Geotechnical Earthquake Engineering

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Module - 5

Lecture - 19

Wave Propagation (Contd…)

Let us start our today's lecture of this NPTEL video course on geotechnical earthquake engineering. On this course, we are now going through our module five, which is wave propagation.

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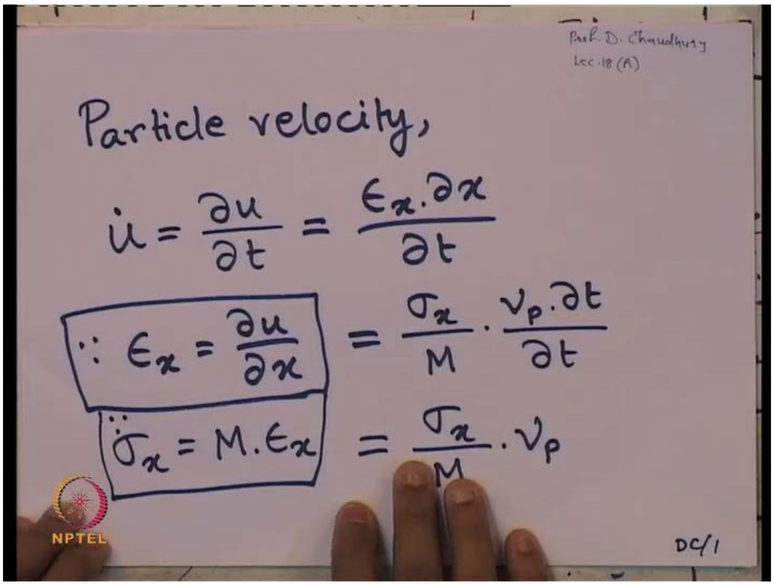

Let us quickly recap what we have learnt in our previous lecture. In the previous lecture, we have seen for one-dimensional wave propagation in an infinite rod; for the case of longitudinal wave, what is the particle velocity; how we can estimate the particle velocity from the displacement function; and using the strain displacement relationship as well as the stress versus strain relationship, we arrived at this relationship. And the v p is nothing but the primary wave velocity or p wave velocity or longitudinal wave velocity, which is expressed as root over M by rho.

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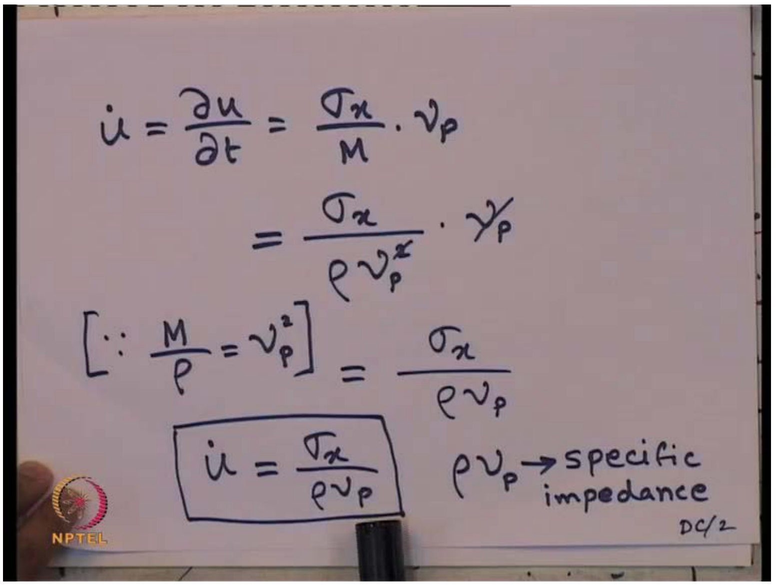

So, this is the relationship as we have seen earlier. As I was mentioning, v p is nothing but root over M by rho; M is nothing but constrain modulus; rho is the density of the material. And this is the relationship between the particle velocity with respect to stresses. And this parameter – the multiplication of density with respect to the velocity of the wave of this p wave – that function rho v p is known as specific impedance. So, that is the specific impedance of the material.

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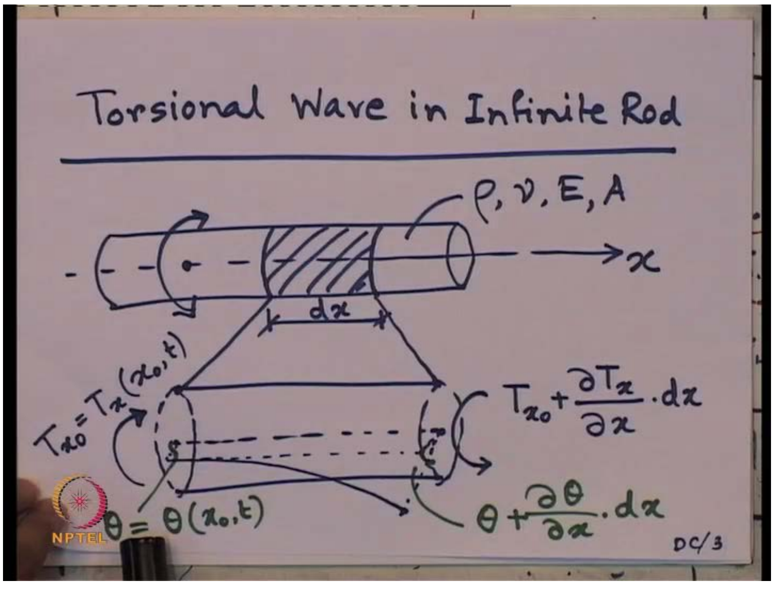

Then in the previous lecture, we have derived what will be the governing equation of motion for a torsional wave when it travels through an infinite rod in one dimension. So, this is the derivation we have seen, the difference of torque at both the ends of this infinitesimal element of the rod with the different magnitude of rotational displacement theta at this end and at this end.

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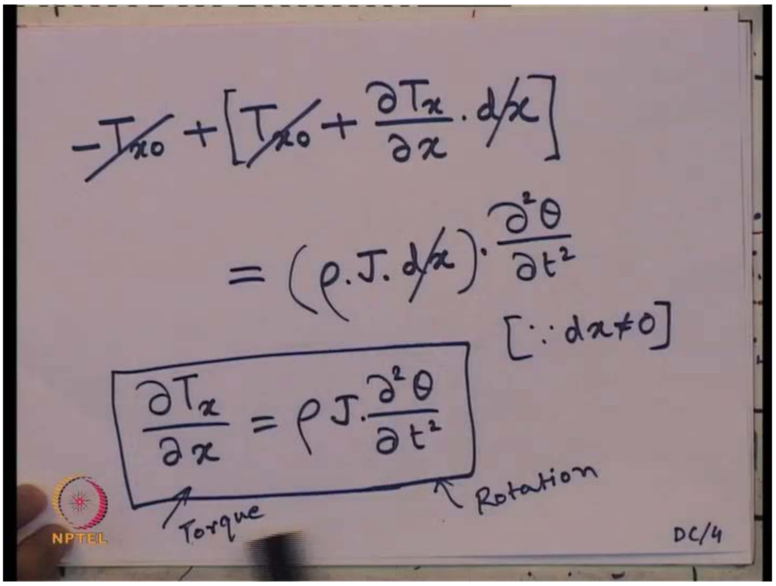

Then, after considering the equilibrium, we had derived at this relationship of torque versus rotation in this form; where, J is the polar moment of inertia.

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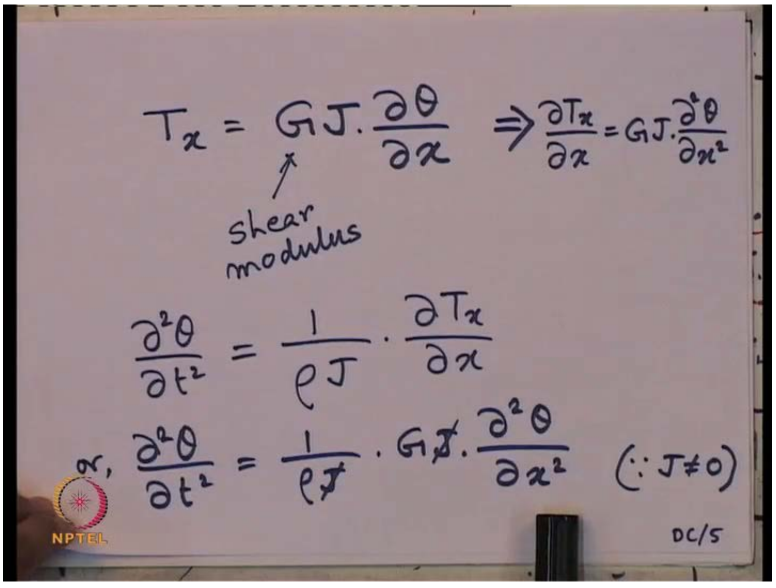

After that, we had applied the relationship between torque and rotational displacement through the shear modulus and polar moment of inertia in this form.

