Structural Analysis - II
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Lecture – 30

Good morning. Till the last lecture, we had been looking at the matrix flexibility approach, which essentially included the matrix force method. Today, we are going to be starting off on what I call as the matrix stiffness method, which is essentially the matrix analysis approach for the displacement method. Let us review what the displacement method talks about.

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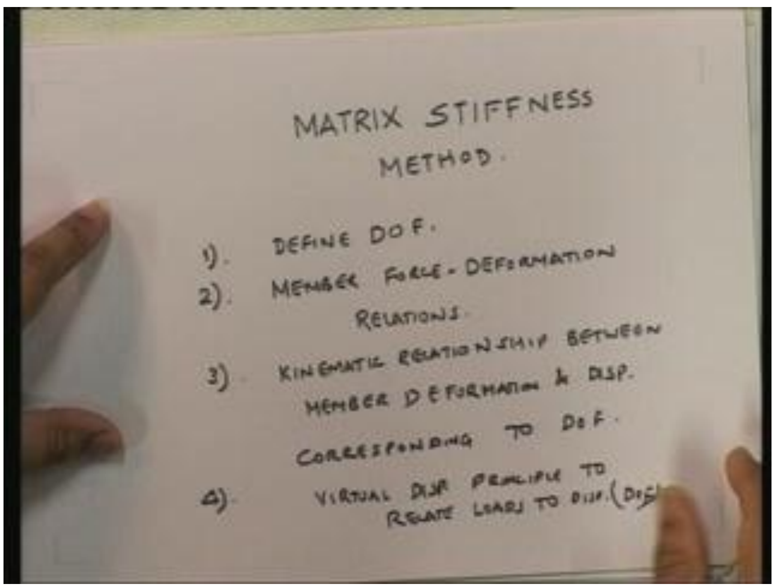

The displacement method essentially goes as follows: define degrees of freedom – that is done very easily; then, the next approach is the member force deformation relations; third is the kinematic relationship between member deformation and displacement corresponding to degrees of freedom; four is the virtual displacement principle to relate loads to displacements for degrees of freedom. I think this is the overall scope of the displacement method and this will essentially be the stiffness method; the reason behind it will shortly be understandable.

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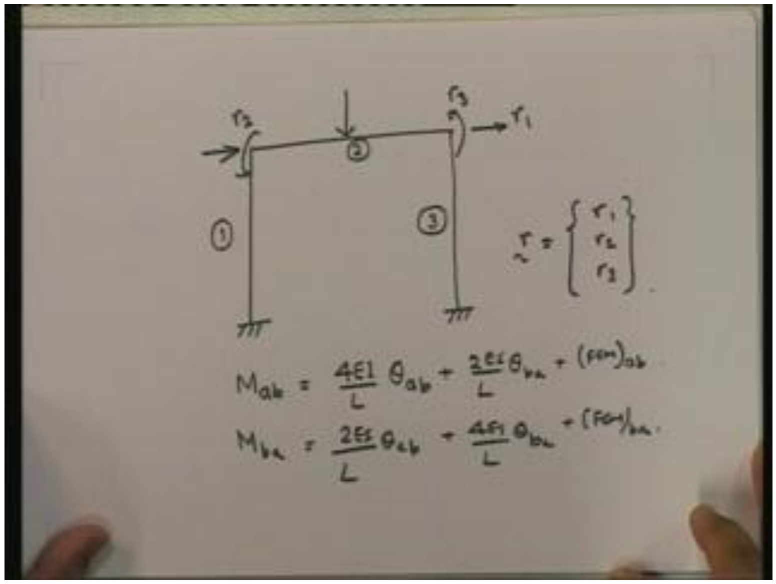

Here, let me call this member 1, member 2, member 3. This is just an illustrative example. Let us say you have this and you have these loads (Refer Slide Time: 04:49). I am not putting down right now what the loads are etc… we are just looking at it in defining the entire problem. How many degrees of freedom do you have in this particular case? Assuming axial rigidity, we know that you have three degrees of freedom. We will call that r_1, r_2, r_3 – these are the degrees of freedom. Therefore, I can say that the r vector is equal to r_1, r_2, r_3.

In the matrix method, everything is either a vector or a matrix. Therefore, I am writing down the displacement vector corresponding to the degrees of freedom in this way and those are my degrees of freedom. The next step is to relate the member force deformation relations and what are those? I will write the slope deflection equations down using the rotations from the chord, so in that case I do not have EI so it is going to be 4EI by L theta$_{ab}$ (this is from the chord to the tangent) plus 2EI by L theta$_{ba}$ plus (FEM)$_{ab}$. These are the slope deflection equations (Refer Slide Time: 07:03). I am going to write this in a different form: I am going to use the same notation that I have been using.

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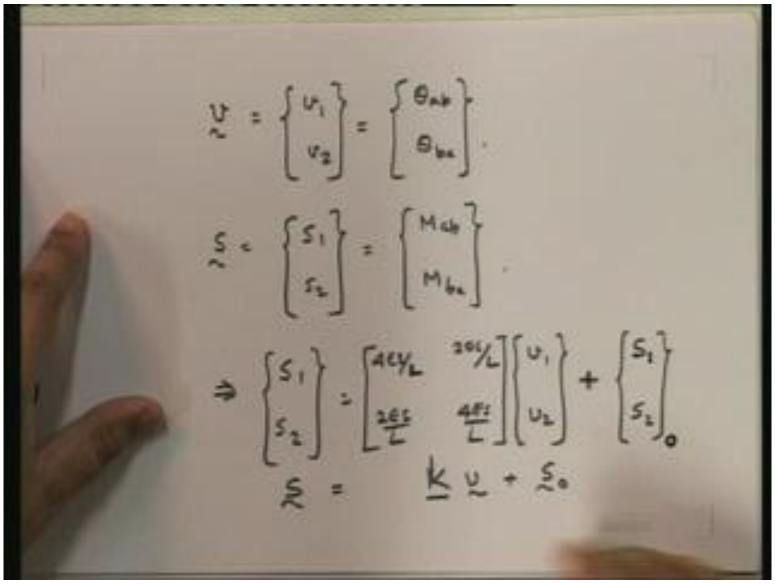

I will say that vector v, which is equal to v_1 and v_2 is equal to theta_{ab}, theta_{ba}. Similarly, S which is equal to S_1, S_2 is equal to… (Refer Slide Time: 07:51) If I write them in this fashion, you will see that I can write S_1, S_2 as… and I am going to call this fixed end moments as…. These are the member end moments for the kinematically determinate structure, so this is a kinematically determinate structure, so that is 0 because in the displacement method, the base structure is the kinematically determinate structure, where all degrees of freedom are restrained – the fixed end moments essentially comes from that. Now, it is interesting to note that this can then be written in this format: S is equal to K v plus S_0, where K is equal to….

