Structural Analysis II

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Lecture – 08

In the last few lectures we have looked at how to get the static and kinematic indeterminacy of a structure. Then we have reviewed the basic concepts of the principle of virtual work. After that, we have actually looked at applying the virtual force method in getting the response of statically indeterminate structures using the force method. The whole point is that you should have already been exposed to the force method. I just spend some time looking, reviewing and especially using the principle of virtual work to set up the equations.

Today, we are taking up a truss example and going through the entire procedure. I promised last time that I should be looking at realistic example problems and today that is what I am going to be doing. I am going to take a realistic problem: a truss problem, then I shall go through with the entire analysis procedure and get the answers so that you would be able to understand all the steps that go through in using the force method for a truss example. Next lecture, I am going to be taking up a beam and a frame example so that we can walk through not only the procedure itself, but the steps in the procedure, so that you can get whatever it is that you have been asked to find out. Let us now look at the truss example.

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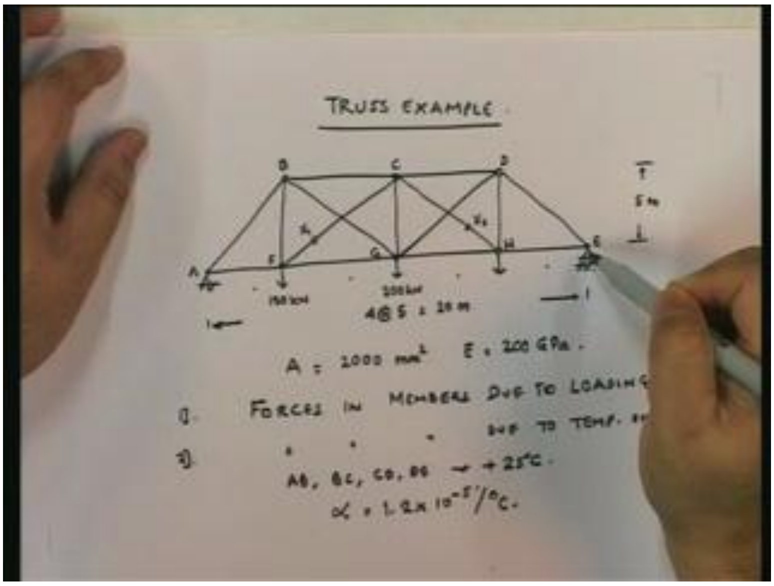

This is a typical truss bridge that you are likely to see if you ever travel by train, on a railway bridge. Of course what I have done is: typically you have many panels in a railway bridge and I have just taken four panels 1, 2, 3, 4; each panel is 5 meters length and therefore the total span of the bridge is 20 meters. If you look at the height of the bridge this is 5 meters; this 5 meter length is a typical length that you would see in most bridges in India. The only thing that I have done is have taken the panel the BCFG panel and I have put two diagonals in it and taken the CDGH panel and I have put two diagonals in it. These are the end portals, these are the verticals, these are the diagonals and this is the bottom chord and the top chord (Refer Slide Time: 04:18 min). So BCD is the top chord, AFGHE is the bottom chord and

BFCGDH are the verticals. AB and DE are the end rakers and the diagonals are BG, FC, CH and GD.

You are given that there is a loading. 150 Kilonewtons at F, 200 Kilonewtons at G, 150 Kilonewtons at H. So this is the loading and you are also given that all the members have an area of 2000 millimeter squared and the E value - this is steel - so it is 200 GPa. The question is: find the forces in members due to the loading alone and the second part is find the forces in the members due to a temperature effect only. The temperature affects the top members AB, BC, CD and DE; these are subjected to plus 25 degree Celsius increase in temperature and alpha is given as 1.2 into10 to the power of 5 which is a coefficient of thermal expansion for a steel member. This is the entire problem and this is what you have to find out.

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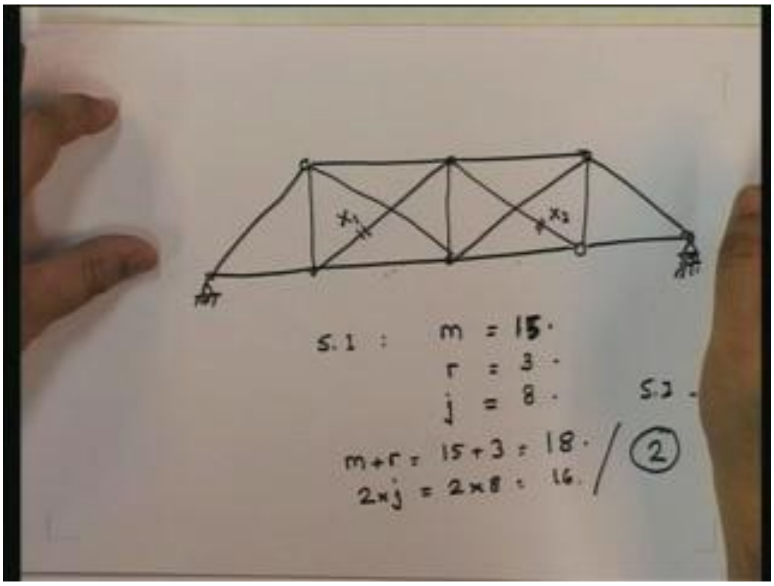

