Unsaturated Soil Mechanics
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Week – 05
Lecture – 15
SWCC and HCF Models

Hello everyone, we were looking at the estimation of hydraulic conductivity function by multi step outflow technique. We have we have also discuss the limitations of the technique and as well as the usefulness of the technique. The technique is useful for the estimation of both soil-water characteristic curve and hydraulic conductivity function to maximum suction value of 1500 kilo Pascal, if you use 15 bar high air entry porous disk.

There are other various techniques people often use on and off such as steady state flow techniques. So, we often use steady state flows and transient flows through saturated porous media for the estimation of saturated hydraulic conductivity. Similar techniques are also used in the literature for the estimation of unsaturated hydraulic conductivity at various suction values by controlling suction.

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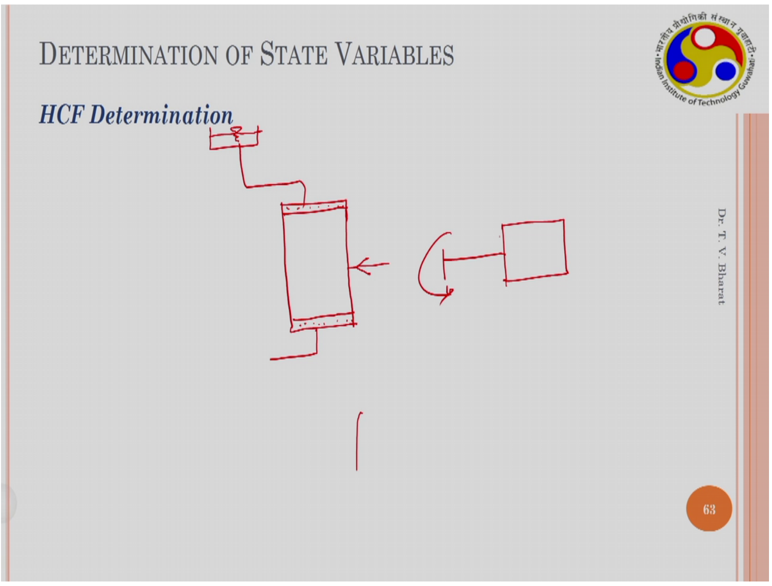

So, you may have a soil column, it is connected to two different reservoirs. So, now here we maintain certain head, and we allow the water to take place go through it. However, in unsaturated hydraulic conductivity technique, the airflow is also considered and high air entry porous disk are used at the top and bottom of the soil column. And here also similar to the axis translation technique or multi step outflow technique, the air pressure is controlled, and water pressure is certain in certain value of water pressure is maintained, and suction is controlled in the soil.

And under this controlled suction, the flow is allowed to take place. Such techniques, we understand that such techniques are applicable for coarse grained soils. So, in that in coarse grained soils, it is easy to control the air pressure and maintain the water flow to take place through the soil. Otherwise if it is a clay soil, you will we will not get appreciable amount of water through the soils at the outlet. Similarly, often centrifuge techniques are also used where you have a soil sample placed in the centrifuge, and which is rotated at certain angular velocity.

So, when it is rotated, a centrifugal force is applied on the soil mass. So, then the water flow will be faster, because compared to the saturated flows. Unsaturated flow through unsaturated soils, the flow rate will be very less. So, here in a centrifuge the flow rates can be improved or increasing the acceleration. So, such stress are also often used in the literature. However, such techniques are also applicable for coarse grained soils, because in fine grained soils, soil will start settling and consolidation takes place.

Therefore, the such techniques are limited for coarse grained soils only. So, therefore hydraulic conductivity function estimation for all soils may not be possible, and even for coarse grained soils the determination in the laboratory is very expensive. So, therefore often the hydraulic conductivity is determined from the soil water characteristic curve itself. Once the soil-water characteristic curve is properly determined in the laboratory by using different techniques, using different ways that we will discuss very soon, the HCFs can be predicted, so such prediction models are very commonly used in the flow through unsaturated soils.

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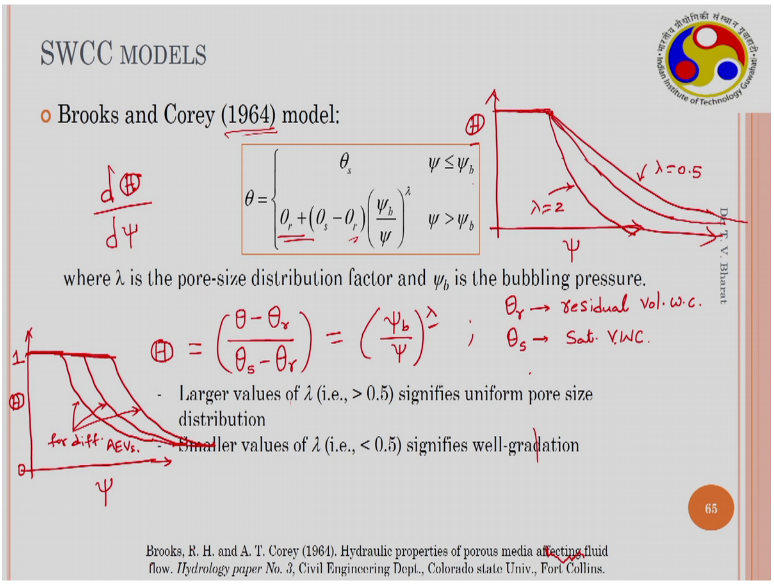

Let us look into several SWCC and HCF models available.

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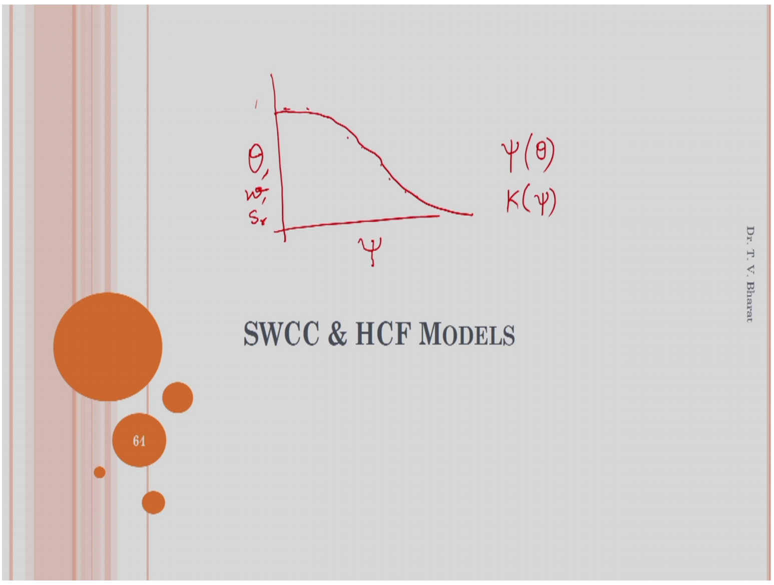

We also required to model the soil-water characteristic curve data, so that is when we get volumetric water content verses suction or so the theta verses psi or gravimetric water content verses psi or degree of saturation verses psi. This data when we obtain in the laboratory such as this data points. We need to obtain a smooth curve such as this curve for the modeling purpose because of the soil-water characteristic curve and hydraulic conductivity function or required input data for the flow through unsaturated soils.