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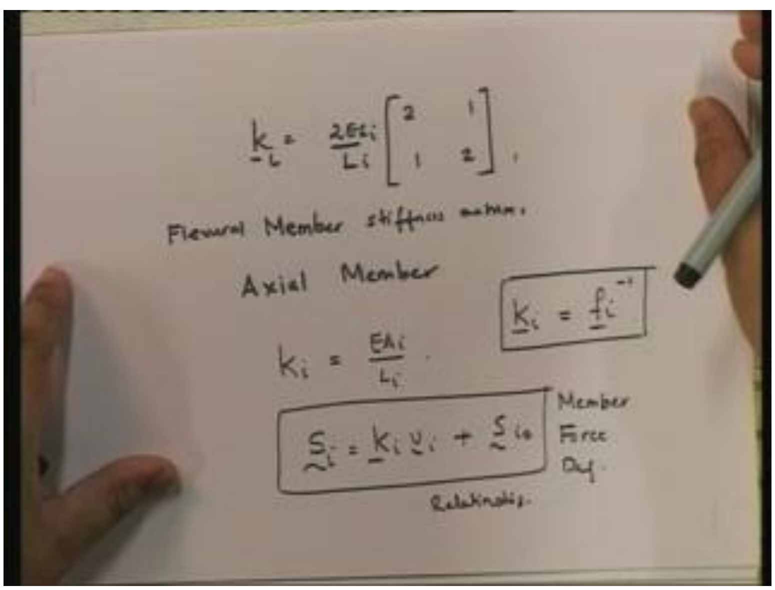

K is 2EI by L into (2, 1, 1, 2) – this is of course for a member i and this would be this (Refer Slide Time: 09:50). This is the stiffness matrix – of course flexural member stiffness matrix. If you have an axial member then you will see that K_i is equal to $(EA)_i$ upon L_i. The interesting point to note is that K_i is an inverse of the flexibility matrix. We will see that that is true for the flexural member; for this it is obvious because invert of this is L upon EA which you already know; for this (Refer Slide Time: 10:59), you just need to invert it 2 by 2.

This is actually the way we had set up the slope deflection equations, if you remember, right at the beginning **so** there is nothing new in this. This relationship at the member level exists as long as you define the degrees of freedom to be $theta_{ab}$ and $theta_{ba}$, which are known as deformation degrees of freedom. I am not going to go into those details because that comes in a much later course where you relate different degrees of freedom for a member. We are being consistent and we will continue with this approach of defining the degrees of freedom.

Essentially, what we have done is; S_i is equal to K_i into v_i plus S_{i0} this gives us the member force deformation relationship. What is the next step? The next step is our kinematic relationship, so let me take the example that I have.

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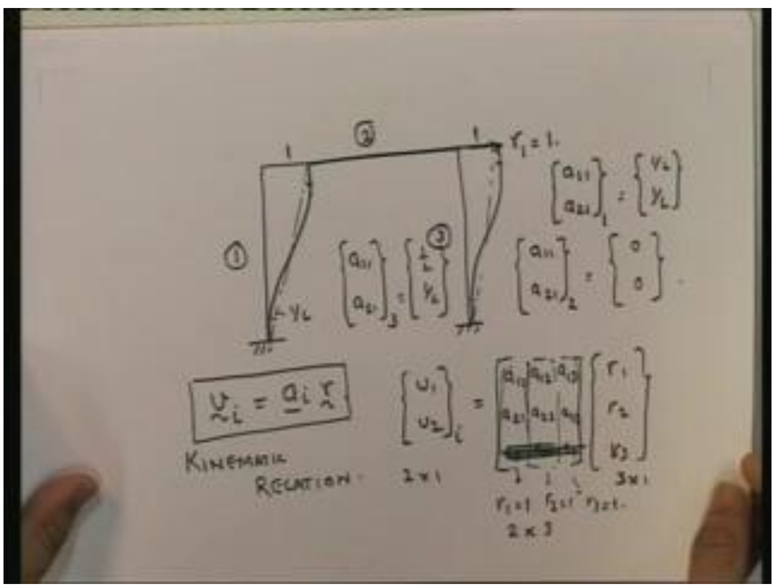

Just broadly looking at it, how do we relate the member force deformation relationship? The way you do it is; first and foremost exactly the same way that you got the load member force relationship in the flexibility approach. You put r_1 is equal to 1 and the other two equal to 0; if you look at the displacement pattern, this will be 1, 1; if we look at this, this is 1 by L, same thing here. If I am going to call this member 1, member 2, member 3, the way we do it is we say v_i is equal to a_i into r – this is the relationship we are looking for each one. The way we get it is… again, this will become v_1, v_2 for each member is related to… here, we have three degrees of freedom r_1, r_2, r_3 and so what we have is a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23} and a_{31}, a_{32}, a_{33} – each one we get by putting r_1 is equal to 1, this one we get by putting r_2 is equal to 1 and this one we get by putting r_3 is equal to 1 (Refer Slide Time: 14:52). This is known as the kinematic relation.

Therefore, what we do is we put each one and for each one, we find out its column. Here, what is v_1? It is just a_{21}, this is a 3 by 1, this has to be a 2 by 3, so you get this to be 2 by 1 (Refer Slide Time: 15:44), so you get a_{11} and a_{21}. Let me find out a_{11} and a_{21} for the first one. What is it? From the chord to the tangent, so that is positive 1 over L and a_{21} – from the chord to this thing, anticlockwise positive 1 by L. What is a_{11}, a_{21} of 2? It is 0, and 0. What is a_{11}, a_{21} of 3? You will see again from the chord to the tangent, so it becomes 1 upon L, 1 upon L. We have got the first column. For the next column, what do we need to do?