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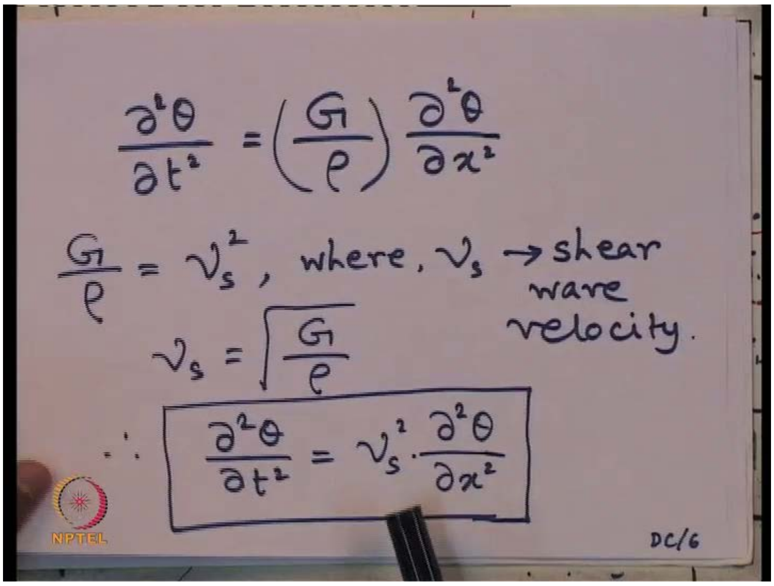

And on further simplification, what we had arrived finally, is this one; where, this parameter again we have defined as square of shear wave velocity. So, shear wave velocity is nothing but root over G by rho, where G is the shear modulus of the material and rho is the density of the material. And hence the governing equation of motion in terms of the acceleration, this is the rotational acceleration; and this is the second order differential of the rotational displacement with respect to the x-axis system or space coordinate system related through the shear wave velocity of the media for this torsional wave.

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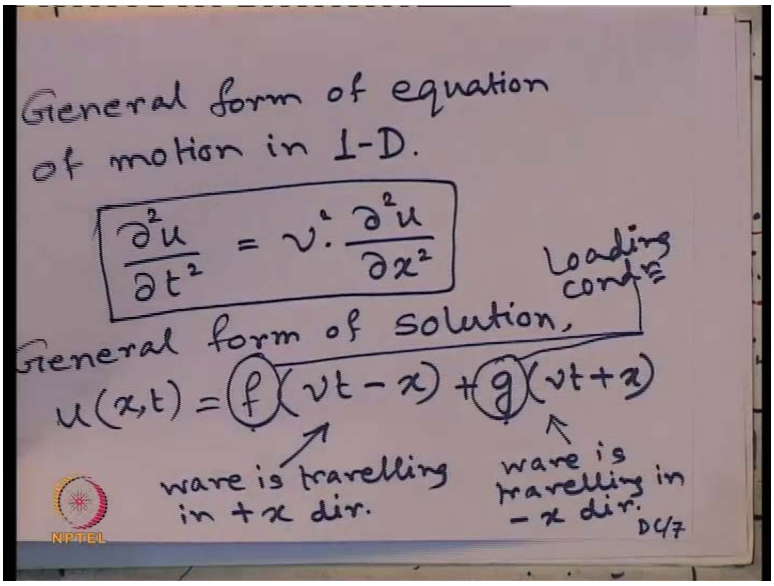

So, general form of governing equation of motion for any wave travelling in this one dimension can be represented by this equation. So, for longitudinal wave, it will be v p; for torsional wave, it will be v s and corresponding displacement. For longitudinal wave, it is u; for torsional wave, it is theta; that is, the rotational component of the displacement. And, we have seen, for this type of governing equation of motion, we have the general form of solution expressed in this format. Now, this general form of solution again we can get a particular solution depending on the loading condition. So, these functions depend on what type of load is applied to the system.

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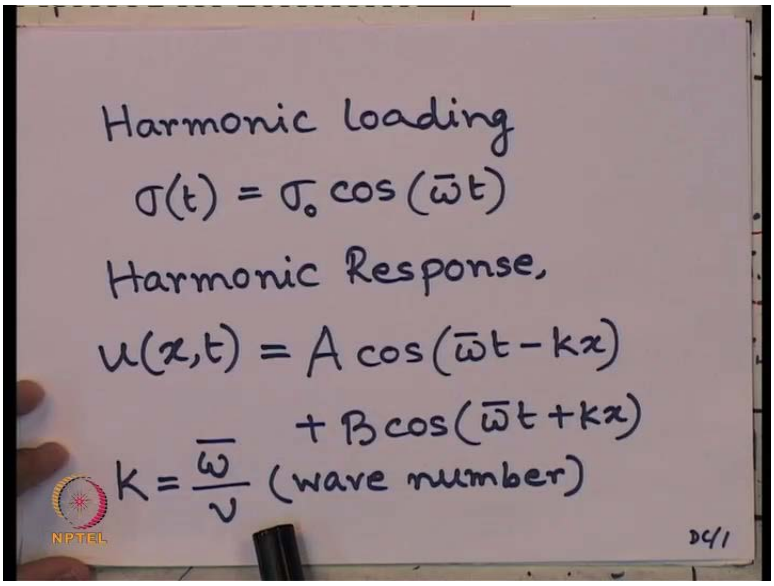

We have also seen for harmonic loading, this is the response what we have obtained; where, we have defined this parameter K as a wave number, which is nothing but the ratio of the circular frequency to the velocity of wave in that medium.

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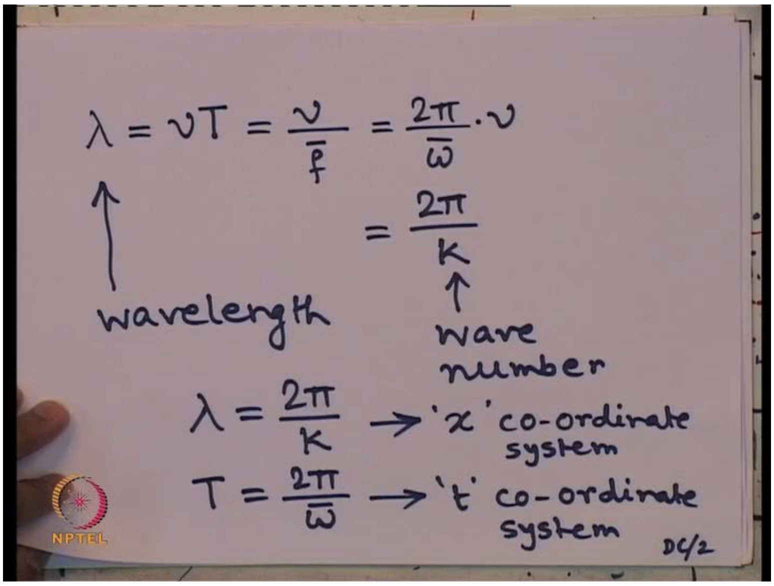

We had also defined another parameter, which is known as wavelength – lambda, which is related to the wave number in this format. And, we have seen the similarity between the relationships of this wavelength lambda with respect to wave number K. And, in the time co-ordinate system, we have seen the relationship between time period with respect to the natural circular frequency.

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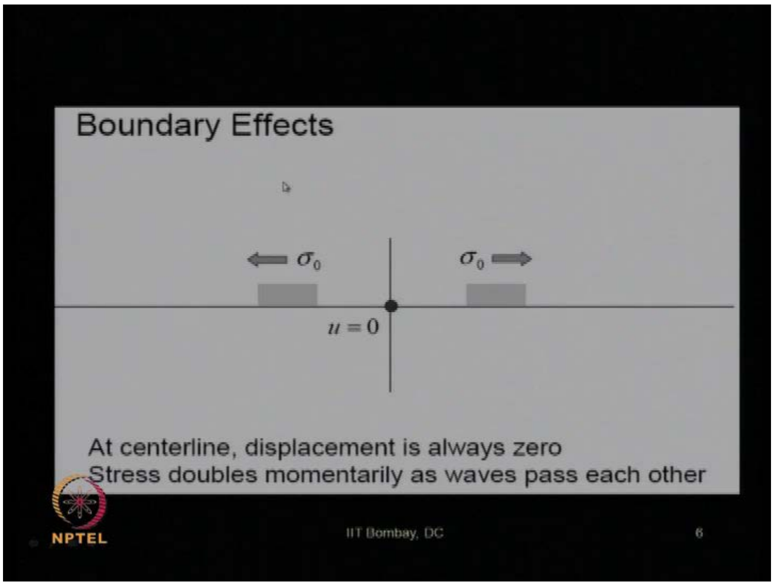

After that, what have seen in the previous lecture, we can see the slide here; we have considered various boundary conditions like this, where centerline, where displacement will be 0.