If you look at this again, let us first find out what the static in determinacy of this structure is. For that let me draw the structure again. We have to find out the static determinacy of truss. First and foremost, find out the number of members, it is 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. The number of members is 12. Number of support reactions are 1 2 at the hinge and 3 at the … One at the roller, so total 3. Now we have to find out how many joints; 1, 2, 3, 4, 5, 6, 7, 8. How many members do you have? 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15; so we have 15 members. I missed the 3 verticals when I was doing the computation; so there are 15 members. Let us look at it. How many unknowns do we have? For each member - one force. The actual force in the members is unknown and the number of reactions are unknown. The total number of unknowns are: m plus r which is 15 plus 3 which is equal to 18 and number of equations are two for each joint, so it is 2 into 8 16. So, the static indeterminacy of this problem is 2.

You see we are going through all the steps, in that we have first the computation of the static indeterminacy of this structure. Once we have the static indeterminacy, then we find out the number of redundants. Redundants are those which if we put equal to 0, then we have a stable statically determinate structure. Let us look at that. In this particular case, this is my X_1 and this is my X_2.

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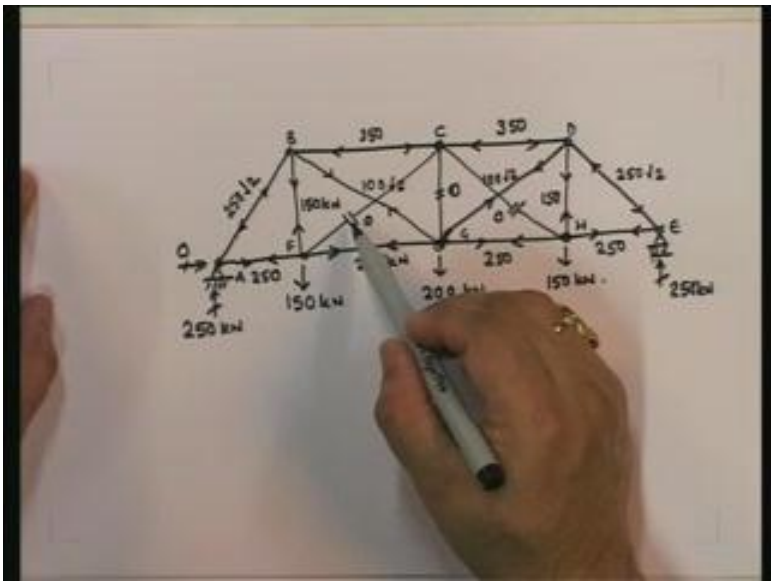

These are my members, therefore if I look at my base structure, actually this member and this member are the same size because the panel size of this and this is the same. So this is my base structure. Now, I am solving the first part of the problem which is forces in the members due to loading. So let us put the loading: 150, 200, 150. This is the loading and this is a statically determinant structure. I need to find out all the forces in all the members due to the loading alone.

If you look at it, we see sigma f_x is equal to 0 gives me this, and sigma f_y is equal to 0 and taking moments about any point you will see the symmetric loading. So therefore without much ado, I am going to write down the reactions. If you look at the reactions then this is 250, 250 here and 0 is the horizontal truss here. So once we have done that, we can actually start solving for the structural loading and therefore if you look at it, I can start at this point (Refer Slide Time: 11:22). Let me just put it down. This is A, B, C, D, E, F, G, H. Taking the method of joints at this point we will see that this is equal to (because the vertical component of it has to be equal) this and therefore the horizontal component of this is equal to this (Refer Slide Time: 11:50)) so this is 250. Now, if we go over here and take moments of joints here, this turns out to be 250 and this turns out to be 150.

Once we have that, then again I can actually go everywhere, if I do this here this (Refer Slide Time: 12:25) will also turn out to be 250 over root 2, this is going to be equal to 250. Similarly this will be 250 and this will be 150. Once we do that we can find out the forces in these members also; without much ado; I am just going to go ahead and do this. If you look at this, the vertical component of this member has to be equal to this minus this (Refer Slide Time: 13:18). If you really look at it this will become 100 root 2 and if we do that then this becomes 350; if you take moments at this particular point, this member turns out to be a zero force member, this one is 350 and similarly this is 100 root 2. So I have got all the forces and all the members.