So, therefore we require a smooth functional form between psi and theta, and k verses psi. So, these functions are often required for the hydraulic, for the flow through unsaturated soils. So, therefore we required to model the measure data from the laboratory are in the field. So, we have several SWCC models available in the literature of unsaturated soil mechanics, such as a Brooks Corey model, which is proposed in 1964.

So, this is theta is equals to theta s, when psi is less than or equal to psi b. Psi b is the babbling pressure. So, this is a bubbling pressure, which is nothing but the air entry value, the air entry suction. So, when it is air entry suction or less than air entry suction, theta is equals to theta s. When the cross the suction crosses bubbling pressure, then you have theta r plus theta minus theta s minus theta r times psi b by psi power lambda. Here lambda is a pore size distribution factor and psi b is the bubbling pressure or air entry value.

So, in this equation the lambda values generally vary between zero point small values may be 0.1 or something. And it can go to very large values like 2 or so. Larger values of lambda signifies uniform pore size distribution and smaller values of lambda signifies the well-gradation. So, therefore this equation also can be written as, the bottom one can be written as theta minus theta r by theta s minus theta r is equals to psi b by psi power lambda, so which is nothing but which is referred with the bigger theta, which is called normalized volumetric water content. So, here theta r is the residual water content residual volumetric water content. And theta s is a saturated volumetric water content.

So, therefore when we plot a big theta that is normalized volumetric water content, and on x axis you have psi suction then for different values of air entry value. And for the another air entry, so this is how it varies. The normalized volumetric water content varies from 0 to 1, because when theta approaches saturated value, saturated volumetric water content that is equals to porosity. Then this whole thing will becomes 1, so there is 1. So, as a when the bubbling pressure is when the suction is less than or equal to the air entry value or the bubbling pressure, so theta is equals to theta s, therefore that is equals to 1.

And if psi or suction is more than more than the bubbling pressure, then the normalized volumetric water content decreases from one, so this is how it is varies. These three are for different air entry values for different AEVs. Similarly, for different lambda values the normalized volumetric water content verses psi is varies in this manner. The AEV remains same, so this is how it varies. So, AEV remain same, but lambda values are different.

This is this maybe for lambda equals to 0.5, and this maybe lambda equals to this is for lambda equals to 2. So, lambda equals to 2 are very large values indicate uniform pore size distribution. When you have uniform pore size distribution, you have a steep soil-water characteristic curve. And the residual water content is achieved at very small value of suction, because all the pores are uniform. So, water retention is not that significant, because it immediately loses its water as a suction increases.

On other hand, when you have well-graded soil, you have several different types of pores. You will generally have smaller pores in that particular case. So, therefore the it extends, the residual water content is expected to be existing at very high values of suction. So, therefore the slope is not steep, and it extends to very large values of suction. So, this can be simulated very well using this particular expression given by Brooks and Corey, and which is very simplest possible model, because it has only one parameter to estimate. So, only lambda needs to be estimated, while fitting the data.

So, the bubbling pressure can be usually when you plot in terms of normalized volumetric water content and suction, generally the bubbling pressure can be identified or observed generally that is known. So, therefore only one fitting parameter is available. Often we may not know the residual water content value.

And therefore, in that particular case the two parameters need to be estimated from the data. So, this is very simple technique that is advantage of this particular model. However, the major disadvantage is that the discontinuity at air entry value, which has a discontinuity. It reaches 1 up to the air entry vale and beyond that it decreases, here there is a discontinuity. If you take the slope d theta by d psi d theta by d psi, then this value is not defined at this particular point. So, therefore you cannot obtain a smooth function using Brooks Corey method.

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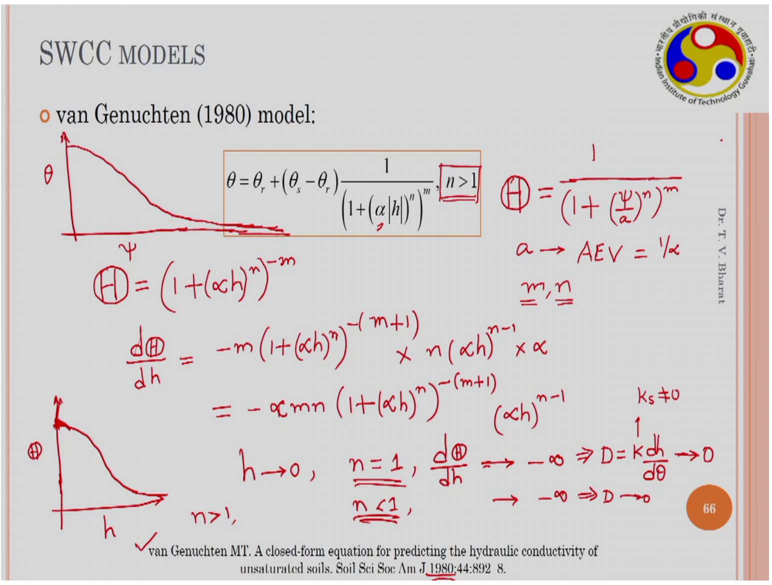

And we have another method called van Genuchten model, which is very popular for the estimation of soil-water characteristic curve from the measured data of theta verses psi. So, the expression is theta is equals to theta r residual water content plus theta s minus theta r similar to the Brooks Corey model times 1 over 1 plus alpha h whole power n, so here n should be more than 1. So, here this mod indicates the h should be positive value should be substituted here or alpha h should be positive. And because the expression is in terms of matrix suction head and matrix suction head is negative value, so you will get complex numbers, if n is real number.

So, this is normalized volumetric water content is equals to 1 by 1 plus alpha, often this is written as psi by a whole power n, and this whole power m. Often this is written in this particular form in our geotechnical engineering. This initial expression was available in the soil science literature. When it is used in geotechnical engineering, we are acquainted or we often use a soil water potential or suction directly. Therefore, the expression is modified into this, here a represents air entry value air entry value, which is 1 over alpha when you compare with this expression. And a is related to air entry value, but it is not equal to the air entry value.

So, therefore, this is also a fitting parameter. Often it is shown on several soils and using the modeling also that a is not equal to the air entry value, which is related to air entry value. When you have a large air entry value and a value is higher. When you have a smaller air entry value for a given soil, even a is small. So, qualitatively these two can be related, however there they are not quantitatively equal. So, the other parameters are m and n. So, n is related to the pore size distribution of the soil, m controls overall symmetry of the soil-water characteristic curve. So, when theta r is also not known, then you have four parameters to estimate or determine. So, we will see how generally these this parameters are determined from the laboratory estimated data of theta verses psi.

Theta s generally known because when you conduct a test initially at slurry state or whatever the state the soil is in at fully saturated state the porosity of the soil is known. Knowing the density of the soil, and water content one can estimate the dry density of soil from that one can estimate the void ratio. When void ratio is known that is, it can related to the porosity. And porosity at fully saturated state is the volumetric water content at saturated or saturated volumetric water content, so that is how one can estimate the theta s in the test.

However, theta r estimation is little difficult. And sometimes, you can use theta r is equals to as small as possible for clays, say 0.01 or something that people often use. And here, the n value should always be more than 1. So, this is a constraint coming from the equation. So, let us try to understand, why n should be more than 1. If I write in terms of alpha h only and h is used in a positive terms, then this is the expression that I can derive.