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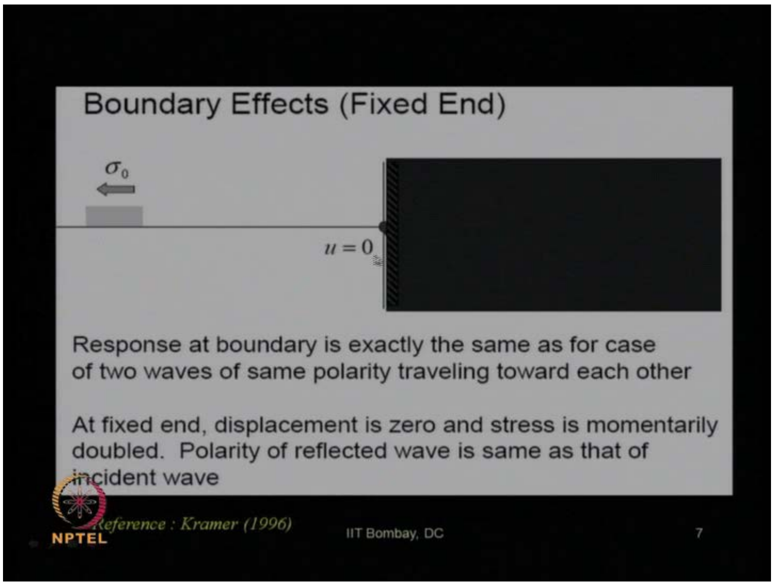

Then at the fixed end, where the displacement has to be 0.

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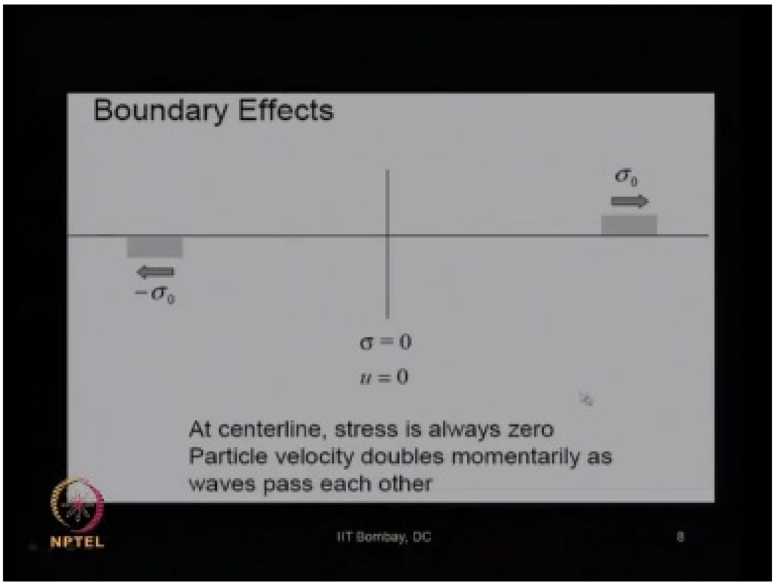

Also, a boundary effect, where both says stress and displacement has to be 0.

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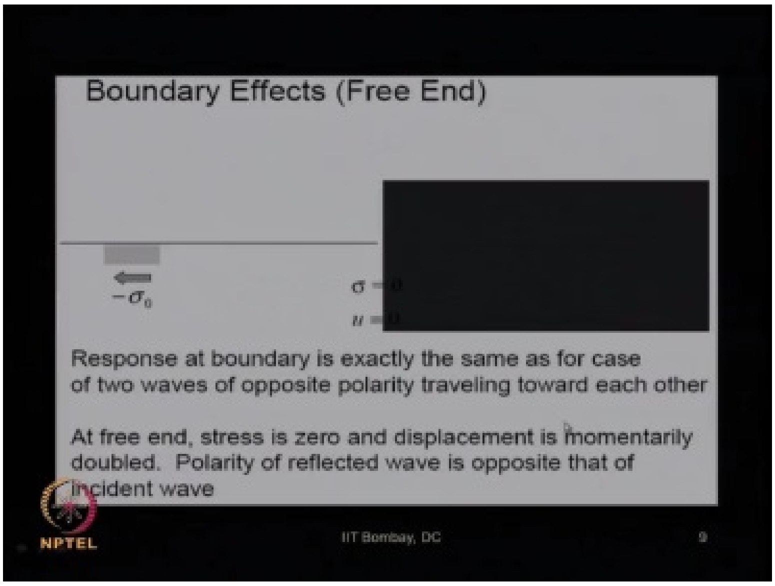

Then at the free end, where stress has to be 0.

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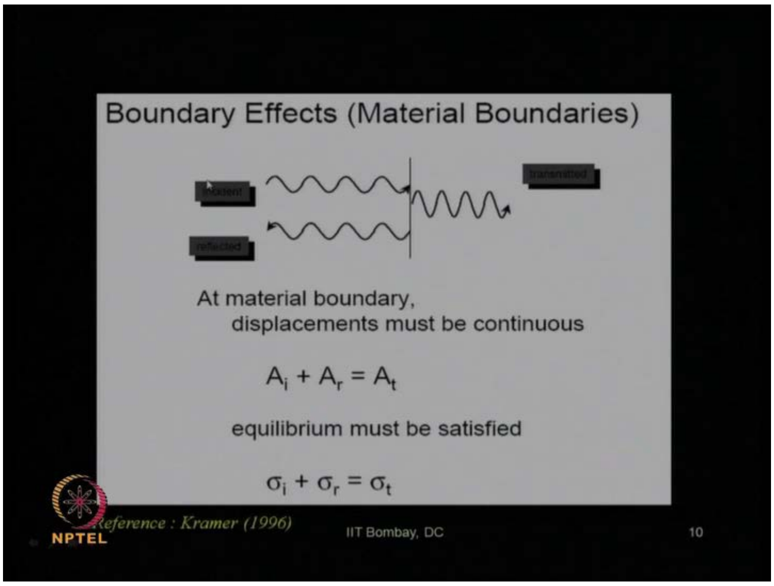

Then, we have seen the boundary effect, that is, material boundary when there is a layered body, how this incident wave; then, transmitted wave and reflected wave – they are related to each other through this example.

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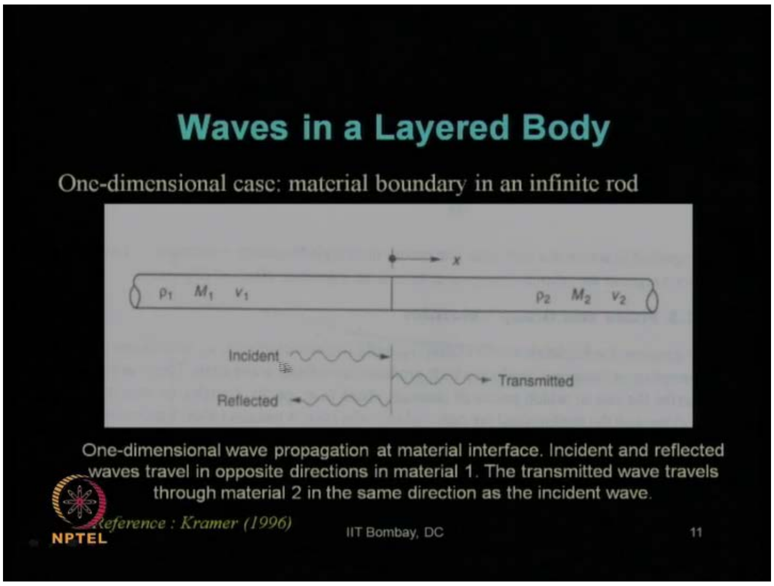

That is, for one-dimensional case, when wave is travelling in this one dimension in a layered body like this, we have seen this is material 1; this is material 2. We have defined the x co-ordinate system in such a way that this is the positive x-direction and this boundary itself is the x equals to 0. And, incident wave comes from material 1. Some of that goes in the material 2 as transmitted wave; and, some of the wave comes back in the same medium as reflected wave.

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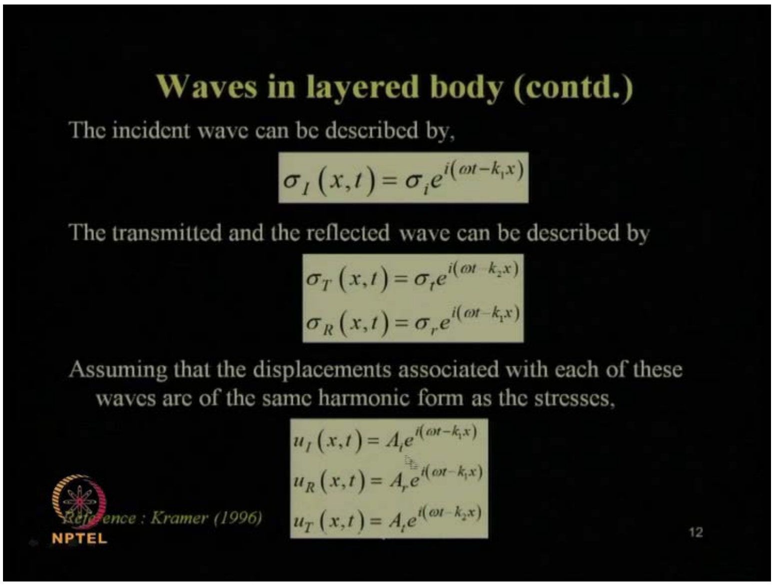

Then, we have seen how these stresses of incident transmitted and reflected wave can be represented in a harmonic fashion; also, how the displacement functions can be represented through this form.