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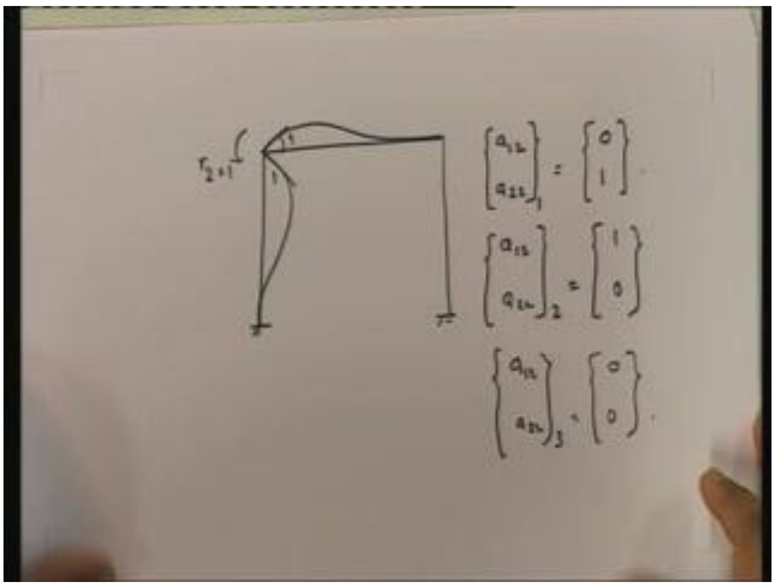

Put r_2 is equal to 1; r_2 is equal to 1 says this is 1, this is 1 (Refer Slide Time: 17:18), this is r_2 is equal to 1. What will my a_{12} and a_{22} for the first one look like? It is going to be 0, 1; a_{12}, a_{22} of 2 is 1, 0; a_{12}, a_{22} of 3 is 0, 0.

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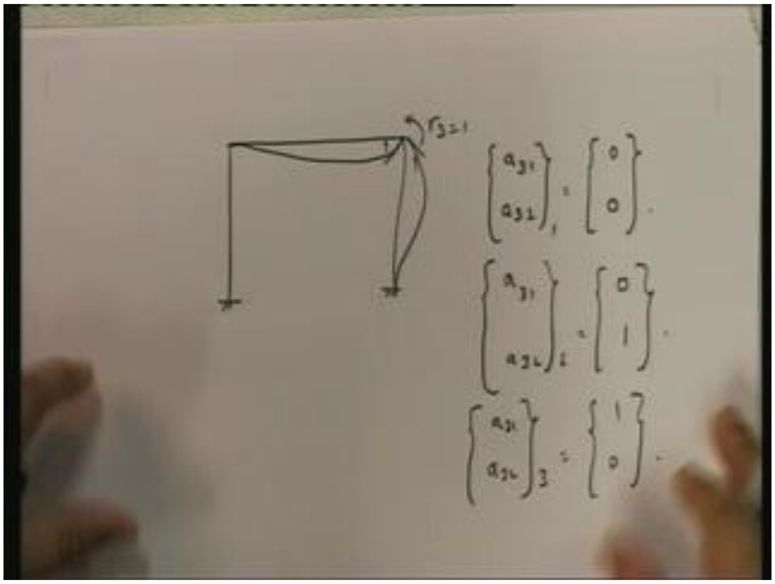

Finally, put R_3 is equal to 1, so this is 1 and this is 1 (Refer Slide Time: 18:29) and therefore, a_{31} and a_{32} of 1 is equal to 0, 0; $(a_{31}\ a_{32})_2$ is equal to 0, 1 and a_{31}, a_{32} of 3 is equal to 1,0.

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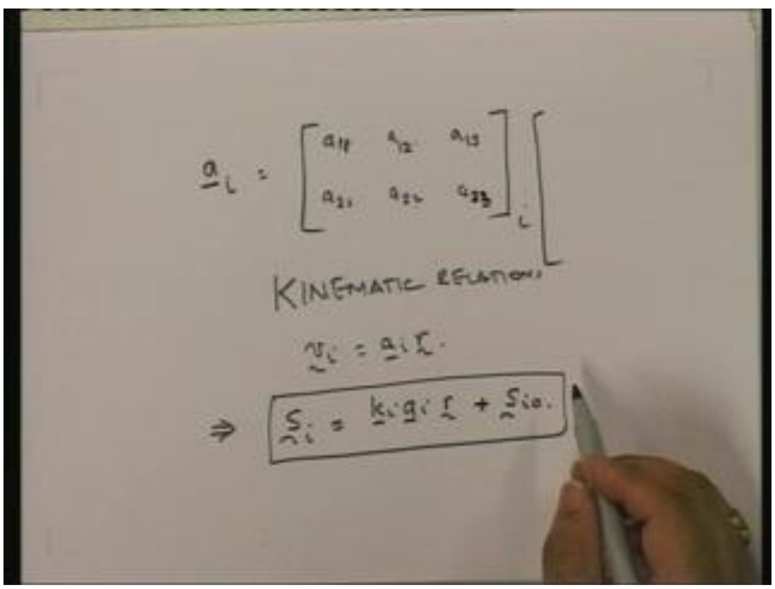

In this way, what we have done is we have evaluated a_i for each i. I have made a mistake here: this is (Refer Slide Time: 19:42) one three, two three, one three, two three and one three, two three. The first one relates to the degrees of freedom of the member and the last one refers to the degrees of freedom at the global level. Therefore here (Refer Slide Time: 20:07) this is two three, first one – member level, second one and so on. We have evaluated this for every member and that gives us the kinematic relations which essentially means v_i is equal to $a_i r$.

If we now put this into our entire equation by substituting, we get S_i is equal to K_i a_i r plus S_{i0}, S_{i0} being the fixed end moments. So now this is incorporating the kinematic relationship into the member force deformation relationship, so you get the member forces in terms of the….

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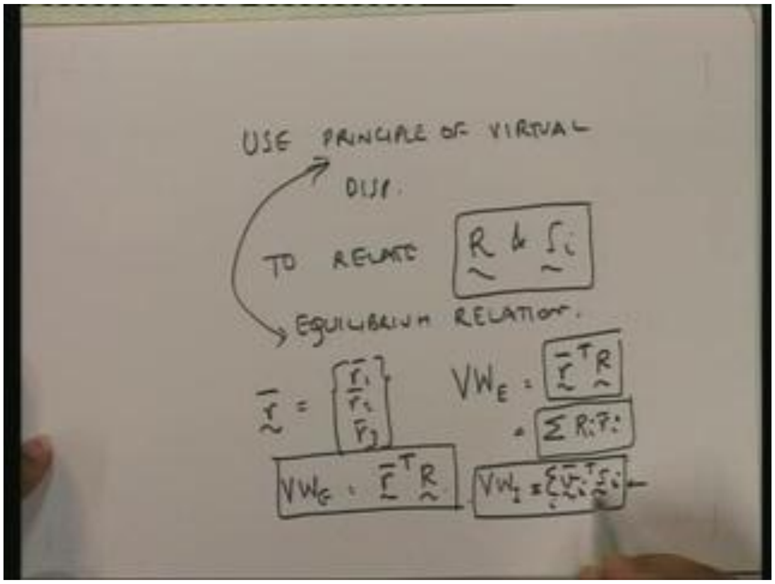

The next step is virtual work and in virtual work, what do you do?
You use the principle of virtual displacement to relate R and S_i – this relationship would typically be a kind of an equilibrium relation and the principle of virtual displacement actually replaces the equilibrium relation. How do we do it? What we do is, we actually apply a virtual displacement pattern, an arbitrary virtual displacement pattern that can be given in this fashion. The virtual displacement pattern is given in this manner (Refer Slide Time: 22:53) this is the virtual displacement pattern.