Now the thing is what is the next step? Remember that these members are cut but they still exist. Remember that we know that the forces and these members are 0 because we have cut them. However, members still exist so I should actually put it down, 0 and 0. However, one other point, again I am going back. How do we get the additional equation? We have 16 equations and 18 unknowns. We actually need two more equations so that we can solve for X_1 and X_2. How do we do that? By actually finding out the displacement of these two points relative to each other given the loading, and also given X_1 and X_2. Once we find those out then we can find out that the compatibility condition over here will be that this displacement has to be equal to 0. The compatibility condition here is that the displacement has to be equal to 0 so that gives us two additional equations. Once we have those two additional equations, we can solve for them, get X_1 and X_2. Once we get X_1 and X_2, all the member forces can be found out. Now how do we find out the displacement at that point? Now I want to introduce one concept and that is, that if you note, that all I need to do, just look through this:

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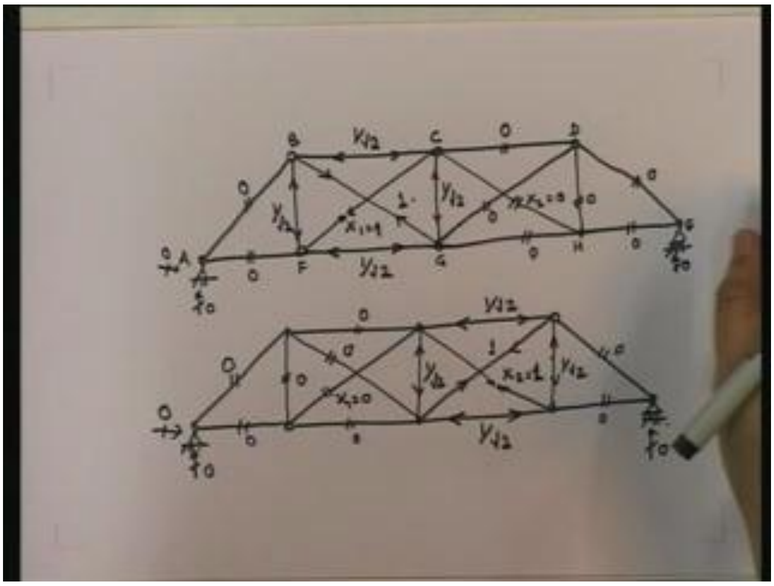

I need to actually solve two other problems. Let me go through it and then explain to you what I am doing. What I have done here, is that I need to look at two additional analyses. If you look at this, what does this represent? Here, what I have done here is corresponding to redundant force, I have put a unit force here and corresponding to the second redundant force this is 0. The second structure has the force corresponding to the first redundant as 0 and the force corresponding to the second redundant as 1 and I claim that if, I solve these two additional equilibrium problems I should be able to analyze the structure.

Let me explain how that is. Let us look at this; one of the first things is the compatibility. For the compatibility I have to find out the displacement at this point due to the loading, plus due to X_1, plus due to X_2. Actually, the displacement at this point I need to find out under this loading, under this loading of course not equal to 1 equal to X_1 and under this loading where X_2. If you add this and this and this with this not 1 but X_1 and X_2, then you see that is the actual structure that you have, and this point is referred (Refer Slide Time: 18:55). What you are trying to do is again use superposition. Find out the displacement at both the cuts due to all the loads and then sum them all up and that will give you the actual displacement.

Now, the loading we have done; if I need to find out this displacement and I am using the principle of virtual displacement what would I do? I would use a virtual force system where this was equal to some arbitrary value; I might take it equal to 1. If I wanted to find out the displacement here, what would I have to do? I would have to take a unit force corresponding to this and then find out the work done. So this would be the virtual force system to find this displacement. This would be the virtual force system to find this displacement under this loading.

Furthermore, under this loading I need to find out the displacement here. So under this loading you see this problem. Again it is the same loading excepting that is not equal to 1, there it will be multiplied by X_1. That will give me the displacement at this point. The real force system is this for finding out this displacement here. This is the virtual force system for finding out the displacement here. Then to find out the displacement here, due to this loading - this is the virtual force system Now let us look at this particular case. In this particular case under this loading, if I have to find out the displacement here - then this is the virtual force system. If I have to find out the displacement here - then this itself is the virtual force system. You see the point that I am trying to make is, that as long as I have these three: this one, and this one and this one equilibrium; I can actually get all the variety of things. Therefore, I am going to solve for this and I am going to put is equal to 1 here; because if it is the actual load then all I have to do is find out the forces here and multiply by X_1. That is not just the forces but even the displacements I get here I multiply by X_1 and I get the actual displacements due to X_1.

First, let me solve these structures and then we will go on to see what additional points we have to note. I am going to make a general kind of development on the theory that we have already done where the displacement plus displacement plus displacement is equal to 0. I am going to just put that down in proper format so that you get the complete way of solving all problems. First let me analyze this problem. Under this loading it is obvious, since there is no external loading, sigma f_x and sigma f_y and sigma m is equal to 0 as around any point shows that these reactions are 0. If these reactions are 0 then these member forces automatically are equal to 0. If we look at this here also, this is the force right? So this force is equal to 1. If I take the vertical component of this at this point it will be taken by this, so therefore this is going to be equal to 1 over root 2. The horizontal component of this will be taken by this, so this is 1 over root 2. Then again if I look at this point, this is equal to 0, this is equal to 0 and then since this is equal to 0, this will be equal to 0. So this is equal to 0, because if you look at this since this is equal to 0. At this particular point we have to be careful to see what are the other forces we get.

