello everyone.

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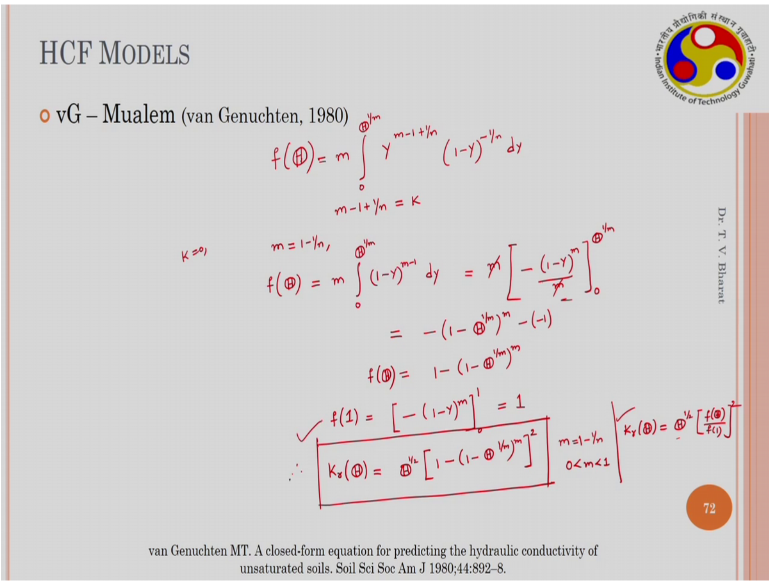

So as we have seen the function f of upper case theta is equals to m times integral 0 to upper case theta power 1 by m, Y power m minus 1 plus 1 by n times 1 minus y power minus 1 by n d y. So, this the particular solution of incomplete beta function and van Genuchten in 1980 felt that, this particular form does not have an analytical solution. So, therefore, he derived particular solutions for this particular equation.

So, to derive particular solutions he assumes m minus 1 plus 1 by n is equals to K and 1 particular solution when K is equals to 0 is when K equals to 0 this is m equals to 1 minus 1 by n. So, substituting K equals to 0 into the above equation results in f of uppercase theta is equals to m times this m minus 1 plus 1 by n is 0. Therefore, y power 0 is 1. So, these results in m times integral 0 to theta uppercase theta power 1 by m times 1 minus y power minus 1 by n d y. So, when we integrate this m times, the integration for this is minus 1 minus y. So, this for inside and for this we substitute instead of 1 minus n m minus 1 here so, because all the parameters are with in terms of m.

So, when we substitute and do the integration this is simply 1 minus y power m divided by m, the integration is the limits are from 0 to uppercase theta power 1 by m. This m gets cancelled and this is nothing, but minus 1 minus uppercase theta power 1 by m, power m and minus when the 0 is substituted this is again minus 1. So, therefore, this is 1 minus 1 minus uppercase theta power 1 by m whole power m. So, this is f of uppercase theta. Now, f of 1 is simply the same thing that is from here minus 1 minus y power m power here, the limits are 0 to 1.

So, when you substitute this is 0 then minus, when you substitute the 0 this is minus 1. So, minus of minus 1 this is 1. So, therefore, the K r relative hydraulic conductivity has a function of uppercase theta is equals to f of uppercase theta by f of 1 times uppercase theta power half, times 1 minus 1 minus uppercase theta power 1 by m power m this whole square. Because, K r of uppercase theta equals to uppercase theta power half times f of uppercase theta by f of 1 whole square.

This is from Mualems model therefore, this is the solution, but; however, there is a condition that m should be equals to 1 minus 1 by n and m should be between 0 and 1. So, these are the conditions for this particular form of equation. Similarly, we can derive for Burdins model.