Then, what is the external virtual work going to be equal to? You will see that this is equal to the work done and if you look at this, this is nothing but summation of R_i r_i – this is basically force into the displacement corresponding to it; this I am writing in matrix form in this manner – this is all I am doing: the displacement corresponding to each degree of freedom multiplied by the load corresponding to that degree of freedom; this (Refer Slide Time: 23:39) can be written in exactly this form. Therefore, the external virtual work is virtual displacement into real loads.

What would be the internal virtual work done? You will see that the internal virtual work will be… each member will be undergoing… and again, this one (Refer Slide Time: 24:15) is effectively is summation of…, so this is summed up over all the members – i is a member – so this is the internal force.

Now, the question here is how is this related to this? There are some additional terms I will bring a little bit later. This does not include all the internal work terms. Especially if you have member loads there are certain loads that are not included in the internal work if you put it in this. But right now, I am just bringing it in a broad framework; later on, I will bring in all the details.

Understand that this is the internal virtual work, this is the external virtual work done at every member level; this again you will see is nothing but you know v_{1i} into S_{1i} plus v_{2i} into S_{2i} which is essentially the moment into the virtual rotation at that particular joint summed up over all the members. Now, the important thing over here is that obviously if you if the system is in equilibrium which is under the loads, the structure deforms and so sets up forces.

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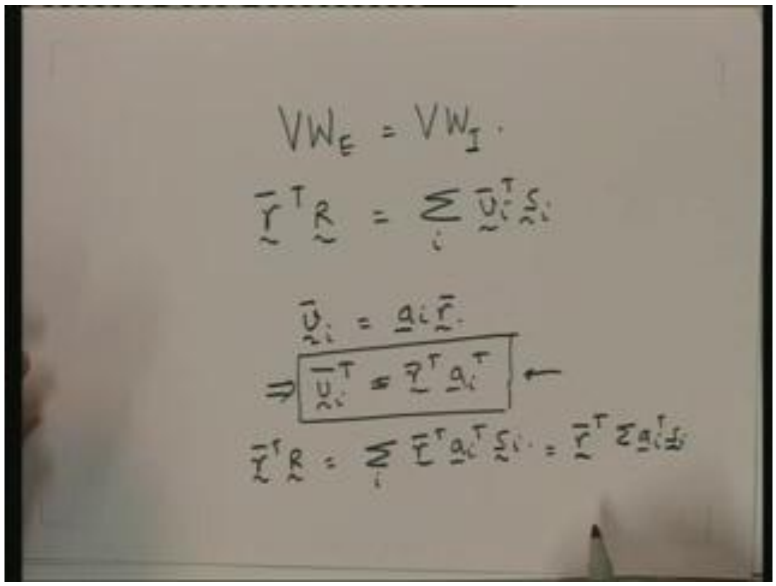

What happens is the virtual work principle says VW_I is equal to VW_e. Now, since VW_I is equal to VW_e, it stands to reason that this into this (Refer Slide Time: 26:02) is equal to summation over i into this, but this does not help us because this is something, this is something else, but can we relate the virtual rotations at the member levels from the displacement? Sure, we can. We already know that v_i is equal to $a_i r$ – this is of course when we have real, but whether we have a real displacement or a virtual displacement, it does not matter, so this is also equally valid: the virtual rotations are given in terms of the arbitrary virtual displacement in terms of the kinematic relationship.

The kinematic relationship remains the same whether the displacements are real or virtual because the boundary conditions are exactly the same and so we go through exactly the entire steps all over again. Therefore, if you look at it, this implies that v_i transpose is equal to r transpose a_i transpose. Now, how we got from here (Refer Slide Time: 27:27) to here is basically matrix algebra. When you take transpose, you interchange the order, you can go back to any matrix algebra book to understand this.

This relationship we are going to incorporate in this. Ultimately, what does this become? It becomes r transpose R is the summation over all the members r transpose a_i into S_i. Now, this does not depend on i, so I can actually rewrite this as r transpose summation a_i transpose S_i. Therefore, what we ultimately have is that the virtual work equation lands up being this – this becomes the virtual relationship, virtual work equation. Note here that this (Refer Slide Time: 28:44) has to be equal to this for any arbitrary r.

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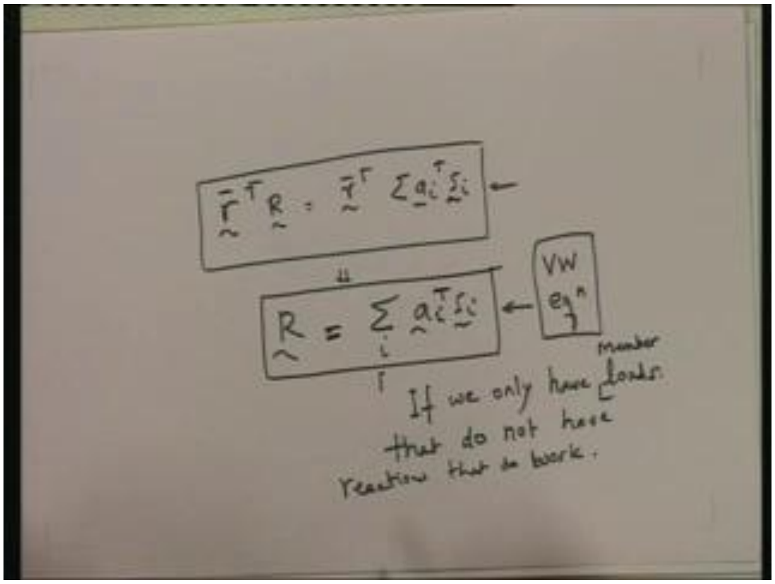

Since the r prime transpose is appearing in both the cases, we see that this implies that R is equal to summed up over all the members a_i transpose S_i. This looks like any equilibrium relationship but it is actually a virtual work equation. It looks like I have got R in terms of S_i, but we have used virtual displacement and never forget that we have to take into account all the work done – up till now in the internal work, we have only included this aspect but later on we will see that there are other terms that come into play. We will see that it is important to …

Right now, I am just writing down the basic equations and therefore if we put that together, this is valid (Refer Slide Time: 29:54) if we only have member loads that do not have reactions that do work – joint loads are not a problem when subjected to a virtual displacement pattern; this is very important. Later on we will include this effect. Therefore, this is true. Now, all I am going to do is substitute the S_i in here and what do I get?