So, the normalized volumetric water content can be written as 1 plus alpha h power n whole power minus m. When you differentiate this expression dh, which is a minus m times 1 plus alpha h whole power n power minus m minus 1, therefore I can write it as minus m plus 1 and times n alpha h power n minus 1 times alpha, which can be written as minus alpha m n times 1 plus alpha h power n whole power minus m plus 1 times alpha h power n minus 1.

If you see this expression as h approaches 0, and if you consider, n value to be equals to 1. So, this slope d theta by dh would approach minus infinity. So, the slope of the soil-water characteristic curve, if this is a soil-water characteristic curve. If this a soil- water characteristic curve at this point, where h is very close to 0. So, the slope of the equation is approaches minus infinity, so that means, this will never go to 0, but it approaches to minus infinity, so because of which what will happen is the diffusivity.

Diffusivity is defined as K times dh by d theta approaches 0, so which is not physically correct. The diffusivity becomes 0, as K approaches Ks, which approached Ks, which is not equal to 0. So, therefore diffusivity should become 0. So, this happens, because when you consider n equals to 1, and similarly when n equals to less than 1 also, when h approaches 0. So, this whole expression approaches minus infinity. So, therefore diffusivity approaches 0. If n is greater than 1, then this whole expression e is well defined, therefore generally the n is restricted to be more than 1 in this particular model.

So, van Genuchten model is very well received and the work by van Genuchten at 1980. So, this particular work or journal paper received more than 20000 Google scholar citations. So, this is this work is very very well received. And even till date, the van Genuchten model is widely used for you know representing the soil-water characteristic curve data. And often used for in the modeling of unsaturated flows.

The major advantage of van Genuchten model, when compared with Brooks Corey model is, it can provide a smooth SWCC curve. So, there is no you know it is a very smooth function or there is no discontinuity anywhere. However, the drawback with van Genuchten model is that so the volumetric water content can never goes to 0 at any given suction. If you look at the theta versus psi, this decreases, and because of the nature of the equation asymptote, this will never approach to 0. So, this is a major issue using van Genuchten model.

All these models are very simple, one can generate different curves by varying the model parameters like a, m, n. You can assume some values for these parameters and one can play with this on spreadsheet. And you will understand, how each parameter influences, the nature of the SWCC curve. So, you would see that, this would never becomes 0. It only approaches 0 at any given suction value, which is physically not correct, because thermodynamic point of view, this has been observed that when suction value is at 10 power 6 kilopascal that is 1000 mega Pascal, the water content should become 0. However, use in van Genuchten model that does not happen.

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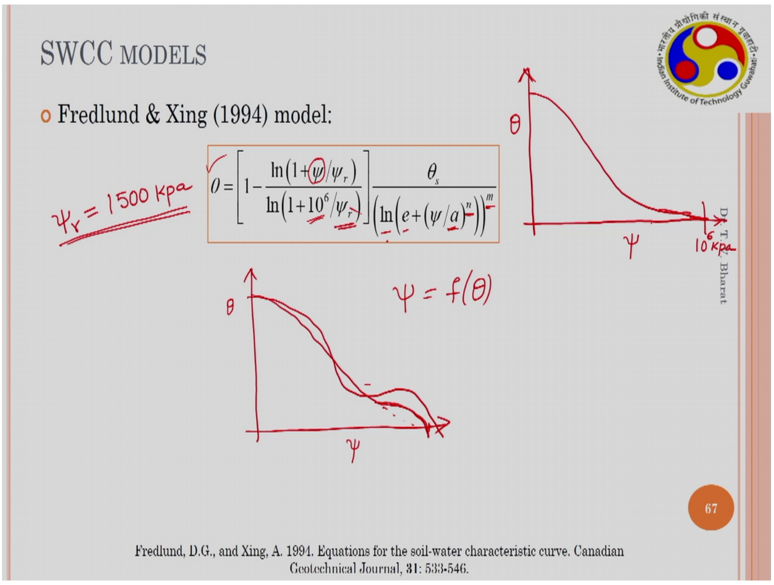

So, we there is another popular model proposed by Fredlund along with his co-investigator Xing proposed another SWCC model in 1994. In this model, the expression is theta equals to 1 minus log 1 plus psi by psi r divided by log 1 plus 10 power 6 by psi r times theta s divided by log e that is exponential of 1 plus psi by a whole power n whole power m. So, this is kind of similar to your van Genuchten equation.

However, you have log here and exponential form here exp of one here. And you have one coefficient or variable here before this one. Here psi r is the suction corresponding to residual water content. And in Fredlund and Xing, they often mention that theta r determination is also difficult, therefore that also adds up as one of the fitting variable. So, however in Fredlund and Xing model, you have another fitting variable that is psi r suction corresponding to your water content.

This value is often modeled along with other fitting parameters like a, n, and m or psi r is substituted with 1500 kilopascal in some of the Fredlunds works. In some works psi r is considered to be 3000 kilopascal also. So, in some papers psi r is also used as a variable, and which is also fitted along with other parameters. So, psi r you can also assume, which is equals to 1500 kilopascal. And such fixing of psi r value is done often in some of the models like geo studio etcetera.

So, in this particular model interesting part is that, here you have 10 power 6 value. Therefore, all psi value should be substituted on kilopascal. And this 10 power 6 kilopascal value indicates that at psi is equals to 10 power 6 kilopascal. When it is plotted theta versus psi 10 power 6 kilopascal, this curve is forced or the water content is forced to come to 0. So, when you substitute psi equals to 10 power 6 kilopascal that is at suction equals to 10 power 6 kilopascal, so this value becomes 1, so 1 minus 1 is 0. So, theta is forced to come to 0, at theoretical value of at 10 power 6 kilopascal 1000 mega Pascal's.

So, therefore based on the theoretical observation or thermodynamic point of view, they observed that at 10 power 6 kilopascal the water content should go to 0. Therefore, based on that they modified the van Genuchten model to bring in that feature that the water content goes to 0 at 10 power 6 kilopascal. So, even though the model parameters here n, m, n, a, n, m are often understood to be flexible. The restriction we made on van Genuchten equation and van Genuchten model like n greater than 1 may be applicable in this particular case also.

So, the advantages here also, you can generate a smooth SWCC similar to van Genuchten equation. And extra advantage maybe that the water content is forced to 0 at very high suction values. Therefore, for clay soils where the suction range extends to such values, because for clay soils the suction range extends to 10 power 6 kilopascal. However, there is a major disadvantage also often observed by our group in fact that because of this particular feature, often it is seen that. When theta verses psi is plotted, the plots often go, and then because it has to go to 0 here.

Often bimodal curves are obtained using Fredlund Xing model. So, this is the soil-water characteristic curve is the which decreases and after that which is forced to when you plot this. So, often this is seen that this value decreases, because it has to approach. It should have gone, the curve should have approached directly here, but instead of that because it has to go to 0 at 10 power 6 kilopascal.

Often the curve behaves in a different manner. And often bimodal curves are obtained using Fredlund Xing model, often like this or sometimes it is like this such bimodal behavior is observed, which does not have any significance, however because of the model restriction that theta should go to 0 at psi equals to 10 power 6 kilopascal such discrepancies observed in the models.

