Structural Analysis - II
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Lecture – 03

Good morning. We have moved on to our third lecture in this course. In the first lecture, we looked at static indeterminacy and how to obtain the static indeterminacy of a structure. In the second lecture, we looked at how to get the kinematic indeterminacy of the structure and then, we quickly looked at various types of structures and looked at what their static indeterminacy and kinematic indeterminacy were.

In this lecture, I am going to be looking at the method of virtual work. This sequence of three lectures that we have are essentially to review things which you have already done, just to put them into the perspective of what we are going to be talking about later. The method of virtual work is something that you people have known for a long time. You may have known it as a unit force method to get displacements or you may have known it as displacement approach to finding out forces.

Let us just review the method of virtual work; it is actually not a method, it is a principle. What does the principle of virtual work say? What it says is this: if you have a body that is in equilibrium under a system of forces and if you displace the body or give a virtual displacement to the body, then the work done by all the forces acting on the structure is equal to 0. What are the important tenets? The body has to be in equilibrium under a set of forces and then, if you apply a virtual displacement, the work done by all the forces is actually equal to 0 – this is the principle of virtual work. However, understand one thing: as soon as it says, it is no longer the principle of virtual work because the principle of virtual work has two methods.

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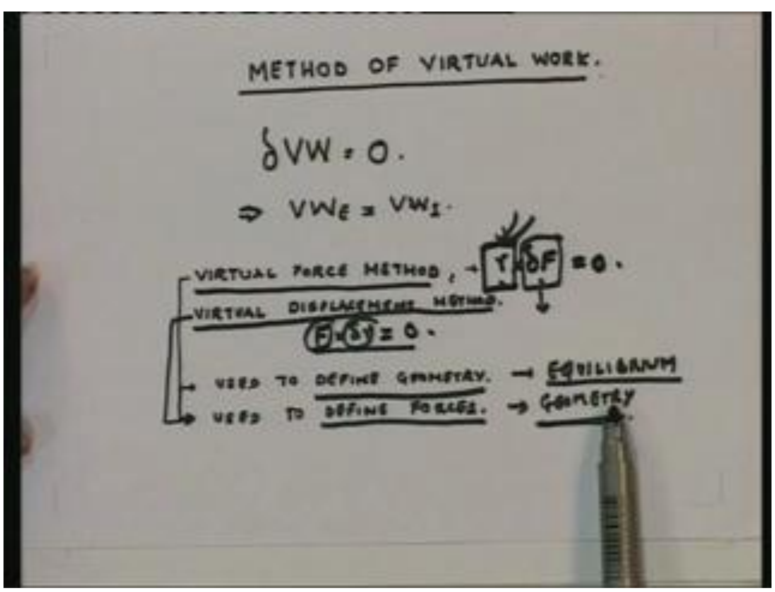

There is the virtual displacement method, which is the common one that we always refer to, that you have a system of forces and you subject it to a set of virtual displacements and then the total work done is equal to 0 (Refer Slide Time: 04:29). However, you have another method called the virtual force method where if you look at the body under a set of forces and it has a displaced shape. If I apply a set of virtual forces on this structure in equilibrium then the work done by these virtual forces undergoing the real displacements of the structure are also going to be equal to 0. This is known as the virtual force method.

What are these methods? Why do we use these methods? It is very important to understand that it is the principle of virtual work, method of virtual displacement, the method of virtual force and so on. It is very interesting to study about these methods. The whole principle of virtual work is actually a very interesting principle which changes a geometry problem into an equilibrium problem and an equilibrium problem into a geometry problem. In other words, the principle of virtual work provides a kind of… taking a real structure into a virtual zone in which geometry is reflected by equilibrium and equilibrium in the real structure is represented by geometry in the virtual structure. You will see how.

What do we use the principle method of virtual force for? What we use it for is to actually find out displacements – this is what we are trying to find out (Refer Slide Time: 06:36). In other words, finding out displacements means we are trying to define geometry – the displaced geometry of the structure, but what do we do? We use the virtual force method. What does the virtual force method do? You apply a set of virtual forces, find out all the forces in the structure due to this action of virtual forces and then multiply the real displacements and deformation in the structure and the total work done is 0. What this gives me is the displacements by actually solving an equilibrium problem (Refer Slide Time: 07:19); so I use equilibrium to define geometry and in the next method, we want to actually find out the forces by applying displacements.