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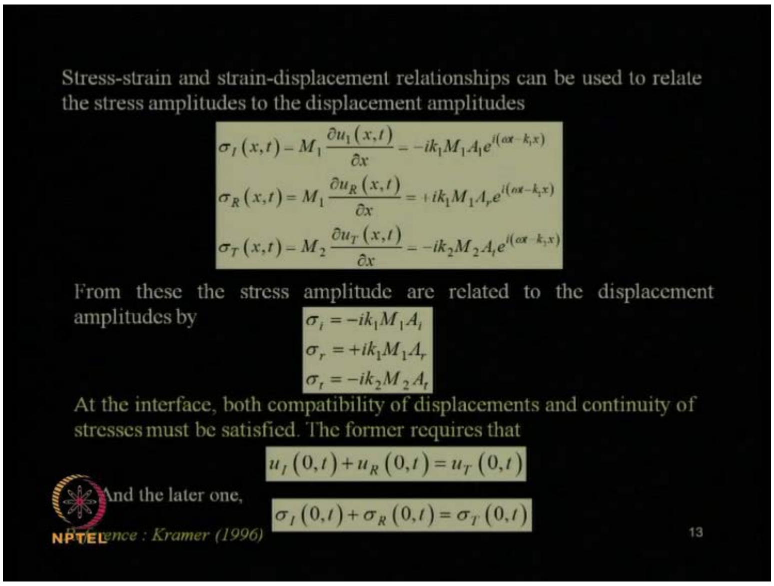

Then, applying the known stress-strain relationship through this modulus; that is, when we are talking about longitudinal wave, we have to take constraint modulus like this. So, we have obtained what are the relationships between amplitudes of these stresses and displacement. So, this is the relationship of amplitude only – stress amplitude with respect to the displacement amplitude.

Also, we had seen in the previous lecture that, at the interference of two materials, this boundary condition has to be satisfied; that is, the displacement compatibility and equilibrium of stress. So, these two conditions has to be maintained. Hence, this incident wave displacement at x equals to 0 plus the reflected wave displacement at x equals to 0 should be equals to the transmitted wave displacement at x equals to 0. So, that is from the displacement compatibility. Whereas, from equilibrium of stress condition, we will get stress due to that incident wave at x equals to 0 plus stress due to the reflected wave at x equals to 0 should be equals to the stress due to the transmitted wave at x equals to 0.

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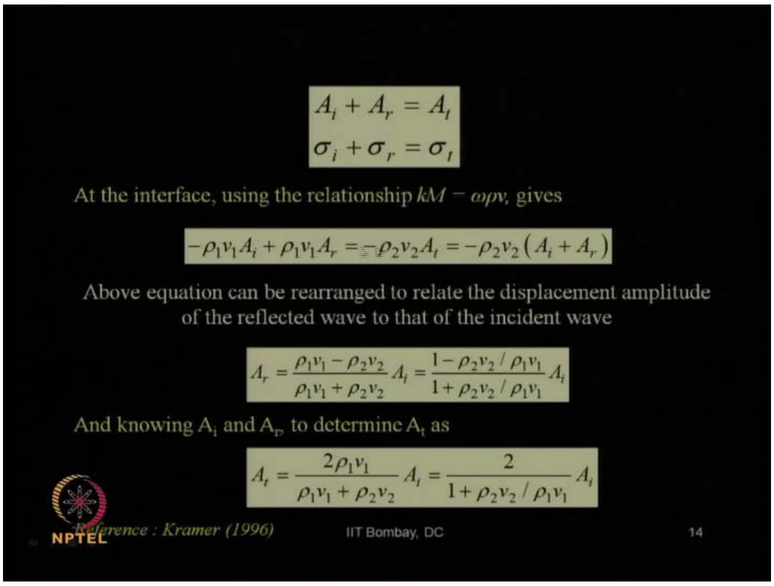

Now, let us continue this in today’s lecture further. So, what we can write from the previous known relationship that, if we equate the amplitudes, the displacement amplitude of incident wave plus displacement amplitude of reflected wave should be equals to the displacement amplitude of transmitted wave. In the similar form, this is from the displacement compatibility relationship. And, from the equilibrium of stress relationship, we get the amplitude of incident wave plus amplitude of reflected wave should be equals to the amplitude of this transmitted wave.

Now, at the interface, using the known relationship of this one, what we can write further? By putting the values of this A i and A r with A t, we will get this relationship. So, on further rearrangement, if we do the further rearrangement of this relationship of displacement amplitude to the reflected and incident wave with respect to the transmitted wave, what we will get? Suppose this reflected wave amplitude we are interested to know, because in most of the cases, what will be known? The incident wave properties will be known. From a media, we will know what is the stress and displacement of incident wave. We have to now find out what are the stresses and displacement in the reflected and transmitted wave. That is what we are more concerned about in practice. So, this is the co-relation.

If you put this simple relationship in this equation, then on further simplification, you will get like this. And, knowing this A i and A r to determine as… So, A t also you can determine with respect to A i in this form, because A r we have represented in this format. Hence, you can represent the transmitted wave amplitude also in terms of this incident wave amplitude, displacement amplitude.

Now, let us look at this parameter very carefully. This is rho times v; that is, density times the velocity. What is that? We have already mentioned. That is nothing but in the precious lecture, we discussed. It is to specific impedance. This is nothing but specific impedance of material 2. And, this is the specific impedance of material 1.

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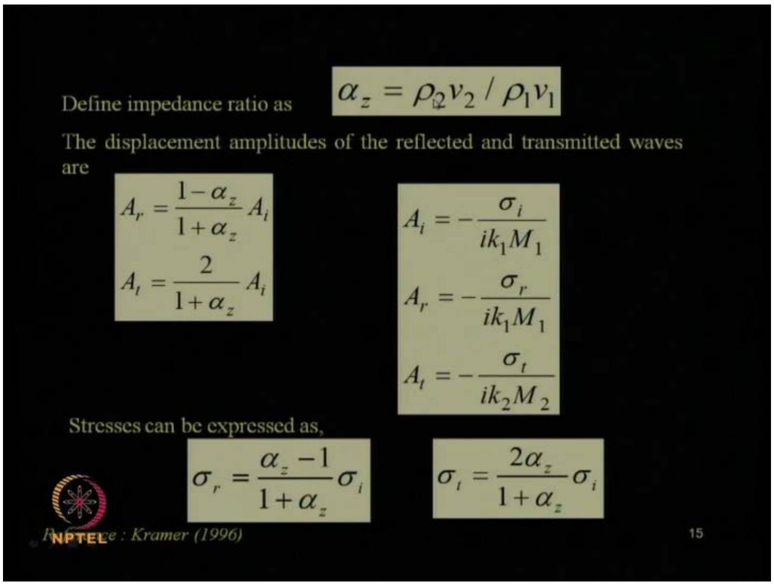

So, that ratio, which is called as impedance ratio; so what is impedance ratio? The definition of impedance ratio is nothing but it is the ratio of specific impedance of two materials. And, when we are taking that ratio, always in the numerator, there will be the material, which is in the transmitted zone; that is, the incident wave, where from the wave is coming that will be in the denominator. So, that is why, you can see this rho 1 v 1 for material 1 is in the denominator; that is the incident wave region; that is the specific impedance of the material, where the incident wave is coming from. And, rho 2 v 2 is the specific impedance of material 2, where the transmitted wave goes in. So, that is the definition of impedance ratio. This is a very important parameter because this defines through the material parameter of two materials. This gives us the idea, how much displacement amplitude of transmitted wave and how much stress amplitude of transmitted wave will occur for a particular incident wave. Why this is important? As I am mentioning, because suppose when earthquake occurs at a large depth; so, basically, from base rock level or bed rock level, that wave will start travelling where that earthquake focus is existing through this form of seismic waves.

Now, when this wave travels and finally, comes close to the grounds surface, we should know that after travelling through various layers of material, that is, from rock to stiff soil to soft soil typically, what will be the changes in the displacement amplitude and stress amplitude of the stresses, because that is most important; we need to find out. So, there this use of impedance ratio is very important, which we will be easily able to obtain from knowing the density and that particular velocity of the wave. Suppose we are talking about shear wave velocity, we have to take shear wave velocity of two materials. If we are talking about the primary wave velocity, we have to take corresponding primary wave velocity in the two medium like that. So, we have to find out the impedance ratio of the two materials. But, remember, the impedance ratio for primary wave or shear wave, etcetera will remain same for between two given materials. Why it will not vary between primary and secondary? Because if we have remembered the relationship between primary wave and secondary wave that is, through a constant relationship between the ratio. So, it is the ratio of two similar parameters. So, that is why, it gives the similar value of impedance ratio whether it is a p wave or s wave.

Now, what we can see from this (Refer Slide Time: 15:38) slide, the displacement amplitude of the reflected and transmitted waves – they can be represented through this impedance ratio in this simplest format. So, these two equations are very important; we should be able to find out what is the displacement amplitude of the reflected wave with respect to the incident wave amplitude, displacement amplitude over this impedance ratio; and, what is the displacement amplitude of the transmitted wave related to this displacement amplitude of the incident wave through this impedance ratio. That is why impedance ratio is so important.