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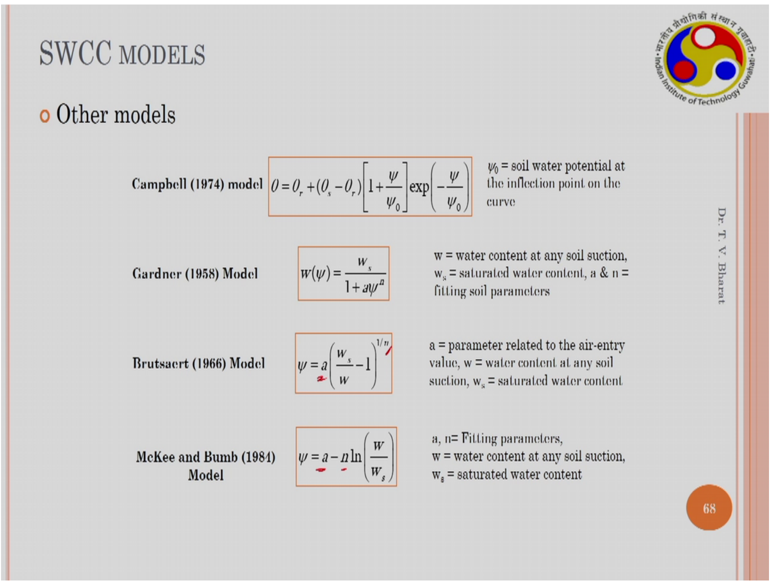

We have several other models like Campbell model, where theta is equals to theta r plus theta s minus theta r times 1 plus psi by psi naught into exponential of minus psi by psi naught. Here psi naught is the soil water potential at the inflection point on the curve, theta r is residual water content, and theta s is saturated water content. And the Gardner's expression, the gravimetric water content is equals to the saturated water content divided by 1 plus a times psi power n. Here a and n are fitting parameters. Similarly, Brutsacrt model where psi is equals to a times W s by W minus 1 whole power 1 by n. Here a and n are fitting parameters or model parameters and others are similar to here.

And McKee and Bumb model psi is equals to a minus n log of W by W s. Here again a and n are fitting parameters, and W is water content at any given suction, and W s is saturated water content. So, not just these models, we have several tens of models that are available in the literature and often used. But, most often used models are only the three, which are discussed little elaborately that is Brooks Corey, van Genuchten, and Fredlund Xing.

We have seen that the Brooks Corey model has a discontinuity at the bubbling pressure or the AEV or air entry value. Therefore, even though it is very simple, often in the modeling of partly saturated flows it is very difficult to use. When it comes to the van Genuchten model, which is three parameter model, which can generate smooth soil-water characteristic curve. However, the water content only approaches to 0, but it does not become 0 event at a very high suction values.

When it comes to the Fredlund model Fredlund Xing model, the feature it has a feature of feature for reducing the volumetric water content to 0 at suction value equals to 10 power 6 kilopascals. And this has a new model parameter such as a psi r psi suffix r, which is a suction corresponding to residual water content, which needs to be either determined along with other fitting parameters such as a, m, and n or which can be fixed to certain value like 1500 kilopascal or 3000 kilopascal.

Another important point here with Fredlund Xing model is are limitation is that this cannot be reversed or it cannot be written as psi in terms of theta. Van Genuchten model can be reversed or inverse form can be written. But, Fredlund and Xing model cannot be written as psi of theta that is a major limitation.

So, therefore with other hydraulic conductivity models, this integration is little difficult that we will see now. When comes to the hydraulic conductivity function models, as I explained that not many experimental techniques are available for the determination of hydraulic conductivity function in the laboratory as well as field. Only the multi step outflow technique is available that too it has a limitation to use more than 1500 kilopascal. So, this is mostly restricted for coarse grained soils. So, hydraulic conductivity function is often determined from the soil-water characteristic curve data only.

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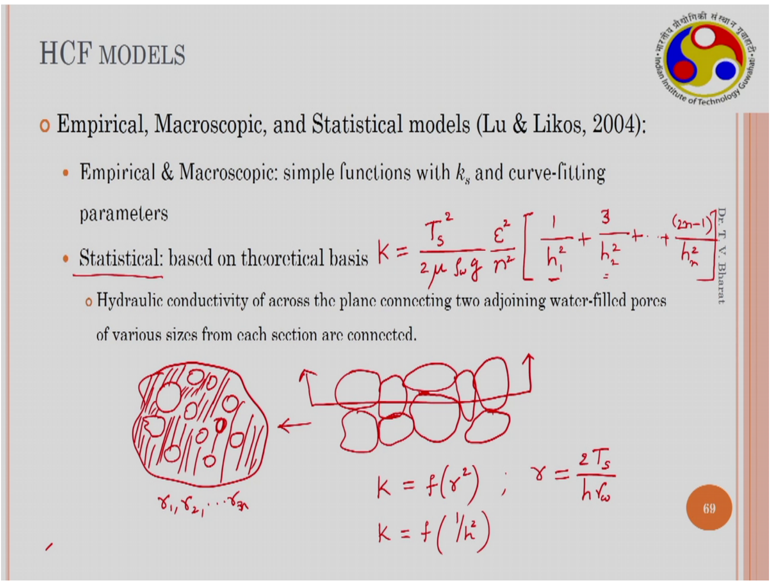

So, for that we need to understand, what different models we have. In hydraulic conductivity function, we have empirical models, macroscopic models, and statistical models. The empirical and microscopic models these are simple functions, they may use K s in the expression and some curve fitting parameters. These are simple expressions like y equal to m x plus b such as linear expressions or something about the use for the hydraulic conductivity functions. And or the using the macroscopic behavior, they may see the similarity between HCF and SWCC based on that they utilize similarities between these parameters, and they used to predict.

Generally, the empirical models and macroscopic models are fitting models for the existing hydraulic conductivity function data. When you have determined using pressure plate apparatus the hydraulic conductivity data, then these models can be used to fit and determine the fitting parameters. On the other hand, the statistical models are advantages, because there is a theory behind it. So, the hydraulic conductivity of they assume. So, in the statistical models, if we assume that this is one cross section where, this is one point where, you have several pores that are available. So, in between you have soil grains.

So, in such scenario, so this is one for example, you have several soil particles, which are placed in this particular manner, there is no overlap. So, this a one particular soil structure. When I consider the cross section at any given place, the cross section is this you have several pores designated by say r 1, r 2, r 3, which are distributed. Here we consider, circular pores that exists. And then, we consider the probability of connecting one soil pore with their adjoining water-filled pore with some probability. And then, we can consider that if there is a probability that, this pore exists. In the next to the other pore in the next cross section, what is the probability of that, because if you if there is a probability that there is a pore exist, then flow takes place through that.

Using Hagen Poiseuilles equation, so the which states that the hydraulic conductivity is a function of r square. And as we know the, if you consider, circular pores r can be approximated as 2Ts by h times gamma w, because this is pressure. Pressure is equals to 2Ts by r, so when we assume that the contact angle is 0. So, therefore hydraulic conductivity is function of 1 over h square or the head suction head.

So, this is how the function is related, when different probability functions are used you may get a expression like K equals to T s square by 2 mu rho w g times epsilon square by n square times 1 over h 1 square plus 3 by h 2 square plus 5 by h 3 square like that and 2n minus 1 by h n square. So, such expression would result for hydraulic conductivity of unsaturated soils. Here, T s is a surface tension, mu is viscosity rho w g, and epsilon is the dielectric, and n is porosity. So, here the different expression for hydraulic conductivity function based on statistical models will have K as a function of 1 over h h 1 square plus 3 by h square 2 h 2 square like that you will have. So, essentially they indicate different pores, sizes you have in the soil mass.