What do we do? We actually apply a virtual displacement pattern, find out the displacements and deformation in all the members, then find out the work done by all the real forces undergoing these virtual displacements and deformations and equate it to 0. In other words, what do we do? With this, we find out forces in the structure; we define forces in the structure and what do we use? We actually use geometry. Because given a virtual displacement to find out the displacements and deformations of all the joints and members, you are required to solve geometry. You see, it is just the flip side, to define geometry we use equilibrium and to define forces we use geometry; that is, in essence, the method of virtual work. It is all very fine to say this, but let us actually use these methods and that will become much more understandable when I actually give you this. Let us use the virtual force method to find out the displacements in a structure, let us do that.

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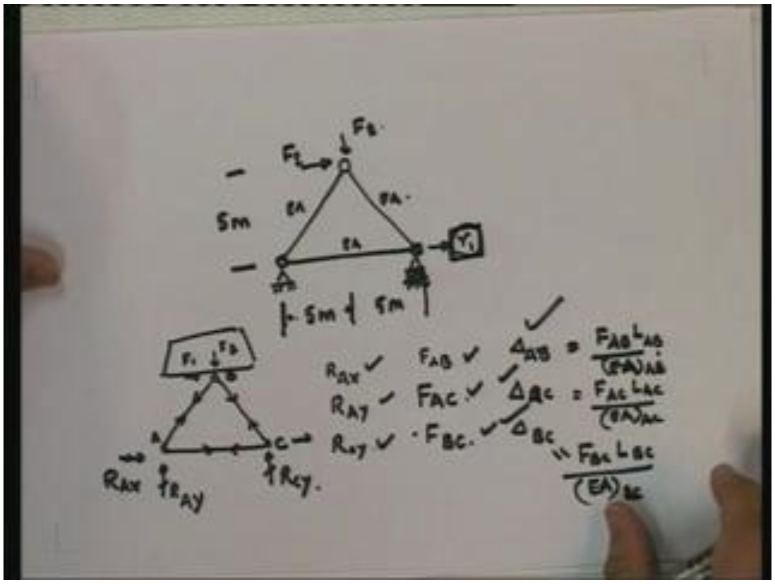

Let me go back to my base truss that I have and now, let me say that I subject it to force F_1 and F_2 (Refer Slide Time: 09:19). My question over here to you is to find out the horizontal displacement of this joint (Refer Slide Time: 09:32) – that is my question to you. These are the real forces acting on the structure and I want to find out what is the horizontal displacement of this point under these forces. How would I do that? Let us go through the steps and that will actually illustrate the concept of how the method of virtual force is used to define the geometry. Simple problem; I am not going to make it very complicated; I am not even trying to analyze the structure.

Given these loads, find out the displacement – that is my question. How do I do that is by using the method of virtual force. What is the first step? Note that this is a statically determinate structure – I have made it purposely that. The first step is to find out all the forces in all the members and supports; I can easily get it. Let me give some numbers, let me put that this is 5 meters (Refer Slide Time: 10:49), this is equal to 5 meters and 5 meters – all of them have the same E and A, all of them are identical. I need to find out this one.

First step: forget this, take the real structure. Under these set of forces, I find out the reactions. Let me call this A (Refer Slide Time: 11:57), B, C. I will find out R_{AY}, R_{AX} and R_{CY}. Note that these are all the support reactions that you develop in the structure. Can I find these out? I can find these out. I do not want to go into it. You have three unknowns, three equations – one is for horizontal, one is for vertical and one is taking movements about any point. You can find out R_{AX}, R_{AY}, and R_{CY} – you know these now. Once you know these, can you find out the forces in these members? I can analyze the truss and I can find out the forces in these members. You use any method: method of joints or method of sections; I do not care how you do it but given the fact that this is a statically determinate structure, I should be able to find out the force in member AB, the force in member AC and the force in member BC – these also I can find out. What are these? These are for these applied loads (Refer Slide Time: 13:28) and I can find these out.

Once you know these, what else do you know? These are the axial forces in these members; I can find out the axial deformation in member AB, in member AC and then in member BC. How can I find these out? I can find these through.… I am not even going into this, it is something that you should know: axial deformation is given in terms of the axial force, the axial force divided by A gives you stress, stress by E gives you the constant strain, strain into L is equal to the deformation – these are things that you have picked up a long time ago and I am just writing them down. Of course, EA of AB, EA of AC (Refer Slide Time: 14:47) and similarly, for this, F_{BC} into L_{BC} upon EA of BC.