And, how to estimate this incident wave displacement amplitude? If you know the loading condition or the stresses, which are getting developed for the incident wave, this is the stress amplitude. This is how the displacement is related to the stress amplitude in this format; that we have already seen the derivation in the previous lecture. Also, for the reflected wave, displacement amplitude is related to the reflected wave stress amplitude in this form. And, the transmitted wave displacement amplitude is related to the transmitted wave stress amplitude in this form through their modulus.

Now, stresses – they can be expressed in the similar fashion like through this impedance ratio like this; that is, the reflected wave stress can be obtained using this relationship for a given stress amplitude of incident wave. Also, the transmitted wave stress amplitude can be obtained from the incident wave stress amplitude through this impedance ratio. So, what does it mean? If suppose we know the stress amplitude of the incident wave of any particular earthquake and if we know the material property of two different layers of media, we can easily estimate all other parameters; can you see? So, only parameter we should know is this sigma i. So, if sigma i is known, you can compute from sigma i A i. Once you know A i, you will get A r A t. Once you know sigma i, you will get sigma r, sigma t.

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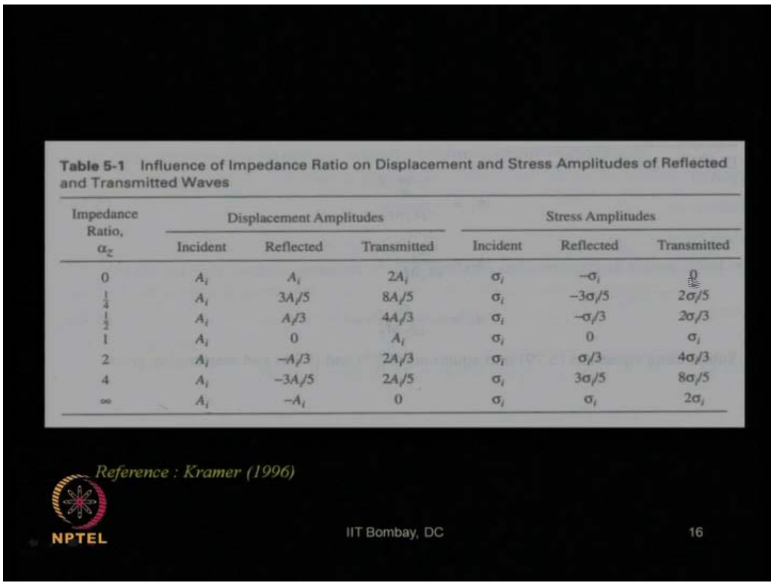

Table 5-1 Influence of Impedance Ratio on Displacement and Stress Amplitudes of Reflected and Transmitted Waves

Impedance Ratio, α_z	Displacement Amplitudes			Stress Amplitudes		
	Incident	Reflected	Transmitted	Incident	Reflected	Transmitted
0	A_i	A_i	$2A_i$	σ_i	$-\sigma_i$	0
$\frac{1}{4}$	A_i	$3A_i/5$	$8A_i/5$	σ_i	$-3\sigma_i/5$	$2\sigma_i/5$
$\frac{1}{2}$	A_i	$A_i/3$	$4A_i/3$	σ_i	$-\sigma_i/3$	$2\sigma_i/3$
1	A_i	0	A_i	σ_i	0	σ_i
2	A_i	$-A_i/3$	$2A_i/3$	σ_i	$\sigma_i/3$	$4\sigma_i/3$
4	A_i	$-3A_i/5$	$2A_i/5$	σ_i	$3\sigma_i/5$	$8\sigma_i/5$
∞	A_i	$-A_i$	0	σ_i	σ_i	$2\sigma_i$

Let us look at this table. This table is taken from Kramer’s book – Table 5.1. What are the influences of this impedance ratio value – that alpha z value on this displacement and stress amplitude of the reflected and transmitted waves. If you see, the impedance ratio equals to 0. What does it mean – impedance ratio equals to 0? Let us go back to the definition. Impedance ratio 0 means this is 0. So, in that case, incident wave displacement amplitude and reflected wave displacement amplitude – they remain same. Transmitted wave displacement amplitude is double of this. That you can easily get by putting alpha z equals to 0 in this equation. And, incident wave alpha i will be reflected wave minus alpha i; and, transmitted wave stress amplitude will be 0. What is this situation? Can you guess? It is nothing but a free boundary, where the stress at the boundary condition has to be 0. That is what it shows.

Suppose if we are talking about shear wave velocity. When this becomes 0, when v comes close to the ground surface, because it cannot travel in the air media. So, in that case, you will get this rho 2 v 2 equals to 0. So, that is nothing but when you come to a boundary and end of a material, there you will get… displacement amplitude gets doubled. But, stress amplitude – there will be 0; that is what.

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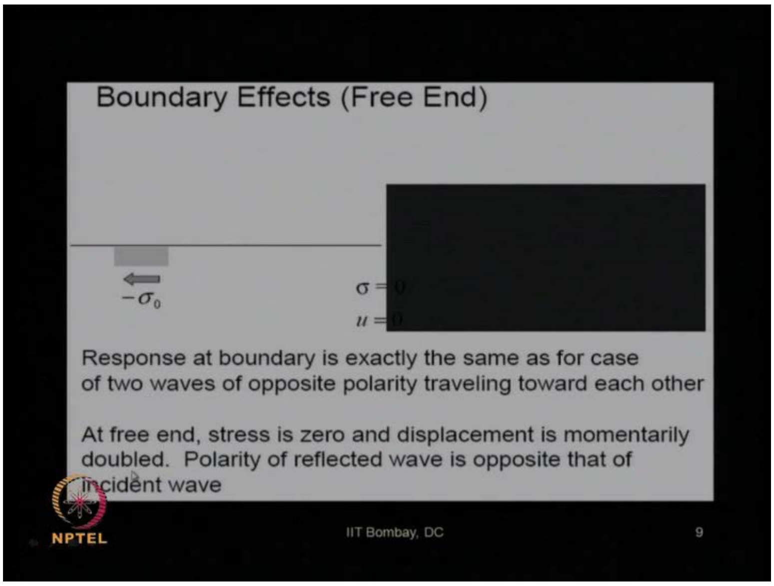

If you remember in our previous lecture, we have discussed this. We have mentioned that boundary effect of free end… What happens at the free end? Stress becomes 0, but displacement is doubled. Can you see this? So, that is what now we have seen through this mathematical expression also, which is valid.

Now, let us check another case. Impedance ratio equals to 1. What does it mean impedance ratio equals to 1? That means this material and this material are same; that is, there is no in fact, any material boundary. The wave is travelling through the same material. So, what is expected? Same incident wave only will continue. So, let us look at the displacement amplitude of incident wave A i reflected is 0, which is quite obvious. Nothing should get reflected. Entire thing should continue as a transmitted. So, that is, the same wave will continue. So, A i will continue. And, let us look at the stress conditions. Incident wave stress amplitude is sigma i. There is nothing called reflected. And, entire thing gets transmitted; that is, the same wave continues; so sigma i. So, these are the cross check whether this derivations of equations are correct or not through this common known conditions.

And, what is the impedance ratio of infinity? What does it mean when it will become infinity? Infinity means if this one becomes 0. Suppose there is a wave say p wave which travels in the air also goes to another media. So, we have incident wave displacement amplitude; reflected wave displacement wave is negative of that; that is, in opposite direction. Negative sign indicates, it goes in the opposite direction. Always remember, all the negative sign indicates the opposite direction. Transmitted – nothing get transmitted. Whereas, for stress amplitude, what happens, the incident wave and reflected wave stress amplitude are same; for transmitted wave stress amplitude, it gets doubled.

Now, what are the common practical forms of impedance ratio for our geotechnical earthquake engineering? If we go back to this definition of impedance ratio, we will see typically when we come from layered soil…

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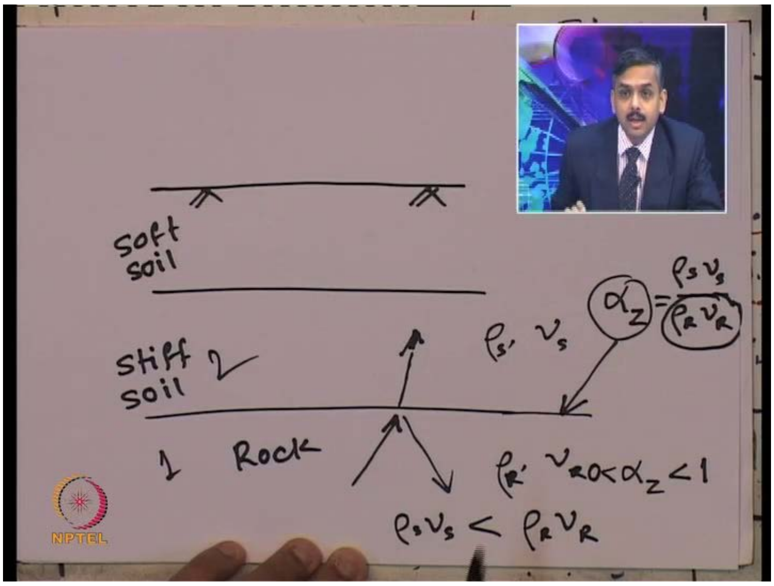

Let us say here say we are coming from stiff soil, rock, soft soil. This is the typical profile. Now, from rock, your waves are coming – incident wave. So, some of that will get transmitted; some of that will get reflected. Now, depending on the values of this rho and v of this material and this material, rho of rock, v of rock and rho of soil, v of soil – which one will have more density? Of course, the rock compared to soil. Now, which one will have more velocity of wave? Of course, again rock compared to soil. So, which material will have more specific impedance? Of course, rock will have more specific impedance than the soil. So, when we are talking about this alpha z between these two layers… We are considering alpha z between these two layers, which is rho of soil v of soil by rho of rock v of rock, because generally, these waves are coming from here. So, this is our material 1 here and this is material 2. So, that is what we have seen in this slide, the definition of impedance ratio is specific impedance of material 2 divided by specific impedance of material 1.