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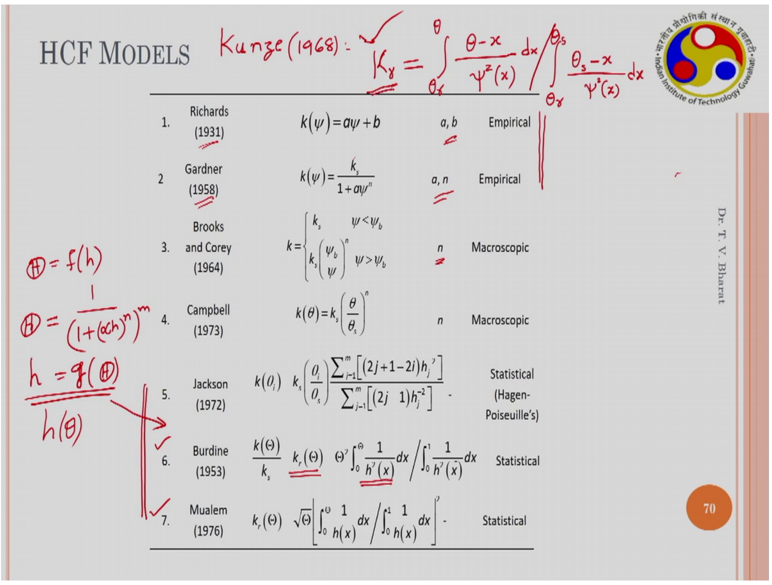

So, if you look at different available models, so the old one is Richards model, which is as old as 1931. So, the model is a linear assumption. The hydraulic conductivity is inversely linearly dependent. So, simple linear equation where a psi plus b, which is used this is a simple empirical equation. And Gardner, he had 1958 had a given expression, which is k s, which uses saturated hydraulic conductivity. So, k s by 1 plus a times psi n, a and n are empirical parameters. So, when whenever you have this empirical equations the fitting parameters need to be determined. These are fitting parameters; the fitting parameters need to be determined by considering best fit between the measured hydraulic conductivity data and the expressions.

And, Brooks Corey method, which is a macroscopic model, because which construct similarity between soil-water characteristic curve and hydraulic conductivity function. And based on the similarity, where you have lambda, which is related to n, so that is how it is used, and so this is a macroscopic model.

Similarly, Campbell propose another model, which is also macroscopic model. So, these three are statistical models, we have many other statistical models. But, here I have given three such models, this is based on Hagen Poiseuilles expressions. Where Jackson model, where it considers k of theta i is equals to saturated hydraulic conductivity times theta i divided by theta s. So, theta i is at any given point i value and theta i by theta s times if sigma j equals to 1 to m 2 j plus 1 minus 2 i times h j over minus 2 that means, 1 over h square, that is how we have used also using Hagen Poiseuilles approximation. This form is divided by sigma j equals to 1 to m, 2 j minus 1 divided by h square. So, this form is derived based on Charles and George expression. And here K is the hydraulic conductivity at any given theta fine.

So, this is another model, which is a Burdine model. And here the expression is in terms of relative hydraulic conductivity, where relative hydraulic conductivity is hydraulic conductivity at any given water content divided by the saturated hydraulic conductivity k by k s, which is equals to theta square times integral 0 to theta is normalized volumetric water conductivity, 1 by h square x dx by 0 to 1 1 by h square x dx. Here x is integration variable.

The Mualem expression is also similar to Burdine model. So, these two expressions Mualem and Burdine is Burdine are very often used in the geo technical engineering as well as in soil science literature also. Here for integrating or for estimating the hydraulic conductivity function from the soil-water characteristic over data. We require a continuous function of h of theta. Here if we have h of theta, so we can directly substitute it here and we get an expression for hydraulic conductivity function.

So, here we do not require any other fitting parameters. What are the fitting parameters we established for the soil-water characteristic curve can be directly used for the hydraulic conductivity function determination. This is also seen for many coarse grind soils these models provide very satisfactory results. Here, if you see the van Genuchten model, if you recollect your van Genuchten model, so that is theta is equals to 1 by 1 plus alpha h power n and whole power m. So, here it is written theta equals to function of h, but this also can be written h as a function of or another function of theta. So, inverse is very easy to do, because that is what is required for the estimation of hydraulic conductivity functions here.

However, if I go back using Fredlund and Xing model, in this expression it is not possible to represent psi as a function of theta, because inverse is not possible, so that is a major limitation of Fredlund and Xing model. So that is a reason, why they use a different expression for determination of hydraulic conductivity function, which is called Kunze model K r theta r to theta theta minus x by psi square x dx divided by theta r 2 theta s theta s minus x by psi square x dx. So, this is a expression used Fredlund. To combine with Fredlund and Xing, SWCC model for the production of relative hydraulic conductivity.

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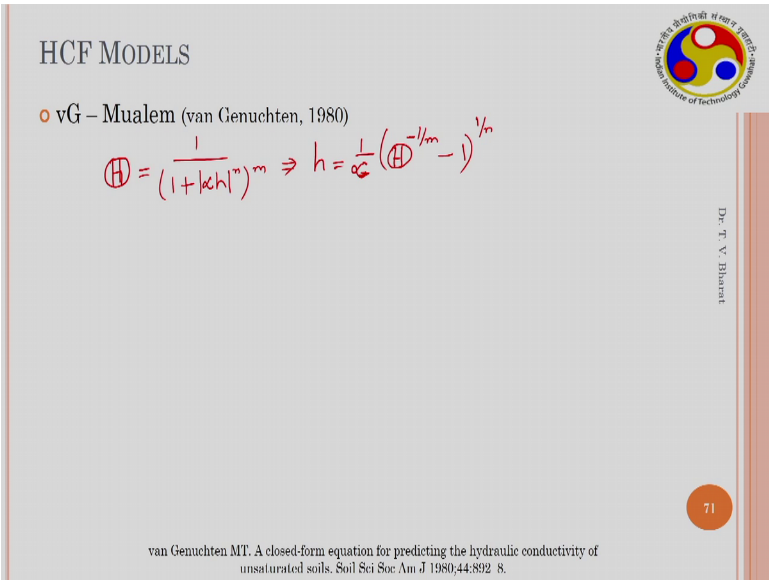

So, let us see one of such derivation for hydraulic conductivity function determination using SWCC, where the vengeance model is combined with Mualem expression, then you get analytical expression for hydraulic conductivity functions. So, once the SWCC parameters are available directly, we substitute this parameters to obtain the hydraulic conductivity function. As we have seen the theta is equals to 1 by 1 plus alpha h over n whole power m, which can be written as in terms of h as h is equals to the theta power minus 1 by m. And if you take the minus 1 that side minus 1, then whole power 1 by n and 1 by alpha, such a simple thing to invert the expression.