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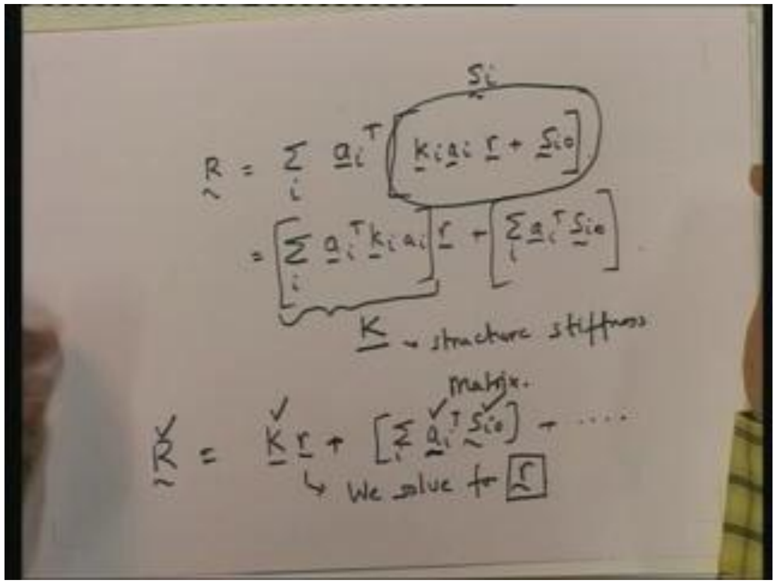

I get R is equal to summed up over all the members a_i transpose and what is S_i? It is equal to K_i a_i r plus S_{i0}. If I rewrite this, this becomes summed up over all the members a_i K_i a_i r and note that since r is this, the summation is only here plus summation over all the members of a_i transpose S_{i0}. This is nothing but the structure stiffness matrix and therefore, R looks like K r plus … We will see later on there are some other terms in here, but this in essence represents my solution. Therefore, what do we normally know? We know this, we can find this out, we know this and we can find this out and therefore we solve for r. Once we solve for r, this is my S_i, so I substitute r and I get my member end forces and that in essence is the member structural analysis.

You have found out the displacement corresponding to the degrees of freedom and you have found out the member forces – this overall is the basis for the matrix stiffness method. Now, I am going to spend quite a few lectures because there are certain issues that I have not… I put dot dot dot and this dot dot dot includes some terms that come in under specific conditions. What I am going to do for the rest of the time in today's lecture is actually take a simple problem and illustrate the methods that we have done. I am going to solve a lot of problems using the stiffness method; you will understand why because in today's computer application where we do analysis by computer software packages, this is the method – the stiffness method is really the method that is used for solving problems.

Till now, I have always done only illustrative problems because they are methods that are useful, but since this method is the one that is used the most in today's world, I am going to solve a lot of problems. I shall start off by looking at simple problems and then get more and more complicated till we have tackled all kinds of problems that you can come up with.

We are going to look at first structures with joint loads, then we are going to look at structures with member loads, then we are going to look at structures where the members are not only subjected to loads but also to temperature and slowly, we will continue on and look at all the variety of problems so that you are exposed to the spectrum of problems. Of course, one of the things that you have to appreciate is that since I will be doing everything by hand, I am only going to use at the most one or two degrees of freedom in a structure. Let us start off by looking at some simple problems and then go on from there.

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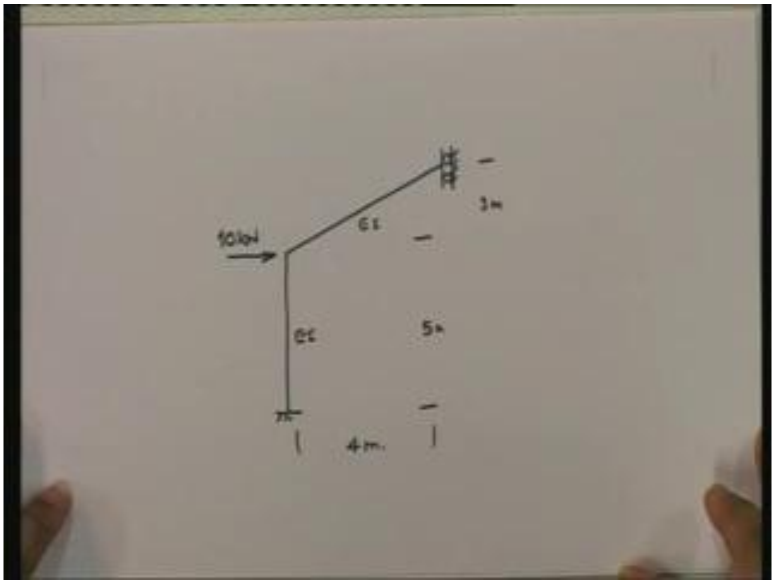

Let me take one of the problems that I have been looking at thematically for a long time. Let me put it this way: this is 5 meters, this is 3 meters, this is 4 meters, this is EI, this is EI and I am going to say that this is being subjected to a load of let us say 10 Kilonewtons (Refer Slide Time: 36:33). We have to find out the displacement at all points, in other words, the deformed shape of the body under this load and also find out the member end moments so that we can draw the bending moment diagram for this structure.

First and foremost, no point, we have dealt with this particular issue long enough to realize that this is a two degree of freedom structure. I have done enough of how to compute the kinematic indeterminacy or the number of degrees of freedom of a structure, so from here on, if you do not understand how I have taken these, how there are two degrees of freedom and why I have taken these two, please go back and review from the past few lectures in this course. These are the two degrees of freedom.