Now let us look at something where we know. Do I know all the forces? I do not know all the forces; there are three unknowns here. Which place should I go to? I will go to this point where I have two unknowns. If I look at this particular one, the vertical component of this is going to be equal to this. Therefore if I put that you will see that this will be equal to 1. Then if you look at this particular one, the vertical component of this is equal to horizontal component of this. So this is going to be equal to 1 root 2. Where do we go? Here we know 3 of them.

If you look at this particular one I know 1, 2, 3 so I have only 2 unknowns so I can solve for them. If I look at the vertical component does it give me anything? No. Can I look at the horizontal component? You know this horizontal component this one does not contribute; only this contributes. So if we look at this, the horizontal component of this is 1 over root 2 and this is opposite, so summation equal to 0. Therefore this force has to be equal to 0.

Now if this force is equal to 0, then only the vertical component of this comes in here and if we look at this, this becomes 1 over root 2. Now let us come here to this point. I know this force; I know this force; I know this force; I know this force. So I just need to check whether this will come out. What will this be? The horizontal component. If I take the horizontal component of this, I get minus 1 upon root 2, so this becomes equal to 0. If you look at the vertical component this is 1 over root 2; everything checks and if I come over here all forces in the members are 0, so therefore it checks. You will always get three checks.

Here I have got sigma f_x and sigma f_y equal to 0 and here I have got sigma f_y equal to 0; these are the three checks that I have in this entire procedure. That means these are the forces due to this application of X_1 equal to 1 and X_2 equal to 0. Similarly, if I apply X_2 equal to 1 and X_1 equal to 0 then we just get exactly the same kind of thing. Only thing is that on this side because this X_1 these panels are non-0 and these panels are 0. Here you will just have the opposite. These panels are 0. I am not going to go through these steps; I will request you to go through these steps yourself. As I said analysis is not something that I am going to be doing. I am just writing down the values; statically determinant analysis you should already be totally used to it. Then this is 0, this is 0 and this is 1 over root 2. This is equal to 0 and this is equal to 1 over root 2. I think that takes care of it; those are the two analyses. Now, having done these three analyses let me write down all the values that I have in a particular pattern. I am going to write it down in a table.

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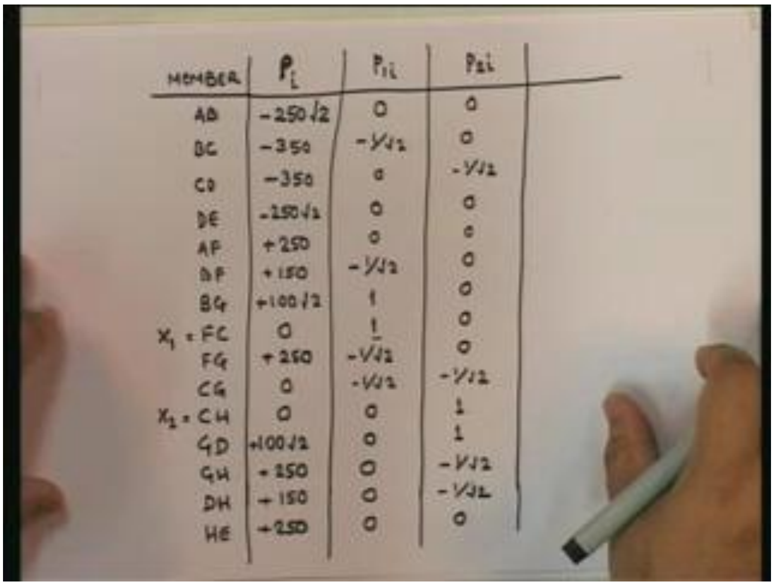

MEMBER	P_i	P_{1i}	P_{2i}
AB	$-250\sqrt{2}$	0	0
BC	-350	$-1/\sqrt{2}$	0
CD	-350	0	$-1/\sqrt{2}$
DE	$-250\sqrt{2}$	0	0
AF	$+250$	0	0
BF	$+150$	$-1/\sqrt{2}$	0
BG	$+100\sqrt{2}$	1	0
X_1 = FC	0	1	0
FG	$+250$	$-1/\sqrt{2}$	0
CG	0	$-1/\sqrt{2}$	$-1/\sqrt{2}$
X_2 = CH	0	0	1
GD	$+100\sqrt{2}$	0	1
GH	$+250$	0	$-1/\sqrt{2}$
DH	$+150$	0	$-1/\sqrt{2}$
HE	$+250$	0	0

First the table will have the member; so members are here, then I am going to write down the load. This I am writing down is the actual force in members due to the loading. Next I am going to write down and I will call this P_{1i} , this is the force that I have got in the members (Refer Slide Time: 28:55) due to X_1 equal to 1 and X_2 equal to 0; these are what I call as $P_{1i.}$ I will explain to you why I call it P_{1i} and then this I will call $P_{2i.}$ I am going to write it down as $P_{2i\ldots}$ Let me write them all down. I have member AB, then I have member BC, then I have member CD, then I have member DE, then I have member AF, BF, BG, FC, FG.