Now, which parameter is more? This parameter is much higher than this. So, now, rho s v s is much lower than rho r v r. That is the typical case for our geotechnical earthquake engineering, which practically we will get at site. So, at site, if we get this situation, obviously, alpha z should be… All practical cases will be less than 1. So, that is the practical range of alpha z. It should be between 0 to 1. Equality of 0 also we have seen; equality of 1 also we have seen. So, most practical site condition in terms of our geotechnical earthquake engineering is alpha z should be between 0 to 1. So, if alpha z is 0 to 1, let us see what happens.

Let us look at this table once again. So, alpha z value is between 0 to 1. So, these are the common ranges. As it is given, 0.25, 0.5… It can be different values based on the known soil parameters or rock parameters. That can be any value between 0 to 1. What happens in these cases? Let us say alpha z is 0.5. For that 0.5, what happens? When displacement amplitude incident wave is A i, reflected is A i by 3. Just putting that alpha z is equals to 0.5, you will get this; will get this also. Similarly, you will get this and this. That is what we are doing here. So, alpha z of 0.5 will give us reflected wave displacement amplitude much lower than what is the displacement wave amplitude for the incident wave. But, let us look at the transmitted wave. It is 4 A i by 3. What does it mean? That means the displacement amplitude of transmitted wave increased from that was there for the incident wave.

Let us look at the stresses. Stress incident wave is this one. Reflected wave is this minus, that is, opposite direction. And, transmitted wave stress is two-third of the incident wave. So, stress is not increasing, it is reducing for most of our practical geotechnical earthquake engineering problem. But, where the big difficulty arises? Difficulty arises with respect to this displacement. That is the reason an earthquake wave, which is not imperceptible or not that much damaging at large depth in rocks media. When it reaches through various soft soil, there will be many fold increase of this displacement amplitude of that wave. That creates more displacement when it reaches or goes through a stiffer to a softer media. That is the reason.

As I have already mentioned in the introductory lecture, Mexico city earthquake of 1985 is an example of this phenomenon of soil amplification; that amplification mostly occurs in the case of when you have a very soft soil. So, from bed rock to if the wave travels through various soil layers and finally, it goes through a soft soil media, you will see displacement amplitude of the transmitted wave in the soft media gets magnified, gets increased much more than what that incident wave was. So, we have to be very careful about these values, which we can easily estimate in the beginning itself knowing the material property of various layers. So, that is why, knowing the material property is very important to estimate the impedance ratio, which automatically will give us what will be the displacement amplitude of any transmitted wave.

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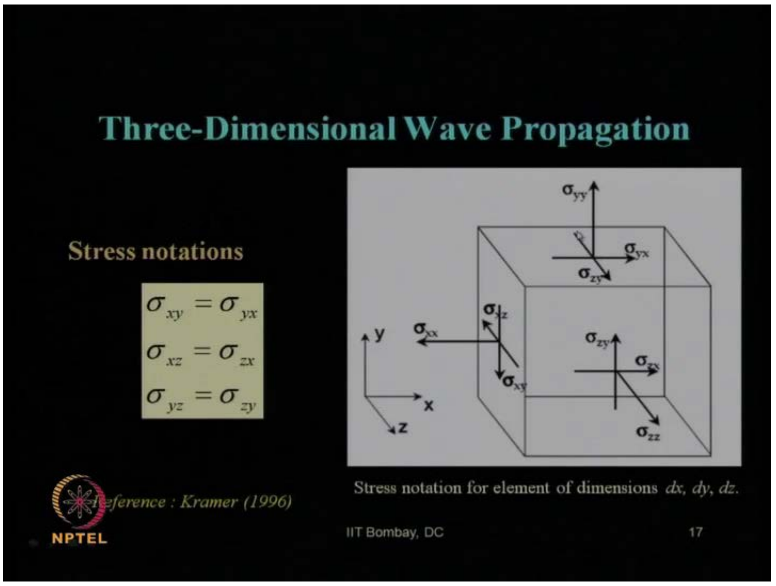

Now, let us come to the three-dimensional wave propagation. So, for three-dimensional wave propagation, let us first take a note of the stress notations. So, this is the three-dimensional element. So far, we have discussed about only one-dimensional wave propagation. Now, we are considering, wave is travelling in all the three directions. So, this is x-coordinate system, y-coordinate system and z-coordinate system. We have the normal stresses: in x-direction, sigma x x; in y-direction, sigma y y; in z-direction, sigma z z; and, the shear stresses like sigma x y. This indicates… As we know, the first parameter indicates the plane; the second one indicates in which direction it works. So, sigma x y means it is acting on x plane in y-direction. This is sigma x z – in x plane again, but in z-direction. So, these two are the shear stresses. Again, on this phase, we can have sigma y x, sigma z y. So, these are… This is actually sigma y z it should be; that is on y plane in z-direction.

Now, considering the stress notations of equivalence; that is, shear stresses of sigma x y should be equals to sigma y x. Why? Because to maintain the equilibrium of the body; otherwise, if they are different, there will be a formation of a couple. So, to maintain the equilibrium, the other phase shear stress has to be the same and in opposite direction; like sigma x y in the opposite direction should act over here; and, this sigma y x should be equals to sigma x y to maintain the equilibrium of the cube or of the soil element in three dimension. Similarly, sigma x z should be equals to sigma z x and sigma y z should be equals to sigma z y. So, that is why, these shear stresses over here: sigma z y in y-direction, sigma z x in x-direction. This is sigma actually z y – it will be y z, because in z-direction. So, direction will come last. But, as we know, sigma y z equals to z y. That is why it is written as sigma z y. Similarly, this is sigma y x in the x-direction, which is equals to this sigma x y.

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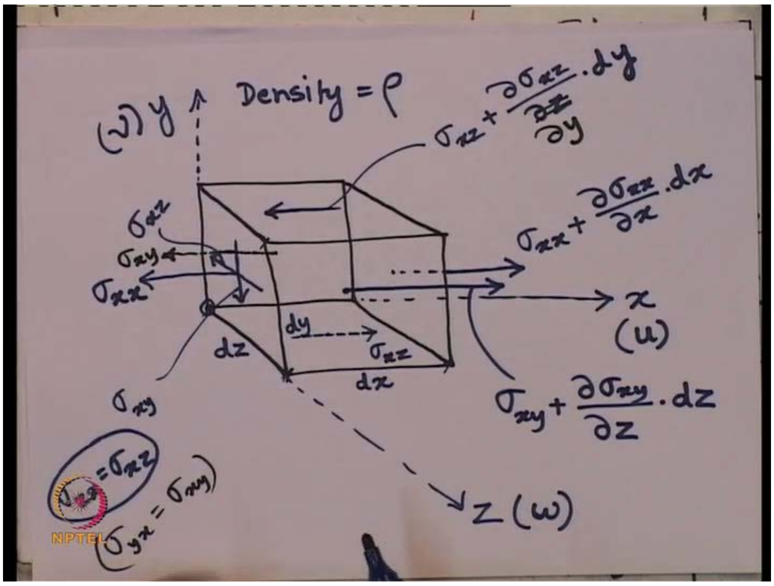

So, knowing this stress condition, let us start deriving; that is, for three-dimensional wave propagation, what will be the basic equation of motion that we are planning to find out. Now, let us take the stresses on the phase. Let us say this is our x-axis; this is our y-axis; and, this is our z-axis. Let us take only one direction first. Then, it will be easy to understand in the similar other directions as well. So, now, in the x-direction, we have at this phase… So, this is basically the origin as you can see. From this phase let us say, the normal stress is sigma x x. And, what are the dimensions of various sides of this small infinitesimal element? Let us say in x-direction, it is dx; in y-direction, let us say it is dy; and, in z-direction, let us say it is dz. So, this to this is dx; this to this is dy; and, this to this is dz. These are the dimensions of the material. And, let us say, the density of the material is known as rho. Corresponding displacements also, let us define. In x-direction, let us say, displacement is denoted as u; in y-direction, let us say, displacement is denoted as v; and, in z-direction, displacement is denoted as w – u, v, w. These are the corresponding displacements in the direction of x, in the direction of y, in the direction of z.