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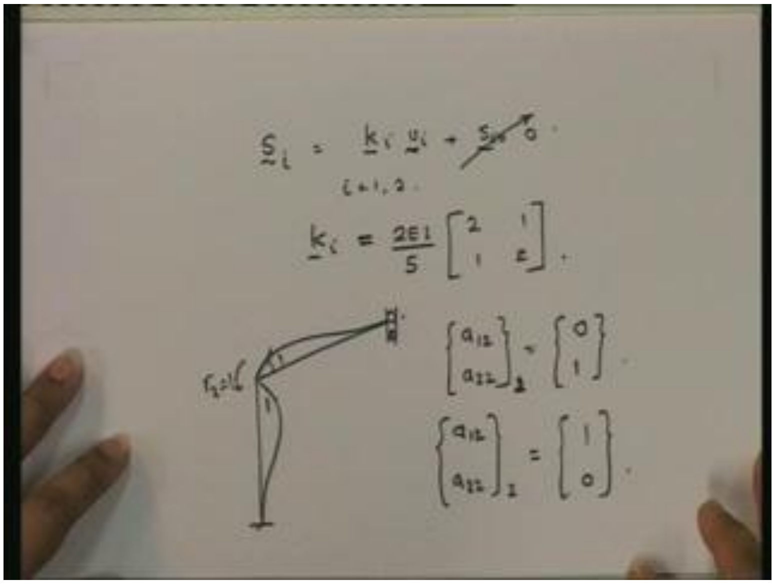

First and foremost is the force deformation relationship for each member. If you look at the force deformation relationship, both S_i are equal to K_i into $v_{i.}$ plus S_{i0} (Refer Slide Time: 38:05), the fixed end moments. If you look at the fixed end moments, since there are no member loads this is equal to 0 and for both of them, for i equal to 1 and 2, since they are identical K_i is equal to 2EI by 5 into (2, 1, 1, 2) – that is my K_i and gives my relationship. This (Refer Slide Time: 38:39) is the force deformation relationship for both the members, they are identical because the member stiffness matrix is identical because EI is the same and the length is the same.

What I am going to do is I am going to do r_1 on a separate sheet of paper but I am going to put r_2 here because you people know that r_2 is relatively simple. So r_2 would imply this and therefore, if you look at a_{12}, a_{22} of 1, it is going to be equal to 0, 1 and a_{12} and a_{22} of 2 is equal to… this is my v_1 – that is theta_{bc} so that is going to be equal to 1, theta_{cb} is my v_2, so this is equal to this. This is the force deformation relationship and this is the kinematic relationships, so we are looking through them reasonably fast.

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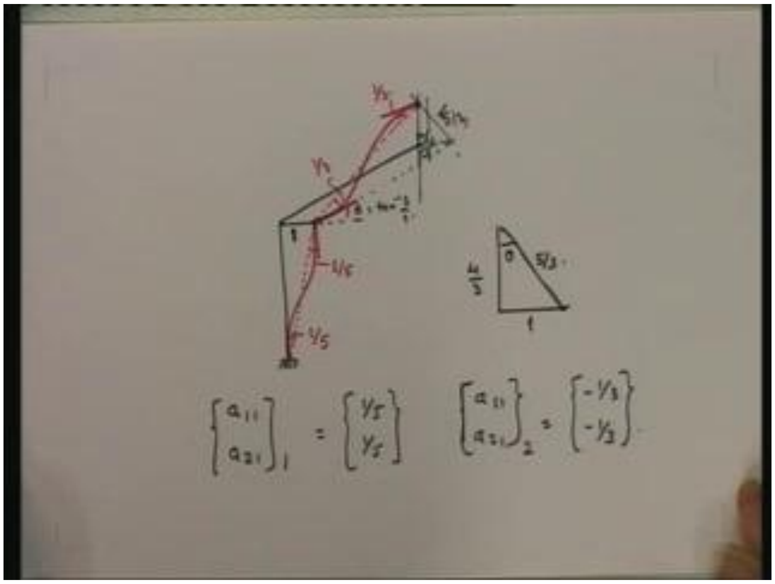

Note that we have already done this, but I want to go through this again; kinematic relationship – there is never any end to kinematic relationships. This point cannot go vertically because this member would have to elongate so this will only go this way so it has come 1 over here. If this member was left to go its own way, this would have gone exactly the same one here, but it cannot, so now it has to move, it is going to be moving along this line. It has to move perpendicular to it so that this is the point where C is. And what is this? (Refer Slide Time: 41:42) This is 90 degrees. Let us see how much this entire thing has gone up by. This one, if you look at it, is going to be…. How we are going to determine this? What we have to determine is that if this is 1, how much is this going to be equal to? Let us look at that. We do not know this, we know this is 1 and we have to find out this – this is this one (Refer Slide Time: 42:25). Let us look at it.

What kind of relationship is this? Since this is theta, it is tan inverse of 3 by 4, so this is this theta and this is the theta; that means that this is 3, this is 4, so this is going to be 4 by 3 and this is going to be 5 by 3. Now, we know that this is 5 by 3. What will the displacement pattern look like? The displacement pattern will look like this. Note this tangent over here (Refer Slide Time: 43:28) has to be this way, the tangent over here has to be this way and the tangent here because r_2 is equal to 0 and here, you have a fixed. Once you have that… and the tangent over here of course has to be this way, the chord is this, the chord is this (Refer Slide Time: 43:50), so this angle and this angle are going to be 1 upon 5 and this is also going to be 1 over 5; this is 5 over 3 and this is 5, so this is going to be 1 over 3 and this is going to be 1 over 3. Therefore, my a_{11}, a_{21} of 1 is going to be equal to – from the chord to the tangent positive 1 by 5, 1 by 5; my a_{11}, a_{21} of 2 is equal to from the chord to the tangent negative – minus 1 over 3, minus 1 over 3.

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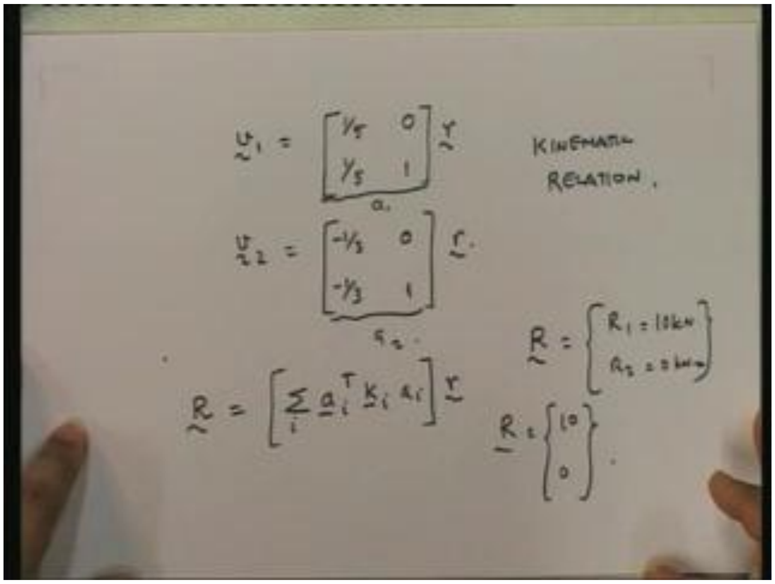

Having got that, now we can put that v_1 is equal to 1 over 5, 1 over 5, this is going to be equal to 0, 1 into r and v_2 is equal to minus 1 by 3, minus 1 by 3, 0, 1 into r – you have got the kinematic relations. From virtual displacement, we know that capital R is equal to summation over i a_i transpose K_i a_i into r; note that here since S_{i0} is 0, this is the only thing (Refer Slide Time: 46:40). Now, what is capital R equal to? This is R_1 and R_2 and if you look at this you will see R_1 is equal to 10 Kilonewton and R_2 is equal to 0 Kilonewton meter. Therefore, R vector is 10 and 0. This is my a_1 vector, this is my a_2 vector, so all I need to do is just go through these steps. Let us go through these steps – it is instructive to go through the steps.