Now I am going to write down all the members here. But you must understand the point that there are a total number of 18 member forces so I have to actually put it down. FG, then I have CG, then I have CH, then I have GD, then I have GH, DH and HE. How many? It is 15, i.e. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15. I have 15 members here and I am going to write down the values of the loads that I get. In member AB I got a compressive force and compressive force for me is going to be equal to negative.

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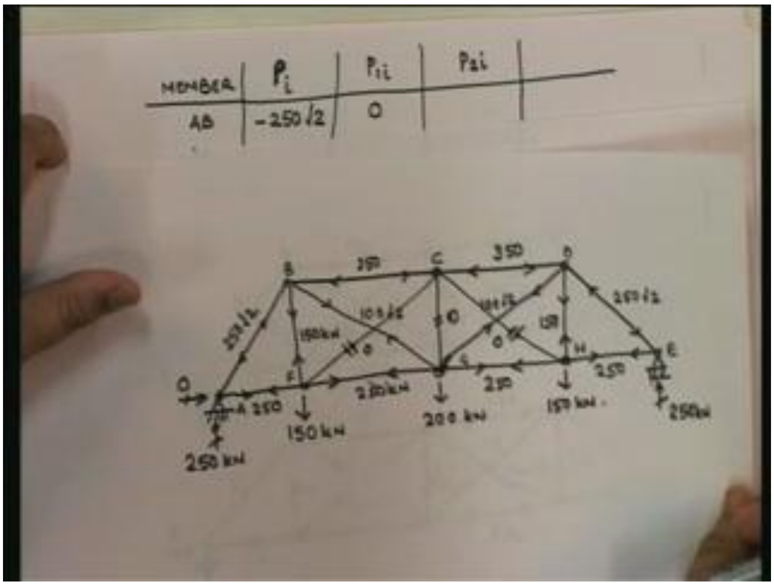

I am just actually trying to explain it to you. What is member AB? 250 compressive, that is all I have done – 250. Then if you look at this AB, in this 0, in this 0 so 0.

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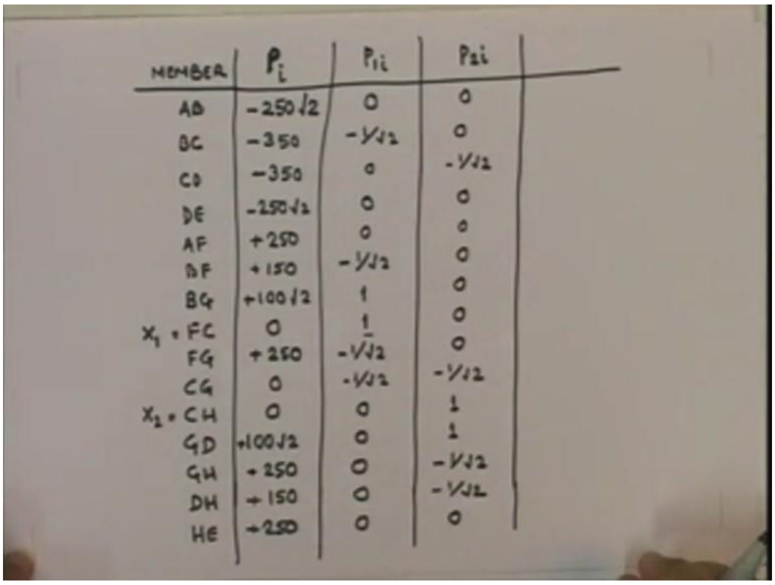

All I am doing here is actually just putting down all the values for you. This is going to be compressive minus 350. Here this is going to be minus 1 over root 2 and in this case 0. This is CD minus 350, then 0, minus 1 over root 2 and then DE is minus 250 for the loading, for this it is 0 and for this it is 0. AF is plus 250 because it is tension and in this case AF is 0, in this case it is 0. BF is plus 150 here, minus 1 over root 2 here and 0 here. BG is going to be plus100 root 2, here BG is equal to 1, here BG is equal to 0. FC is equal to 0 here because FC is X_1, so here it is going to be 1 and here it is going to be 0.

This is FC, the force here is X_1, so obviously it will be 0 here, 1 corresponding to X_1 is equal to 1 and 0 when X_1 is equal to 0. FG is plus 250 here then it is minus 1 over root 2 here and 0 here. CG is going to be equal to 0 here, minus 1 over root 2 here and minus 1 over root 2 here. CH is equal to X_2 so this is going to be 0 here (Refer Slide Time: 34:06), 0 here and 1 here. We look at this for X_1, 0, 1, and 0; for X_2 0, 0, and 1. GD is going to be equal to 100 plus 100 root 2 here. GD is going to be 0 here and is going to be equal to 1 here. GH is going to be plus250 here, it is going to be 0 here and it is going to be minus 1 over root 2 here and finally I have DH. DH is plus150 here, DH is 0 here, it is minus 1 over root 2 and finally HE is plus250, 0 and 0. These are all the members that I have.

Now, understand that this is the real loading; so in the real loading case if I want to find out the displacement corresponding to in FC the separation, then what is my virtual displacement pattern? This one and if I want to find out the displacement corresponding to X_2 in CH, this is going to be virtual force.