Now, if stress when wave is travelling through this infinitesimal element, what is happening? This is the stress – normal stress at one phase; the other phase on this side, the stress will be sigma x x plus there will be change of sigma x x over del x into this distance dx. That is the increment in the change or stress condition of the other side of the element. Why it happens? Because wave travels through this media. So, there is an inertia acting on the system; that inertia will create the stress difference between two phases. Like we have seen in the one-dimensional wave propagation equation also, the similar thing we are observing over here. So, let us look at the derivation over here.

Let us not look at the slide. Here first we will derive; then, we will go the slide. So, now, we are considering the shear stresses. The shear stress in this direction is, this is sigma x y; and, in this direction, we have sigma x z. Now, as I said, let us first concentrate only-direction of wave in the x-direction only. So, we will be interested to know all the stresses acting in x-direction. Now, this is not in x-direction; this is not in x-direction. So, we need not to consider these at this moment, because we are only considering the x-direction stresses. So, what are the x-direction stresses? We will see here; x-direction stresses will be on this phase. There will be this direction. This is sigma. It is in which plane? It is in z plane in x-direction. So, it will be z x. If we can the same notation as we have mentioned, x z also we can write. So, it is basically sigma z in x-direction; so z x. As we know, sigma z x equals to sigma x z. That is why we could write this as sigma x z. Remember that.

What will be the change over a travel of this distance of dy? There will be on this phase in this direction – opposite direction; this should be what? Sigma x z plus del sigma x z changing over the distance of del z into… It is changing over the length of y; it should be y. It has changed through this distance of y. So, what it should be? sigma x z plus del sigma x z that is varying with respect to the distance of y. So, this should be not del z; this will be del y over the distance of dy, because see here it is sigma x z; when it goes to on this phase, it moves by a distance of d y. So, its change will be over the length of this dy. So, del y times dy.

Now, what will be another one in x-direction? In x-direction, if I want to find out one more stress, what that should be? On this plane, on this plane, I have taken… Then, x y – another x y should come from… I have sigma x y over here. So, other phase will be on this front phase and back phase. So, this back phase will have this direction; the back phase – back side in the x-direction. So, what it should be? That is on which plane? That is xy plane. So, that is sigma; I can write xy. It is essentially sigma y x, because sigma y x equals to sigma x y. That is the reason why I could write it as sigma x y. It is on the back side. So, when it travels through a distance of dz on this front side, this phase, it will be this one. What it should be? It should be sigma x y, which is changing del sigma x y over the distance of it travels in the dz-direction. So, over del z times dz.

Now, if I consider the equilibrium of these forces; that is, on the outer phase or after they travel in the x, y, z-directions, the increase in the stresses whether it is normal stress or shear stress in x-direction. So, all x-direction forces now if we consider, what we will see?

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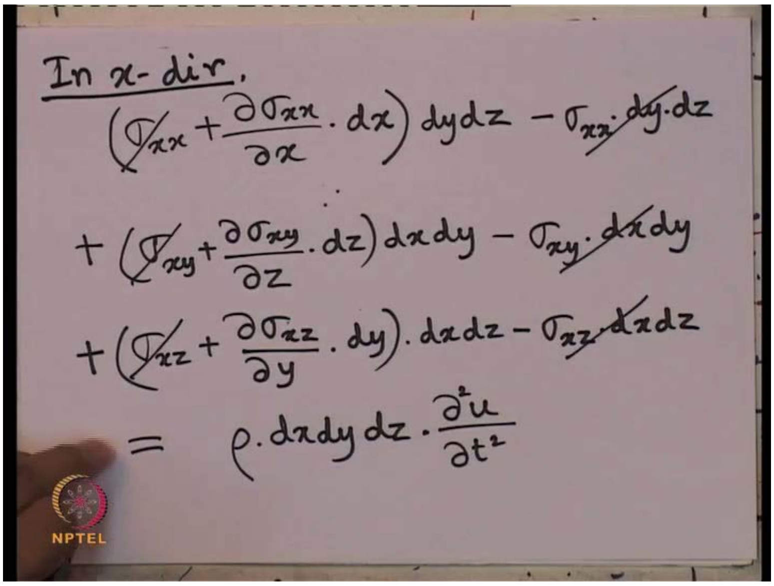

Let us write it down further. We can write down that, sigma x x plus del sigma x x by del x into dx… This is working on which plane? This stress is acting on the plane with cross-sectional area of how much? This is dz; this is dy; multiplied with dy dz. That is the force. That is the positive force in this direction. Opposite force in the other direction is sigma x x times same dy dz. So, minus sigma x x dy dz. So, that is the resultant force in positive x-direction due to the normal stress difference.

Next, I have to write another one, which is sigma x y plus del sigma x y by del z into dz. This shear stress let us look at here. This shear stress is acting on which plane? The cross-sectional area is dx times dy. Look at this plane – front plane. This shear stress is acting on this front plane. So, if I multiply that shear stress with respect to this area – cross-sectional area dx times this vertical dy, we will get the net force in the x-direction. So, dx dy. In the other direction, what is the force, which is acting? This dotted one in black color on the other side of the phase; on this side; on that side. So, that will be minus sigma x y times dx dy.

What else we have? Another force let us look at here. We have… In this, we have taken actually the direction will be different. In the positive direction, it is increasing. So, it should be the incremental portion; it should always in positive direction; and, this should be in this negative direction. So, that is the direction cosines. Also, we can check why it should be in this direction; because this and this should be in this format. And, this – the other side should be in this, so that they are in equivalence. So, with this direction, what we can write now? This is the positive direction in x; this is the shear stress incremental part. That is acting on which plane? What is the cross-sectional area over here? That is dx times dz. So, sigma x z plus del sigma x z by del y times dy times dx dz minus… What is this one acting in the other direction? Sigma x z. So, sigma x z times dx dz.

Now, this net force, which we are getting – this should be equals to… It should be equals to the inertia in x-direction. What should be the inertia? Mass times the acceleration. Now, what is acceleration in x-direction? That is, if you differentiate displacement twice with respect to time, displacement in x-direction is u. We have already mentioned. So, del 2 u del t square is the acceleration. And, what is the mass? Mass is density into the volume. What is the volume of that element? dx, dy and dz. So, inertia has to balance that difference in stresses at two phases of the element in x-direction. So, this is the derivation for in x-direction only. So, what we can write? This, this gets cancelled; this, this gets cancelled; this, this gets cancelled.

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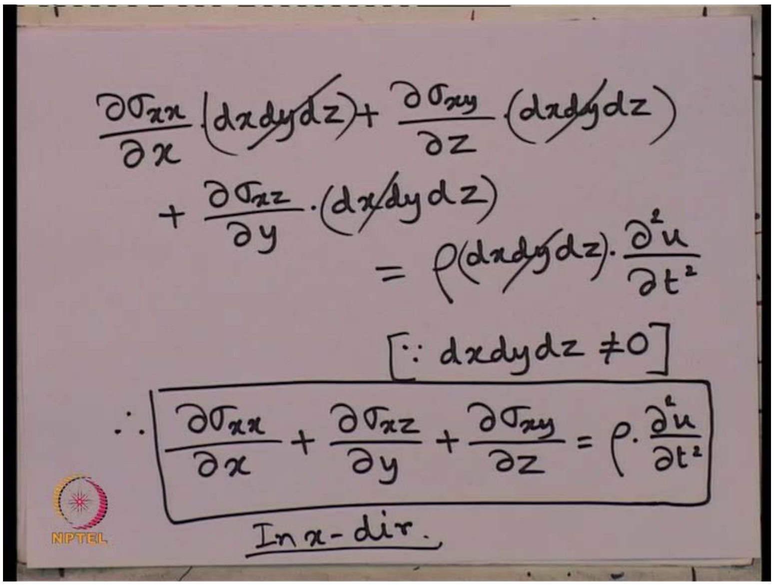

So, on simplification of this equation, we can further write that, del sigma x x by del x into dx dy dz plus del sigma x y by del z into dx dy dz plus del sigma x z by del y into dx dy dz. That is equals to rho dx dy dz times del 2 u by del t square. So, this term I can cancel from both the sides, because this is the volume of the element, which is nonzero; has to be nonzero. Therefore, what we can write? del sigma x x by del x plus del sigma x z by del y plus del sigma x y by del z equals to rho del square u by del t square. So, this is our basic governing equation of motion for three-dimensional wave propagating in x-direction. So, this is in x-direction only. Similarly, if I want to find out what will be the equation in y and z-direction, in the similar format, if we want to derive, the form of the equation will be… Let me write it down for y and z-directions.