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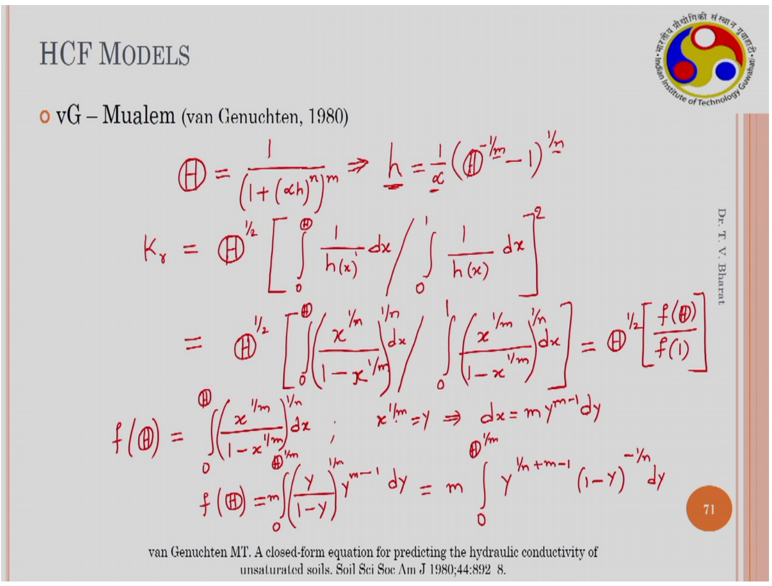

So, here van Genuchten-Mualem. This is model for hydraulic conductivity function estimation. And we can derive analytical solutions for hydraulic conductivity function by combining the SWCC model and the expressions given for hydraulic conductivity functions. One of such integration between SWCC and hydraulic conducted function is van Genuchten-Mualem model.

So, in this model the van Genuchten model we have earlier written, which is theta is equals to 1 by theta is a big data, which is normalized volumetric water content is equals to 1 by 1 plus alpha h. Here h should be substituted in positive thing power n whole power m, which can be inverted to write expression for h as theta power minus 1 by m and then minus 1 goes to other side and whole power 1 by n, and alpha if it this comes this side, 1 by alpha so such as simple expression can be derived, because inversion is so easy.

So, when we invert the expression, we get the h in terms of theta as this. Now, the Mualem model is K r the relative hydraulic conductivity is equals to big theta power half into integral 0 to theta 1 by h of x dx by 0 to 1 1 by h of x dx whole square. So, h can be substituted here to obtain the expression for K r. So, therefore directly when alpha m, n, and r determined by fitting the soil-water characteristic curve data using this model. So, there are directly the parameters can be substituted here to obtain the K r, relate to hydraulic conductivity.

So, here when we substitute this is theta power half and so this becomes 1 by alpha 1 by alpha get cancelled, because that quotient is present on the numerator and denominator as well, so that get cancelled. So, when you write this expression as 1 by theta power m, 1 by m, so then this can be written as theta and using integral variable that is x this can be written as x power 1 by m by 1 minus x power 1 by m divided by integral 0 to theta dx and integral 0 to 1 is the same expression x by 1 by m x power 1 by m divided by 1 minus x power 1 by m dx. So, this can be simply written as theta power half, this is f of theta by f of 1, because it is a same expression. But, the integration is here going from 0 to theta, and integration here going from 0 to 1 that is the only difference.

So, if you obtain f value, then you can obtain f of 1 value also. You can obtain f of theta value, then you can obtain f of 1 value also. So, the f of theta value is integral 0 to theta x power 1 by m by 1 minus x power 1 by m dx. This can be simplified by substituting x power 1 by m is equals to Y. If we do that dx is equals to m Y power m minus 1 dy substituted.

Then this expression becomes f of theta is equals to integral 0 to theta Y power Y times Y by 1 minus Y, and for dx this is m times Y power m minus 1 dy no here this whole power n sorry, I forgot to put the whole power n 1 by n. So, in the expression, so therefore this is whole power 1 by n. So, this can be written as m times 0 to theta Y power 1 by n plus m minus 1 into 1 minus Y power minus 1 by n dy. Here also the phi is phi power 1 by m should be substituted theta power 1 by m should be substituted here. So, here also the integration variable changes from 0 to theta power 1 by m, because here substituting this value, it goes from 0 to y. So, here this is theta power 1 by m, so this is a change.

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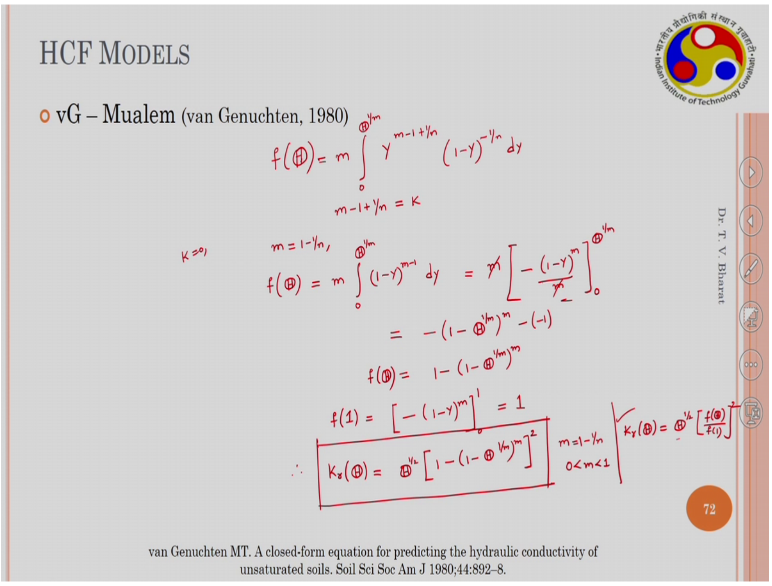

So, then we get an expression for phi of theta, which is m times integral 0 to theta power 1 by m Y power Y power m minus 1 plus 1 by n into 1 minus Y power minus 1 by n dy. This is a particular form of incomplete beta function. In it is initially the van Genuchten thought in 1980 that in its most general form this equation does not have any analytical solution. So, therefore he started deriving particular solutions.

He is shown that for example, when you show m minus 1 plus 1 by n is equal to some values say K. Then you can derive particular solutions by assuming k equals to 0, k equal to 1 likewise. You can derive any number of particular solutions. So, for example, k equals to 0 results in one particular solution that is f of theta is equals to m times 0 to theta power 1 by m 1 minus so as we have seen the function f of upper case theta is equals to m times integral 0 to uppercase theta power 1 by m Y power m minus 1 plus 1 by n times 1 minus Y power minus 1 by n dy. So, this is a particular solution of incomplete beta function.

And van Genuchten in 1980 felt that this particular form does not have an analytical solution. So, therefore he derived particular solutions for this particular equation. So, to derive particular solutions, he shows m minus 1 plus 1 by n is equals to K. And one particular solution, when k is equals to 0 is when k is equals to 0 this is m equals to 1 minus 1 by n. So, substitute in K equals to 0 into the into the above equation results in f of upper case theta is equals to m times this m minus 1 plus 1 by n is 0, therefore Y power 0 is 1. So, this results in m times integral 0 to theta uppercase theta power 1 by m times 1 minus Y power minus 1 by n dy.

So, when we integrate this m times, the integration for this is minus 1 minus Y 1 minus Y, so this is for inside. And for this here we can substitute instead of n, we substitute instead of 1 minus n m minus 1 here, so because all the parameters are with in terms of m. So, when we substitute and do the integration, this is simply 1 minus Y power m divided by m the integration is the limits are from 0 to uppercase theta power 1 by m. This m gets cancelled and this is nothing but minus 1 minus uppercase theta power 1 by m power m and minus, when the 0 is substituted this is again minus 1. So, therefore this is 1 minus uppercase theta power 1 by m whole power m. This is f of uppercase theta.