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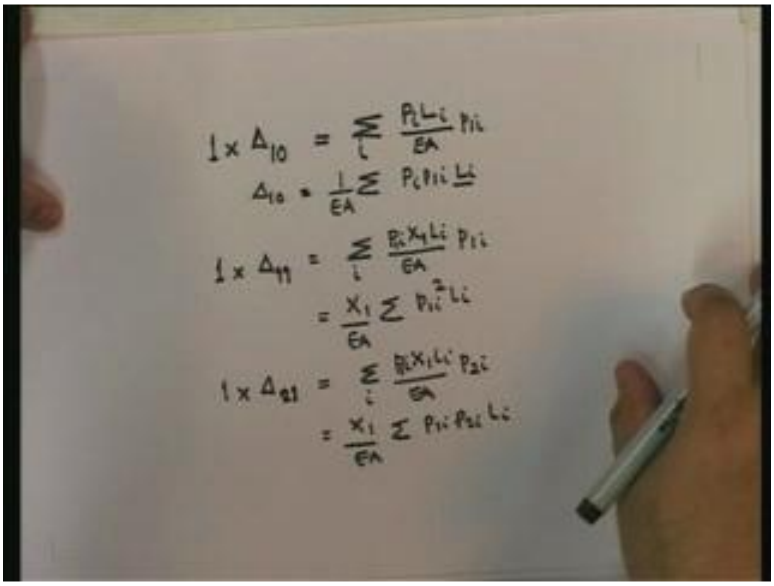

Here, I am going to now put it down on paper what virtual displacement gives me in the first case 1 into delta_{10}. What is this delta_{10}? It is this separation (Refer Slide Time: 36:25). This is my virtual displacement pattern where I have put X_1 is equal to 1. So for that one the virtual work principle is going to be 1; that is the virtual force corresponding to delta_{10} is equal to summation over all the members 'i' and each member deformation. What is the real deformation? Real deformation is the real load multiplied by Li upon EA note that EA here is the same for all the members; this is the real deformation multiplied by the virtual force which is going to be equal to P_{1i}. Is that clear? Understand that delta_{10} is nothing but summation of…. Since EA goes outside; because EA is same for all the members this is going to be equal to P_{1i} L_i. So that gives me delta_{10}. How do I get delta_{1x}? delta_1 due to X_1 delta X_1 is equal to what? Let us look at it this again. Here, the real load is going to be P_1 X_1 because P_1 X_1 gives me the forces in all the members due to the load X_1 the real load. So this going to be this load into Li upon EA multiplied by this one; so the virtual force pattern is going to be $P_{1i.}$ If you really look at this, this turns out to be equal. X_1 and EA are independent of each other so I can take it outside, so I get X_1 upon EA, inside P_{1i} squared L_i.

What do I want to find out? I wanted to find out $delta_{21}$ that is the load corresponding….. The real load is X_1 and I want to find out the displacement corresponding to $X_{2.}$ If I look at this now, I am just going to write these down; I do not need to talk about them anymore. The real load is 1. So the real load is P_{1i} into X_1 into Li upon EA and the virtual load is P_{2i} because there is a displacement that you are finding out, if I put this in, I get X_1 upon EA summation P_{1i} P_{2i} L_i. I can put them down in this fashion.

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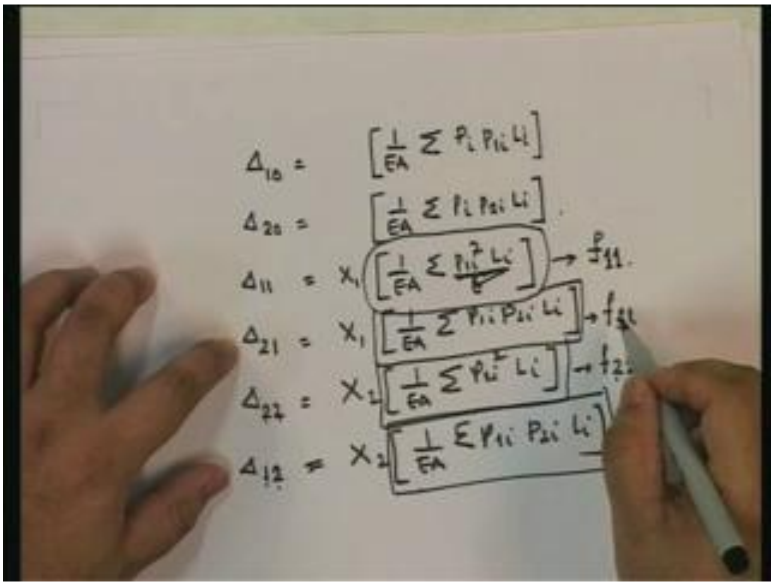

Using the virtual work I got $delta_{10}$ is equal to 1 upon EA P_i P_{1i} L_i. $delta_{20}$ is equal to 1 upon EA summation P_i P_{2i} L_i. $delta_{11}$ this is due to X_1 is going to be equal to X_1 into 1 upon E P_1 square L_i. EA is already outside so the EA is not required. Of course if EA is not the same you can put EA inside also. Then $delta_{21}$ is equal to X_1 into 1 upon EA P_{1i} P_{2i} L_i. $delta_{22}$ is equal to X_2 1 upon EA P_{2i} squared Li and $delta_{12}$ is the displacement at 1 due to X_2 is going to be equal to X_2 1 upon EA summation P_{1i} P_{2i} L_i.