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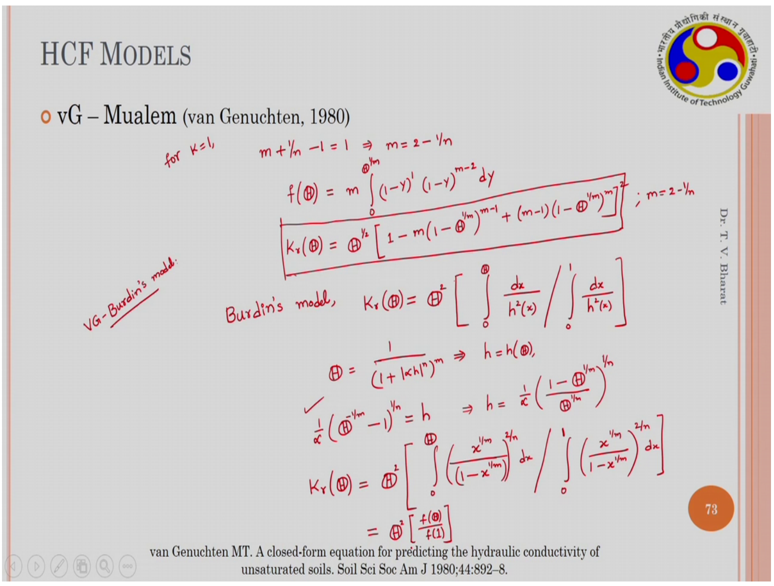

And, similarly another particular solution also can be derived when for k equals to 1. So, this becomes m plus 1 by n minus 1 is equals to 1; that means, m is equals to 2 minus 1 by n. So, when m equals to 2 minus 1 by n is substituted. So, f of big theta are f of uppercase theta is equals to m times integral 0 to uppercase theta power 1 by m, and 1 minus y power it should be m minus 1 plus 1 by n. So, this became 1 times 1 minus y power minus 1 by n for minus 1 by n we can substitute m minus 2 d y.

So, when this is solved using integration by parts we obtain a solution for K r of uppercase theta is equals to theta power half times 1 minus m times 1 minus uppercase theta power 1 by m, power m minus 1 plus m minus 1 times 1 minus uppercase theta power 1 by m power m whole square. This expression for relative hydraulic conductivity in terms of uppercase theta, address normalized volumetric water content. Here the condition is that m should be equals to 2 minus 1 by m. So, this is another particular equation. So, earlier we derived 1 particular equation by assuming K equals to 0 and this is another particular equation by assuming K equals to 1.

Similarly, n number of particular equations can be developed by assuming different values for K. Similarly, using Burdin's model, that is van Genuchten Burdin's model. So, using a Burdin's model K r Burdin model gives that K r of uppercase theta is equals to uppercase theta square times integral 0 to uppercase theta, d x by h square x divided by integral 0 to 1, d x by h square x.

So, by inverting van genuchten equation, the van genuchten equation is big theta equals to 1 by 1 plus alpha h power n whole power m when you invert this expression to write h equals to h of theta. So, this gives theta power 1 by m when you take the m other side and when you put minus 1 by m this is simply 1 plus alpha power n this equals to. So, therefore, minus 1 you can bring the 1 the other side. So, this becomes minus 1 so, then if you take n the other side 1 by n and divide by 1 by alpha is h.

So, then if you substitute for h here then K r of uppercase theta is equals to uppercase theta square times this is 0 to uppercase theta. So, this is d x by h square, anyways you have the same expression on the numerator and denominator the alpha gets cancelled. So, then this can be written as h is 1 by alpha 1 minus big theta power 1 by m by big theta power 1 by m. So, this is 1 by h here. So, are this 1 can be written as 1 minus x power 1 by m whole square or d x I will write it separately here then the this one is x power 1 by m.

So, this whole thing need to be squared, here this 1 by n. So, there is a 1 by n term and square. So, this is 2 by n similarly 0 to 1 x power 1 by m divide by 1 minus x power 1 by m whole power 2 by n d x. So, this expression to solve this can be written as big theta square times f of theta by f of 1, because the functional form is same except the limits are different. So, this is written in this manner.

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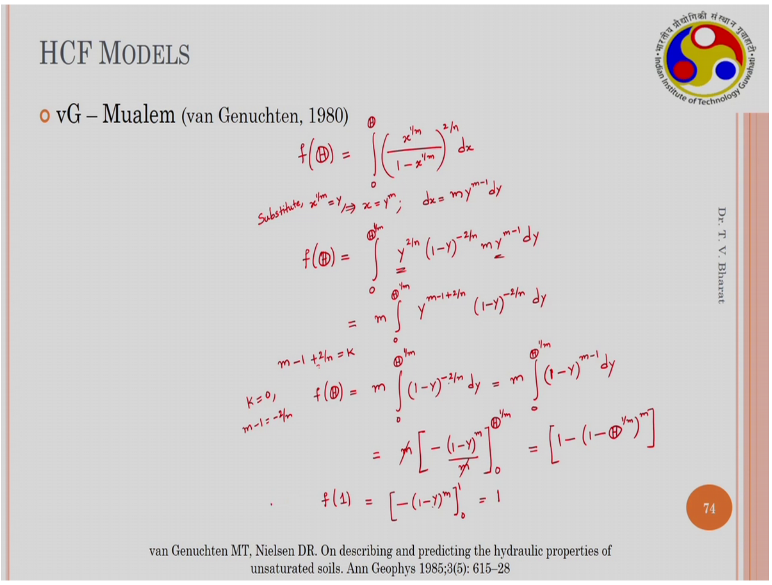