Now, f of 1 is simply the same thing that is from here minus 1 minus Y power m power the limits are 0 to 1. So, when you substitute this is 0, then minus when you substitute the 0, this is minus 1, so minus of minus 1, this is 1. So, therefore the K r relative hydraulic conductivity as a function of uppercase theta is equals to f of uppercase theta by f of 1 times uppercase theta power half times 1 minus uppercase theta power 1 by m power m this whole square, because K r of uppercase theta is equals to uppercase theta power half times f of uppercase theta by f of 1 whole square. This is from Mualem’s model, therefore this is a solution. But, however there is a condition that m should be equals to 1 minus 1 by n, and m should be between 0 and 1. So, these are the conditions for this particular form of equation.

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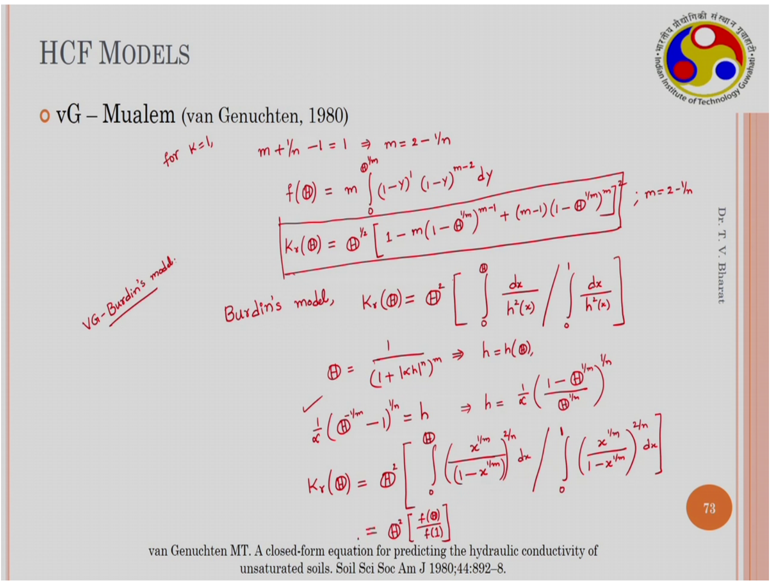

Similarly, we can derive for Burdines model. And similarly another particular solution also can be derived. When for K equals to 1, so this becomes m plus 1 by n minus 1 is equals to 1 that means, m is equals to 2 minus 1 by n. So, when m equals to 2 minus 1 by n is substituted, so f of big theta or f of uppercase theta is equals to m times integral 0 to uppercase theta for 1 by m and 1 minus Y power it should be m minus 1 plus 1 by m. So, this became 1 times 1 minus Y power minus 1 by n for minus 1 by n we can substitute m minus 2 dy.

So, when this is solved using integration by parts, we obtain a solution for K r of uppercase theta is equals to theta power half times 1 minus m times 1 minus uppercase theta power 1 by m power m minus 1 plus minus 1 times 1 minus uppercase theta power 1 by m power m whole square. So, this is the expression for relative hydraulic conductivity in terms of uppercase theta and there is normalized volumetric water content. Here the condition is that m should be equals to 2 minus 1 by n. So, this is another particular equation.

So, earlier we derived one particular equation by assuming K equals to 0. And this is another particular equation by assuming K equals to 1. Similarly, n number of particular equations can be developed by assuming different values for k. Similarly, using Burdines model that is van Genuchten Burdines model. So, using the Burdines model gives that K r of uppercase theta is equals to uppercase theta square times integral 0 to uppercase theta dx by h square x divided by integral 0 to 1 dx by h square x. So, by inverting van Genuchten equation, the van Genuchten equation is big theta is equals to 1 by 1 plus alpha h power n whole power m.

When you invert this expression to write h equals to h of theta, so this gives theta power 1 by m. When you take them other side, and when you put minus 1 by m, this is simply 1 plus alpha h power n this equals to so therefore minus 1, you can you can bring 1 other side, so this because minus 1. So, then if you take n the other side, 1 by n and divided by 1 by alpha is h. So, then if you substitute for h here, then K r of uppercase theta is equals to uppercase theta square times this is 0 to uppercase theta. So, this is dx by h square, anyways you have the same expression on the numerator and denominator the alpha gets cancelled.

So, then this can be written as h is 1 by alpha, this can be written as 1 minus big theta power 1 by m by big theta power 1 by m. So, this is 1 by h here so or this one can be written as 1 minus x power 1 by m whole square are dx I will write it separately here. And this one is x power 1 by m. So, this whole thing need to be squared. Here this 1 by n, so there is a 1 by n term and square, so this is 2 by n. Similarly, this is 0 to 1 x power 1 by m divided by 1 minus x power 1 by m whole power 2 by n dx, so this is expression to solve. This can be written as big theta square times f of theta by f of 1, because the functional form is same except that the limits are different. So, this is written in this manner.

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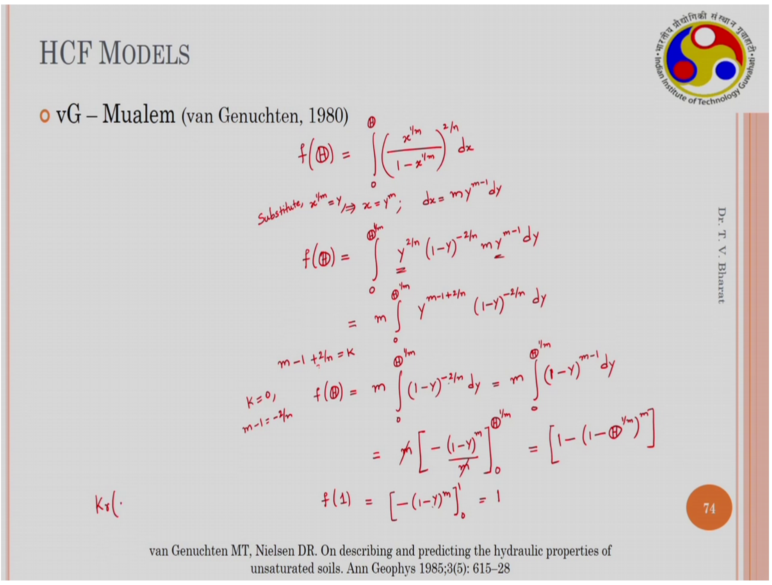

So, now let us solve f of big theta, this is equals to integral 0 to theta x power 1 by m by 1 minus x power m by m whole power 2 by n dx. Similar to the earlier van Genuchten-Mualem equation we again substitute x power 1 by m is equals to y or this can be written as x is equals to y power m. Then dx is equals to m y power m minus 1 times dy, so this expression for dx.

And here, the functional form can be written as 0 to here when this is a earlier when you have x this is 0 to x, now this is Y power m. So, x is substituted to Y power m. So, this is a y power 1 by m times here x power 1 by m is Y, Y power 2 by n times 1 minus Y power minus 2 by n. And instead of dx, if you substitute m Y power m minus 1 dy. Then the integration for this can be obtained by simplifying 0 to theta power 1 by m.