If you do this, let us look at what this term represents (Refer Slide Time: 42:03). This term represents the displacement at 1 due to a unit $X_{1.}$ Displacement at 1 due to a unit load $X_{1.}$ That can be defined as a flexibility coefficient 1-1. What is flexibility coefficient? Understand displacement due to unit load is flexibility. I am putting down f_{11} because that corresponds to $delta_{11;}$ this is the displacement at one due to a unit load at one. What is this? This part is displacement at two due to a unit load X_1 so $f_{21.}$ What is this, displacement at two due to a unit load at one? What is this? $delta_{22}$ load displacement at two due to load at two, so this part is equal to $f_{22.}$ Displacement at two due to a unit load at two and this part is equal to $f_{12,}$ displacement at one due to load at two, so this will be displacement at one due to a unit load at two. All these are nothing but the flexibility coefficients for this structure.

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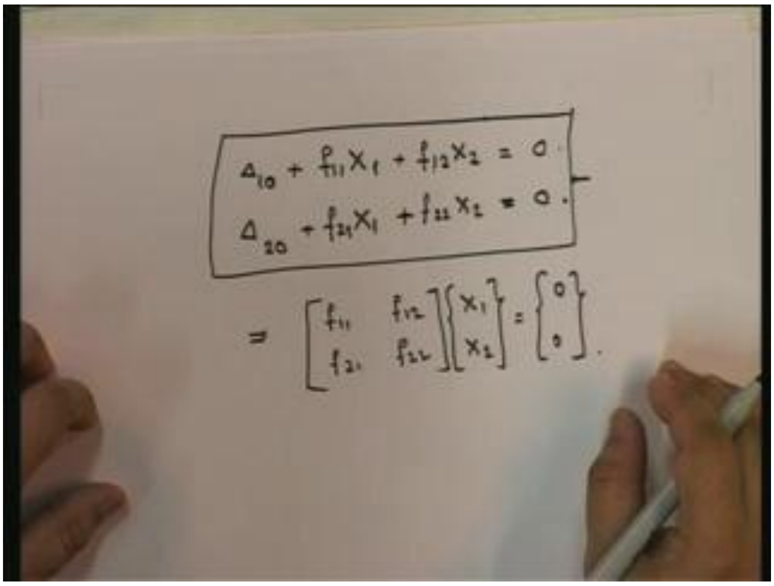

If I put them together you will see that I can rewrite this equation in this manner: $delta_{10}$ this is the displacement at one due to the loading, plus f_{11} X_1 displacement at one due to unit load multiplied by the load itself, plus f_{12} X_2 displacement at one due to the load at two multiply unit load at two multiplied by X_2 and this. If you look at this, what is this displacement at one due to loading, displacement at one due to X_1, displacement and one due to X_2. If you add all of the three up and that is the original structure. This is the displacement in the original structure. What is the displacement in the original structure? Those two points cannot move. Therefore the displacement in the original structure is this. Now let us look at the second compatibility condition. This is the compatibility condition corresponding to the first load. So what I am saying is that at this point due to summation of all the effects these two points cannot go apart because FC is a actually a member.

Similarly, for the other one f_{21} X_1 plus f_{22} X_2 is equal to 0. This is displacement at two due to X_1 because due to unit multiplied by X_1 this gives me the displacement at two due to X_1. Displacement at two due to X_2, displacement due to the loading all of them together add up to 0. These are my compatibility conditions. And I can solve these to get X_1 and X_2. I can rewrite in this fashion f_{11}, f_{12}, f_{21}, f_{22}, X_1, X_2 is equal to 0. For this particular structure I shall write down the L_i. The only thing that I need to do over here is write down what the L_i are for each and then all I need to do is just multiply all those values. Note that once I find out P_{1i} P_1 P_2 L_i I can find out all these coefficients and am going to do those, but first I will write down the L_i.

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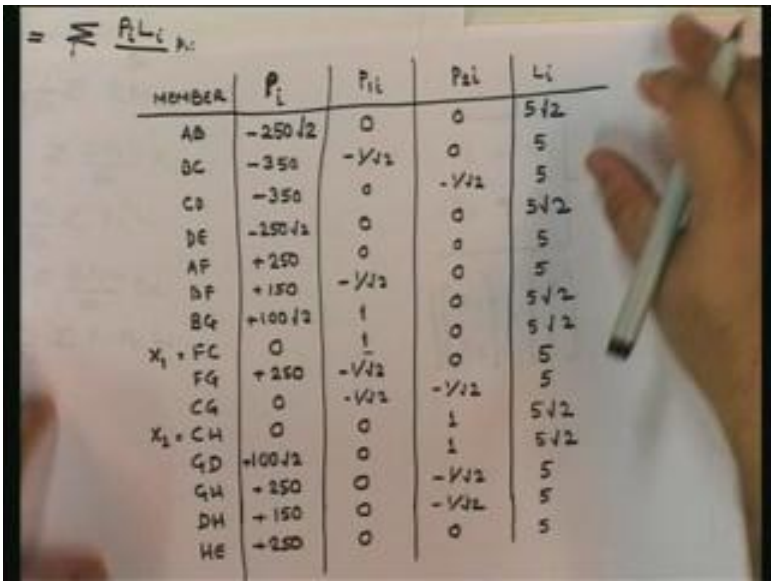