So, now let us solve f of big theta, this is equals to integral 0 to theta x power 1 by m by 1 minus x power m by m whole power 2 by n d x. Similar to the earlier van Genuchten Mualem equation we again substitute x power 1 by m is equals to y, then d x is equals to or this can be written as x equals to y power m, then d x is equals to m y power m minus 1 times d y.

So, this is expression for d x, and here the functional form can be written as 0 to here when this is earlier when you have x this is 0 to x now this is y power m. So, x is substituted to y power m. So, therefore, this y power 1 by m, times here x power 1 by m is y, y power 2 by n times 1 minus y power minus 2 by n. And instead of d x if you substitute m y power m minus 1 d y, then the integration for this can be obtained by simplifying 0 to theta power 1 by m, this y power this 2 terms together m minus 1 plus 2 by n times 1 minus y power minus 2 by n d y.

Again, this is general expression and this is a incomplete beta function. So, van Genuchten felt there is no analytical solutions. So, he derives particular solution by assuming again minus 1 m minus 1 plus 2 by n is equals to K. One particular solution when k equals to 0 is f of theta is equals to m times 0 to theta power 1 by m so, this 1 minus y power minus 2 by m d y.

Here minus 2 by n this is equals to 0. So, m minus 1 is equals to minus 2 by n. So, here this can be written in terms of m 0 to theta power 1 by m 1 minus y power m minus 1 d y. So, integration is m times. So, for 1 minus y power m minus 1 the integration is minus 1 minus y power m divided by m, the limits are from 0 to uppercase theta power 1 by m, m gets cancelled and when you apply the limits this is 1 minus 1 minus uppercase theta power 1 by m power m.

So, this is f of uppercase theta, f of 1 is equals to minus of 1 minus y power m from 0 to 1. So, when 1 is substituted this is 0 and minus when 0 substituted this minus 1 minus minus of minus 1 this is 1.

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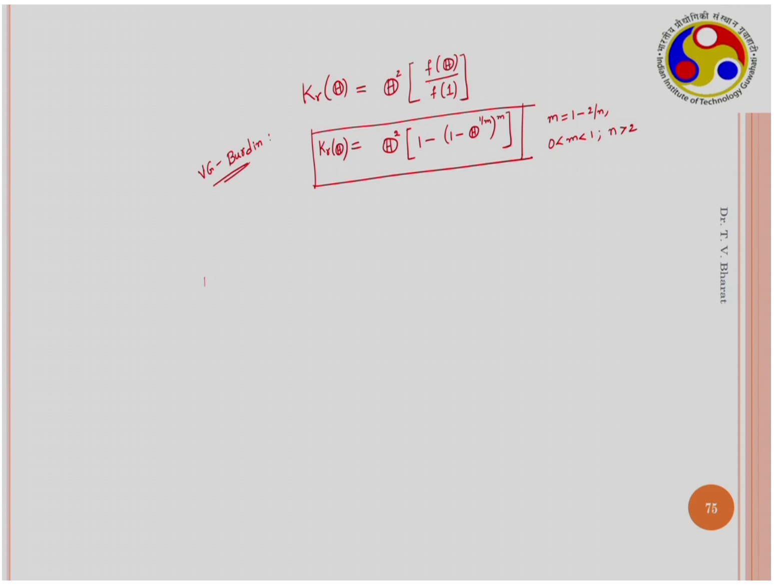

So, when this is simplified, when this is so, when K r is written, K r of theta is equal to theta square times f of theta by f of 1. So, this is equals to uppercase theta square times 1 minus 1 minus uppercase theta power 1 by m whole power m. So, this is K r of theta.

So, this is the analytical solution particular solution for Burdin model van Genuchten Burdin model v r m is equals to 1 minus 2 by m, and m varies between 0 and 1 and n is more than 2. So, these are conditions, this other particular form of v G Burdin model.