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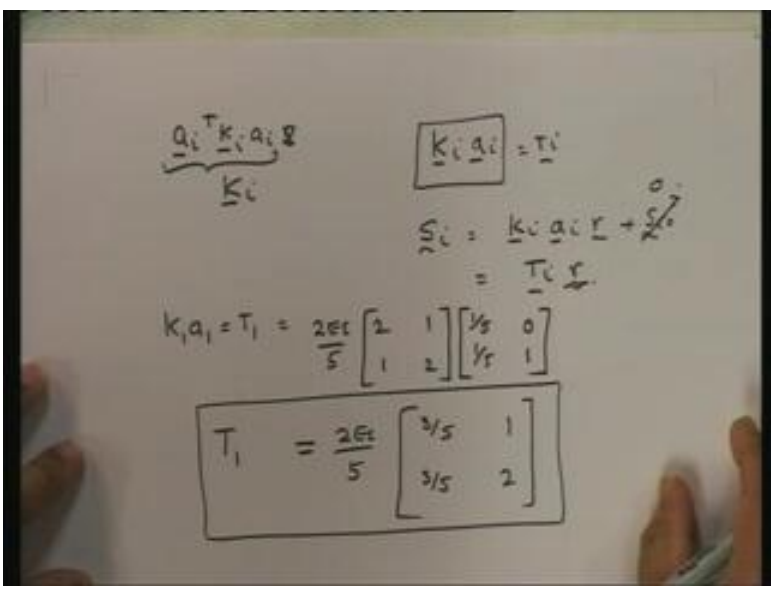

Now a_i transpose K_i a_i r – if you look at it, this part can be actually written as the contribution of the ith member to the structure stiffness matrix and note that K_i a_i also has a specific aspect to it, because you will see that S_i is equal to K_i a_i r plus S_{i0} and in this particular case, this is 0 (Refer Slide Time: 48:08), so S_i…. Therefore, this one I will say is T_i – this gives me directly… K_i a_i is equal to T_i directly gives me S, the member end forces in terms of this, so I am going to actually compute these as a step in the whole process.

Let me first do K_i a_i; K_1 a_1 which is equal to T_1 will be equal to 2EI by 5 into (2, 1, 1, 2) multiplied by this thing for the first, which is 1 by 5, 1 by 5, 0, 1 and this is going to be equal to 2EI by 5 – I keep it outside, inside 2 by 5 plus 1 by 5 is 3 by 5, here 1 by 5 and 2 by 5 is 3 by 5, here 2 into 1, 1, here 2, so this is my T_1.

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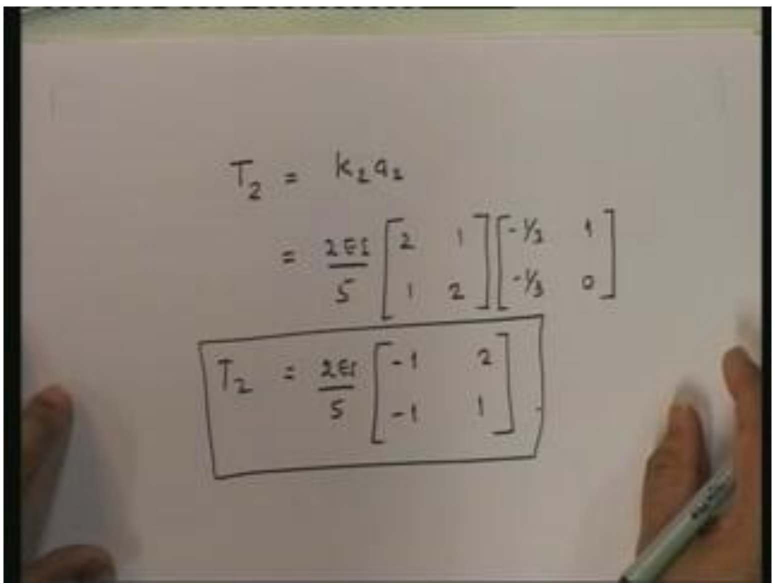

Next, I am going to compute T_2. T_2 is K_2 a_2, which is equal to 2EI upon 5 into (2, 1, 1, 2) multiplied by minus 1 by 3, minus 1 by 3 and then 1, 0; let me look back at my old thing and then I will get back to you – it is 1, 0 so this is equal to 2EI by 5 into minus 2 by 3 minus 1 by 3 is equal to minus 1, minus 1 by 3, 2 by 3 is minus1, here I have 2, here I have 1, so this is my T_2.

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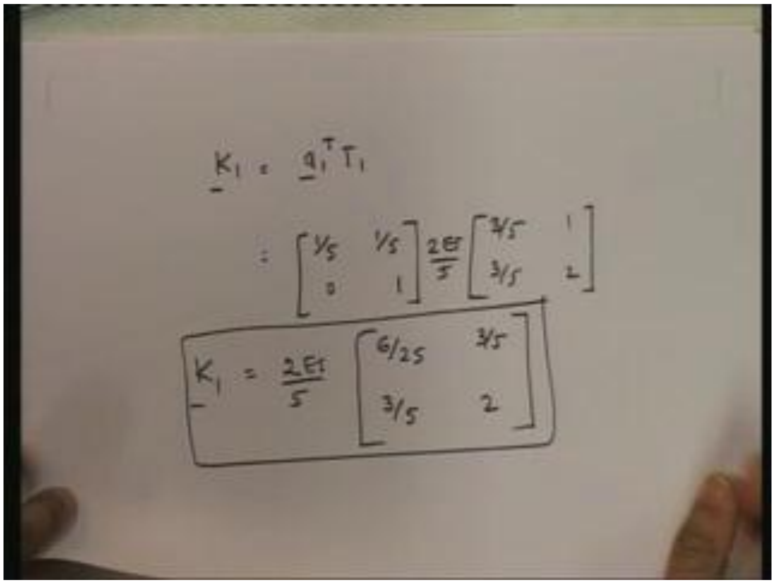

Next is computation of K_1 which is essentially equal to a_1 transpose $T_{1;}$ a_1 transpose is equal to (1 by 5, 1 by 5, 0, 1) times 2EI by 5 (3 by 5, 3 by 5, 1, 2). If you look at this, if I put 2EI by 5 outside, inside I get 3 by 25 plus 3 by 25 is 6 by 25, here 0, 0, 3 by 5, here 1 by 5 plus 2 by 5 is 3 by 5 and this is 2 (Refer Slide Time: 52:18). Note that it is symmetric – it has to be, has to be symmetric. K_1 is the contribution of the first member to the structure stiffness matrix is this.