If you look at the L_i, this is going to be equal to AB-AB is going to be equal to 5 root 2. BC-BC is going to be 5. CD-CD is going to be 5. DE-DE is going to be 5 root 2. AF-AF is going to be equal to 5. BF-BF is going to be equal to 5. And if you really look at it: BG-BG is equal to 5 root 2. FC-FC is equal to 5 root 2. FG-FG is equal to 5. CG-CG is equal to 5. CH-CH is equal to 5 root 2. GD-GD is equal to 5 root 2. GH- GH is equal to 5. DH-DH is going to be equal to 5. HE-HE is going to be equal to 5. So I have got all my displace lengths etc and I can now do all of these computations. I will write down the values for you right now.

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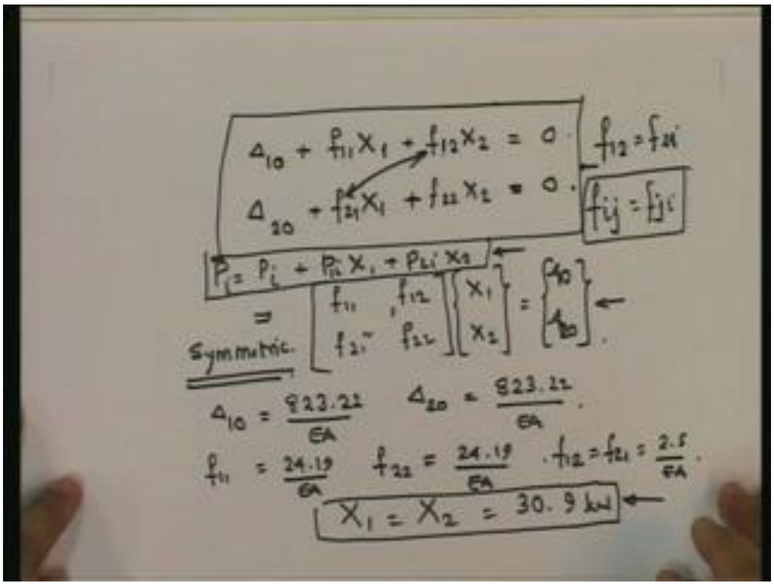

In this particular case $delta_{10}$ is equal to 823.22 upon EA. $delta_{20}$ is identical because this is a symmetrical structure 823.22 upon EA and f_{11-} f_{11} is equal to 24.19 upon EA. F_{22} is also the same: 24.19 upon EA and f_{12} is equal to f_{21} is equal to 2.5 upon EA. This is not equal to 0 this is equal to minus $delta_{10}$ and minus $delta_{20.}$ These terms come in here. This is the equation here. I know f_{11} f_{12} f_{21} $f_{22,}$ I know $delta_{10}$ and $delta_{22}$ and I can solve these equations for X_1 and $X_{2.}$ I will actually write down the values of X_1 and X_2 for you and they turn out to be equal to: X_1 is equal to X_2 is equal to 30.9 Kilonewtons.

Now how do I find out the force in all the members? The force in all the members is going to be equal to P_i plus P_{1i} plus P_{2i} $X_{2.}$ I can find out the actual force in all the members in this fashion. I want to end it by talking about one very fundamental point over here and that is you see f_{12} is equal to f_{21} in this particular case. Is that always true? You will see that what does f_{12} represent? F_{12} represents the displacement at one due to unit load at two and f_{21} represents displacement at two due to unit load at one. The loads in both the cases, in one case it is applied at two and in one case it is applied at one, the loads are the same. So what can we say about the displacements?

Let us go back to the Maxwell's Betti Reciprocal Theorem. If you really look at that, what does that say? It says due to a load at one, the displacement at two is equal to the displacement at one, if you applied the same load at two. That is what the reciprocal theorem. So If you really look at this: The same load is being applied - one unit load. Once it is being applied at one and we are finding out the displacement at two, and in one case we are applying the load at two and we are finding out the displacement at one. Obviously they are going to be the same. Therefore this is a fundamental fact that the coefficients f_{12} and f_{21} are always equal to each other always.

In fact we can make this general; we can say that this is always the case if that is the case and these two are equal to each other. What can we say about this matrix? This is the symmetric matrix and this is always true; the flexibility matrix is always symmetric. This is due to the load alone; we also looked at how to get for the temperature. I am going to leave this for the next lecture, as to, how to compute X_1 and X_2 and then of course, all the member forces; due to the temperature in the members AB BC CD and DE.

Thank you.