Similarly, when k is equals to 1 K is equals to 2 such by substituting several values for several integers for K we get infinite number of particular solutions of the general form, using van Genuchten Burdin similarly van Genuchten Mualem. However, in 1985 after 5 years from this work by Van Genuchten, he realizes that the general form of the equation that is derived has analytical solution.

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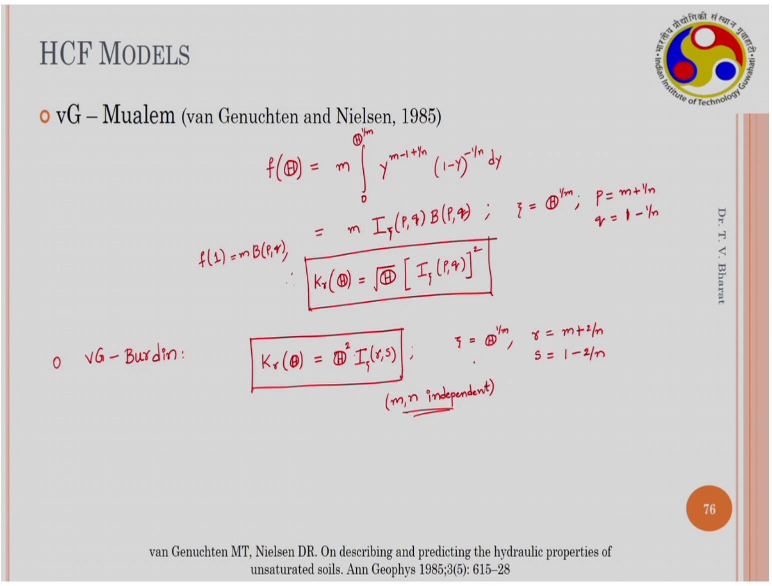

So, when he published work along with Nielsen in 1985, where he provides general analytical solution for these equations. So, for the Van Genuchten Mualem so, the expression is f of uppercase theta is equals to m times integral 0 to uppercase theta power 1 by m, y power m minus 1 plus 1 by n times 1 minus y power minus 1 by n d y.

So, here we assumed m minus 1 plus 1 by n is equals to whether 0 1 etcetera and then we derived analytical solution, but this general form can be expressed as m times incomplete beta function and some zeta parameter of p q and complete beta function B times p q. This can be expressed in this particular form and the incomplete beta function, beta function values are obtained in suitable form that was available during that time, and then using that they got the analytical solution.

Now, in MATLAB and many other programs the incomplete beta function and beta function values can be readily obtained and this can be solved easily. Here this zeta parameter is phi power 1 by m, and here the p is m plus 1 by n and q is 1 minus 1 by n.

So, therefore, the K r of uppercase theta is equals to because f of 1 is m times complete beta function p q. So, therefore, k of theta is equals to square root of uppercase theta times incomplete beta function whole square. So, this is the general solution.

Similarly, using Van Genuchten Burdin, using the same reference van Genuchsten and Nielsen 1985. So, the general expression can be written as k r of uppercase theta is equal to uppercase theta square the incomplete beta function of r s, here this zeta is same uppercase theta power 1 by m. Here r is equals m plus 2 by n and s is equals to 1 minus 2 by n.

So, here there is no restriction between m and n. So, there is no these are not constraint parameters. Earlier whatever the solutions we derived those are particular solutions, where m is related to n. Here m and n are not related m and n are independent in these 2 expressions. So, these are general expressions. Now, we have general solutions for Van Genuchten and Mualem, Van Genuchten Burdin.

Similarly, you have particular solutions by assuming k equals to 0 1 2 and different values. Even though these many number of solutions are available most commonly m and n are dependent equations commonly used for fitting SWCC data. So, this is probably because one reason is that this particular work is not sited that well are there could be another reason is that, when m and n are independent. So, equations over fit the SWCC data and the hydraulic conductivity functions when they are estimated, they deviate significantly from the measured data.

However, when the m and n are restricted even though the error between SWCC data and theoretical data is slightly more than the m and independent case, but the hydraulic conductivity functions are very close to the measured data. So, that is the reason why m and n dependent conditions. So, those are the particular solutions, which we derived earlier. So, these particular solutions are commonly used instead of general solutions.

Thank you.