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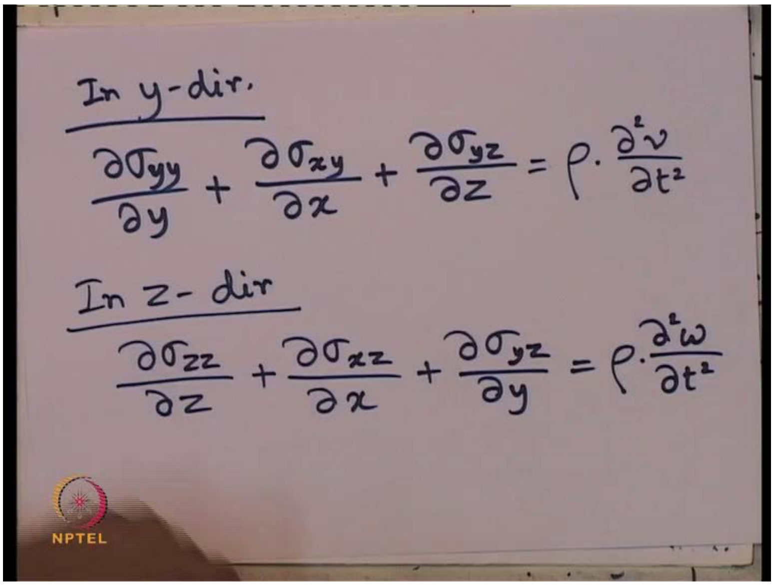

In y-direction, it should be del sigma y y by del y, because this is the normal stress component plus del sigma x y by del x plus del sigma y z by del z should be equals to rho times del square v by del t square. And, we will get here in z-direction, in the similar form, del sigma z z by del z, that is, the normal stress plus del sigma x z by del x plus del sigma y z by del y equals to rho del square w by del t square. So, these are three governing equation of motion, which we will get after solving this three-dimensional wave equation in the solid, which is shown over here.

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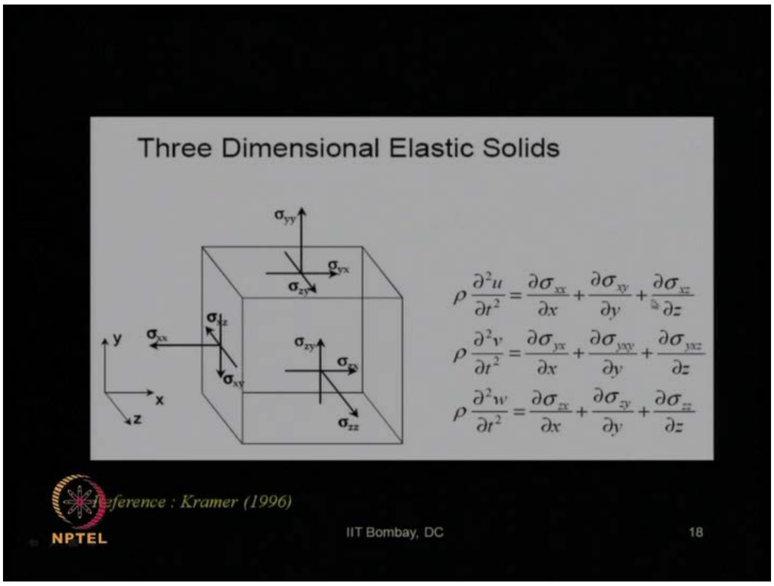

You can see in this slide, three-dimensional elastic solid, which we have derived now. So, this is the final derived equation as I have mentioned just now.

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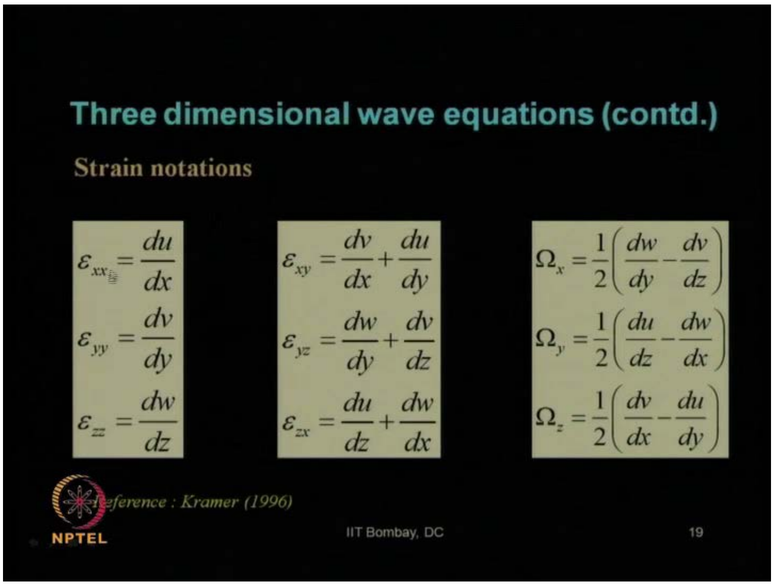

Now, using the strain notations… The strain notations – what we can use? The normal strain in x-direction, in y-direction, in z-direction – those are nothing but epsilon x x is del u by del x; epsilon y y is del v by del y; epsilon z z is del w by del z. These are normal strain. What are the shear strain? These are the shear strain. Epsilon x y – so, del v by del x plus del u by del y; epsilon y z – del w by del y plus del v by del z; epsilon z x

– del u by del z plus del w by del x. And, what are the torsional strains? Torsional strain in x, y and z-direction can be represented by these equations.

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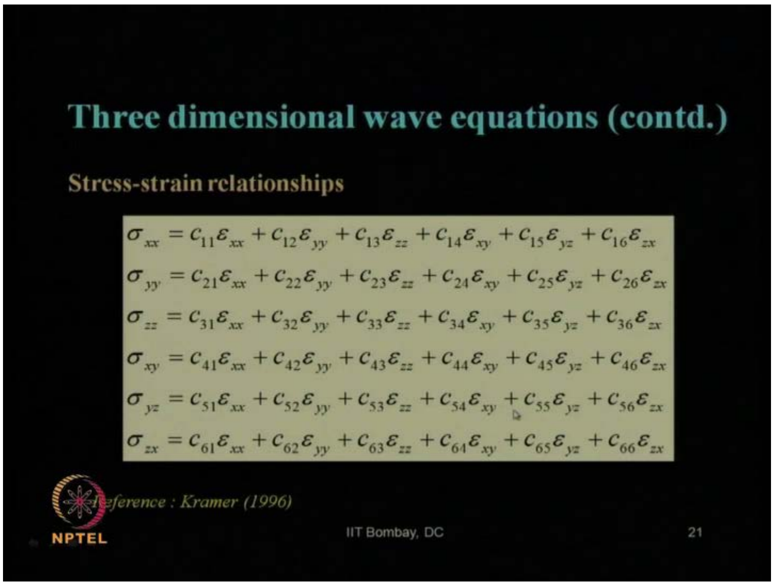

Now, using the stress strain relationship… How we can use the stress strain relationship for this three-dimensional wave equation? This normal stress in x-direction – sigma x x can be represented through strain through some modulus. So, c 1 1 epsilon x x plus some modulus c 1 2 epsilon y y plus c 1 3 epsilon z z plus c 1 4 epsilon x y – this modulus is related to the shear, because it is relating to the shear strain. These are relating to the normal strain. So basically, what you can see, other two shear strains – all these modulus coefficients, that is, stress versus strain can be easily written in the form of a matrix; that is what it shows. So, what is that form of the matrix?

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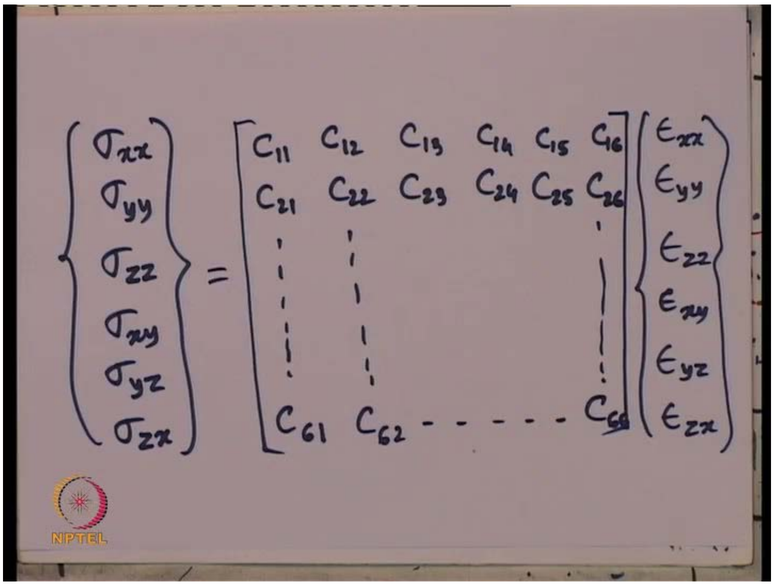

In the simplest form, if we want to rewrite it, the stress strain relationship is something like this; sigma x x, sigma y y, sigma z z, sigma x y, sigma y z, sigma z x. These are the six stresses: three normal stresses and three shear stresses. They are related through some modulus, which is nothing but a modulus matrix through the strain. What are those strain? Epsilon x x, epsilon y y, epsilon z z, epsilon x y, epsilon y z, epsilon z x; so these strains are first three normal strains; and these three are shear strains. They are related through this matrix, what we were showing over here - c 1 1 c 1 2 c 1 3 c 1 4 c 1 5 c 1 6; c 2 1 c 2 2 c 2 3 c 2 4 c 2 5 c 2 6 so on. This one will be c 6 1; this one will be c 6 2. Similarly, goes over here up to c 6 6; like that. So, this matrix is nothing but modulus matrix. So, this stress-strain relationship can be expressed through this modulus matrix. With this, we have come to the end of today's lecture. We will continue further in the next lecture.