This is Y power these two terms together m minus 1 plus 2 by n times 1 minus Y power minus 2 by n dy. Again this is a general expression and this is incomplete beta function. So, van Genuchten felt that there is no analytical solution. So, he derives particular solutions by showing again minus 1 m minus 1 plus 2 by n is equals to K. One particular solution, when k equals to 0 is f of theta is equals to m times 0 to theta power 1 by m, so this is 1 minus Y power minus 2 by n dy. Here minus 2 by n this is equals to 0. So, m minus 1 is equals to minus 2 by n. So, here this can be written in terms of m 0 to theta power 1 by m 1 minus Y power m minus 1 dy.

So, the integration is m times. So, for 1 minus Y power m minus 1, the integration is minus 1 minus Y power m divided by m, the limits are from 0 to uppercase theta power 1 by m, so n gets cancelled. And when you apply the limits, this is 1 minus 1 minus uppercase theta power 1 by m power m. So, this is f of uppercase theta. f of 1 is equals to minus of 1 minus Y power m from 0 to 1. So, When 1 is substituted, this is 0. And minus when 0 is substituted, this minus 1 minus of minus 1, this is 1. So, when this is simplified, when this is K r term is written, K r of theta.

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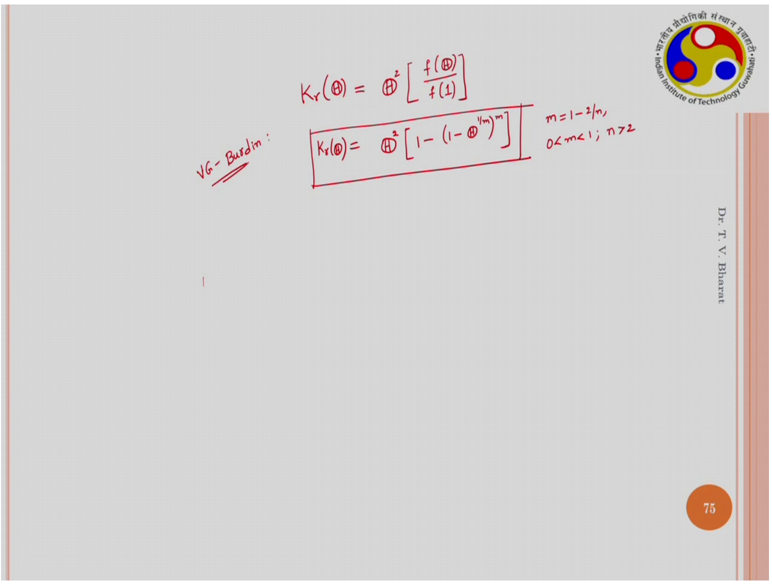

So, when K r is written K r of theta is equals to theta square times f of theta by f of 1. So, this is equals to uppercase theta square times 1 minus 1 minus uppercase theta power 1 by m whole power m, so this is K r of theta. So, this is analytical solution particular form analytical solution for Burdine model, van Genuchten-Burdine model, here m is equals to 1 minus 2 by n, and m varies between 0 and 1, and n is more than 2. So, these are the conditions. This is another particular form of vG- Burdine model.

Similarly, when k equals 1; k equals to 2 such by substituting several integers for K, we get infinite number of particular solutions of the general form using van Genuchten-Burdine, similarly van Genuchten-Mualem. However, in 1985 after 5 years from this work by van Genuchten; he realizes that the general form of the equation that is derived has analytical solution.

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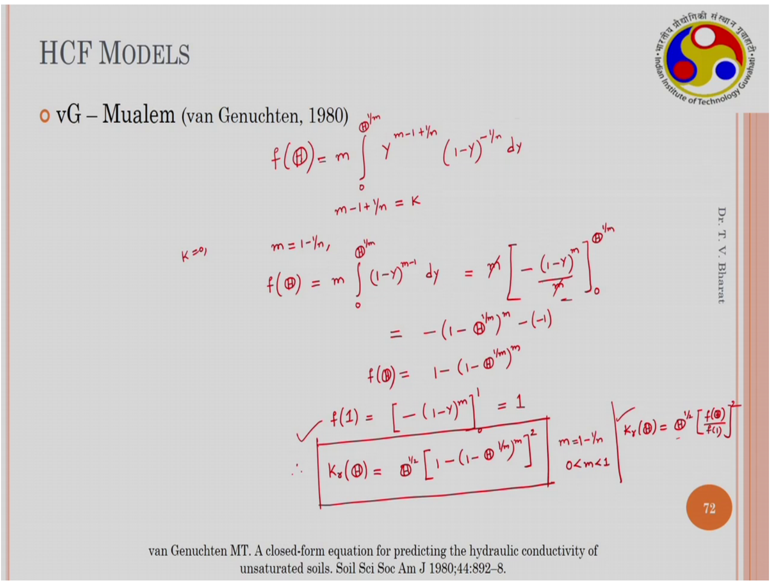

So, then he published work along with Nielsen in 1985, where he provides general analytical solution for these equations. So, for the van Genuchten-Mualem, so the expression is f of uppercase theta is equals to m times integral 0 to uppercase theta power 1 by m Y power m minus 1 plus 1 by n times 1 minus Y power minus 1 by n dy.

So, here we assumed m minus 1 plus 1 by n is equals to either 0, 1 etcetera. And then, we derived analytical solution, but this general form can be expressed as m times incomplete beta function and some zeta parameters of p, q and complete beta function B times p q. This can be expressed in this particular form. And the incomplete beta function beta function values can be obtained table form that was a valuable during that time, and then using that they got the analytical solution.

And now in mat lab and many other programs the incomplete beta function and beta function values can be readily obtained, and this can be solved easily. Here the zeta parameter is phi power 1 by m, and here the P is m plus 1 by n, and q is 1 minus 1 by m. So, therefore the K r of uppercase theta is equals to because f of 1 is m times complete beta function p, q. So, therefore k of theta is equals to square root of uppercase theta times incomplete beta function whole square, so this is the general solution.

Similarly, using van Genuchten; using van Genuchten-Burdine using the same reference van Genuchten Nielsen 1985. So, the general expression can be written as K r of cap uppercase theta is equal to uppercase theta square the incomplete beta function of r, s.

Here this zeta is same uppercase theta power 1 by m. Here r is equals to m plus 2 by n. And s is equals to 1 minus 2 by n. So, here there is no restriction between m and n. So, there is no these are not constrained parameters. Earlier, whatever the solutions we derived, those are particular solutions, where m is related to n. Here m and n are not related; m and n are independent in these two expressions. So, these are general expressions.

As can be seen, now you have general solutions for van Genuchten-Mualem, van Genuchten-Burdine. Similarly, you have particular solutions by assuming K equals to 0, 1, 2 and different values. Even though these many number of solutions are available. Most commonly m and n are dependent equations are only commonly used for fitting SWCC data. So, this is probably because one reason is that, this particular work is not cited that well or there could be another reason is that when m and n are independent. So, the equations over fit the SWCC data, and the hydraulic conductivity functions, (Refer Time: 77:36) estimated, they deviate significantly from the measured data.

However, when the m and n are restricted the even though the error between SWCC data and theoretical data is slightly more than the m and independent case. But, the hydraulic conducted functions are very close to the measured data, so that is the reason why m and n dependent conditions. So, those are the particular solutions, which we derived earlier. So, these particular solutions are commonly used instead of general solutions.

Thank you.