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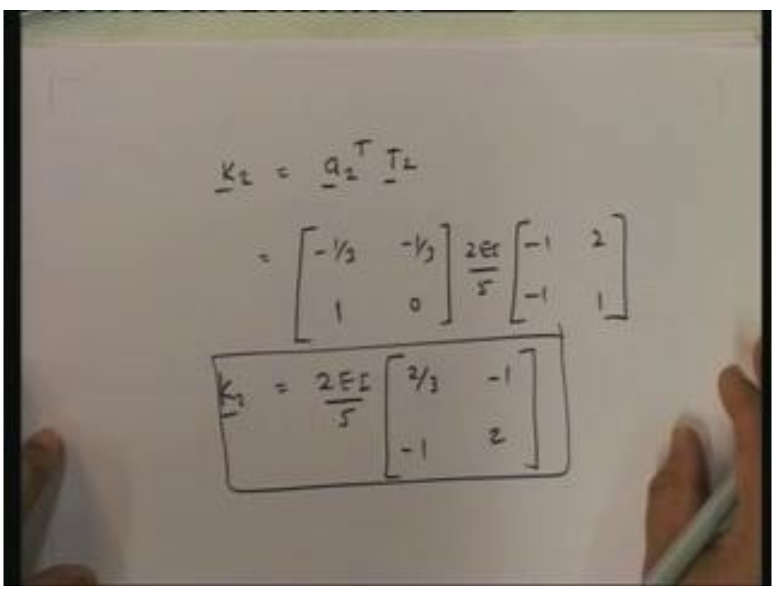

Similarly, K_2 is equal to a_2 T_2, which is equal to… a_2 is (minus 1 over 3, minus 1 over 3, 1, 0) and T_2 is 2EI by 5 (minus 1, minus 1, 2, 1) and I will take 2EI by 5 outside and inside, you get 1 over 3, 1 over 3, so you get 2 over 3, here minus 1, so minus 1, minus 2 by 3 minus 1 by 3, minus 1 and here 2 – that is K_2. Now, if we add K_1 and K_2, what do we get?

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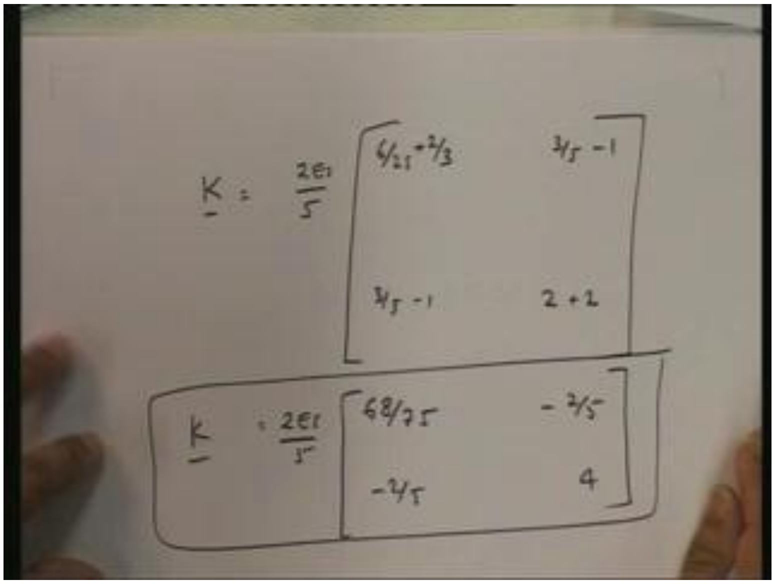

K becomes equal to 2EI by 5 and inside, I have 6 by 25 plus 2 by 3 and here, I have 3 by 5 minus 1, 3 by 5 minus 1 and 2 plus 2. If I look at this, it is 2EI by 5 and this becomes 75, if I take 75, I get 18 and here I get 50, so I will get 68 upon 75, this becomes minus 2 over 5, minus 2 over 5 and this becomes 4 – that is my structure stiffness matrix.

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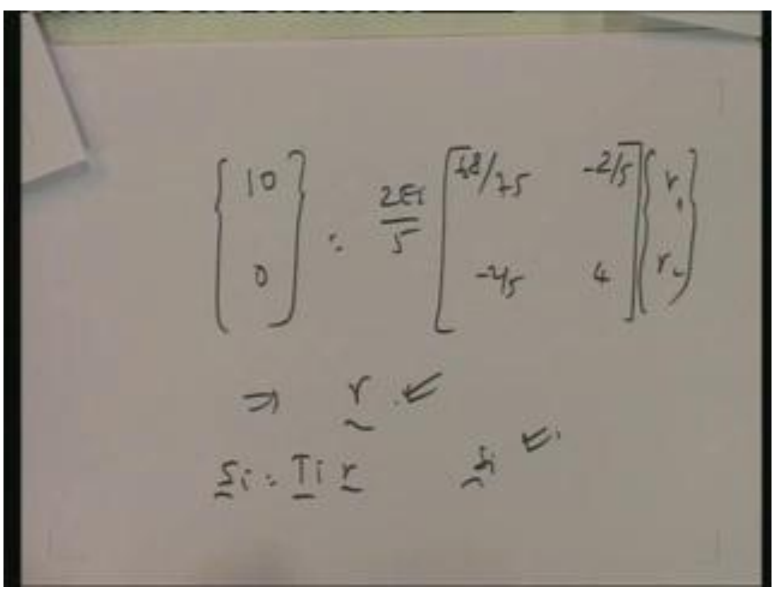

Now, since I know R is (10, 0), 2EI by 5 into (68 by 75, minus 2 by 5, minus 2 by 5, 4) into r_1 r_2; this way, I can solve for r – I just take invert this, multiply this and you can get r and then, my S_i are equal to T_i r; I can find out my S_i and once I find out my S_i, I have solved the problem. You can do the numbers yourself, it is very simple. I have evaluated each and every term and you just need to go through the steps to evaluate them. Once you got your S_i, since there are no member loads, you can draw the bending moment diagram very easily. Once you draw your member end this thing, you can also get the support reactions and everything. This, in essence, illustrates briefly the displacement method.

Go through the numbers yourselves and please solve this problem and get through it. Next time we will see what the answers are to this one.
Thank you very much.