So, in other words, since I am given the EAs, I am given the length (I can find out the lengths), I can find out the forces, I can evaluate these three. Therefore, these are the steps for the real structure – you find these out. Now, the question is these are the real support reactions, these are the real forces and these are the real deformations in each of the members – I can compute these; given this and the undisplaced geometry of the structure, I can always find out all of these. Let us move on.

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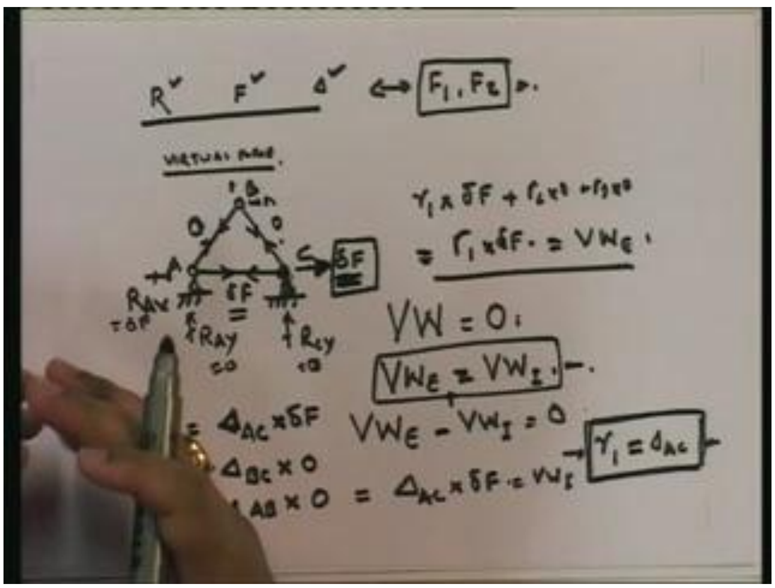

What is the next step?
I have found out all the real forces, real reactions and real deformations in the body; I know R, I know axial and I know real – all of them are due to the loads F_1 and F_2 applied on the truss. What is the next step? I am going to apply the method of virtual force to get my r_1 displacement. What do I do? The method of virtual force says that I need to apply a set of virtual forces on the body. I can choose what I want and the way I choose is, I apply only this, I do not apply any forces here – there is a reason behind it, which we will see. I apply a virtual force; this is a virtual force (Refer Slide Time: 17:01).

How many degrees of freedom does this structure have? The structure, if you remember, has three degrees of freedom, I have already looked at them. What are they? This displacement, this horizontal (Refer Slide Time: 17:34) and this vertical displacement. What I am going to be doing here is, I only need to find out this displacement. Therefore, what is the work done by all the external displacements? They are r_1 (the displacement that I am interested in) into dF.… I have r_2 and r_3 over here, but they do not have forces attached with them, so r_2 is equal to 0, r_3 is equal to 0, which gives me r_2 into 0 plus r_3 multiplied by 0, which is essentially r_1 into dF.

My work done by the external displacements is actually just r_1 into dF; that is the reason why I only apply a force here (Refer Slide Time: 18:20); that is the work done by this virtual force undergoing this real displacement. What are the other things? The other things are that the internal virtual forces will also do work because you have real deformations in the body under the action of forces. What I need to find out is under this force, I need to find out what are the forces in the body. Again, I find out R_{AY}, R_{AX} and R_{CY} – I find those out. I find also out the virtual forces developed in all the members. In this particular case, it is very simple. You can say that this is equal to delta F (Refer Slide Time: 19:33), this is equal to 0, this is equal to 0, you can compute it and you will see that this is equal to 0, this is equal to 0 and this is equal to delta F – I have analyzed it; it is very simple, so I can analyze it; it does not matter, it is not relevant in this particular case; however, I have done it, you can get it. Now, the one aspect of it is that if you remember, the equations say total virtual work is equal to 0. We can also write this in another way: the work done by the external displacements is equal to the work done by the internal displacements.

What are the internal displacements? How can we write it this way? All the internal forces in the structure are essentially reactions to the loads – they actually oppose the loads, so the work done by the external displacements and forces are actually opposite the work done by the internal forces. If you look at this equation (Refer Slide Time: 20:54), I can actually write it in this fashion; if I separate out the internal and external, this becomes this. Why is this minus? If you really look at it, all the internal forces are reactions – they oppose the loads and that is why there is always a negative sign. When I take this, this goes into this, so this is also another way of writing the virtual work equation.

What is the external virtual work in this particular case? You will see that this is the external virtual work. What is the internal virtual work? The internal virtual work is the work done by each deformation multiplied by subjected to the forces. In this particular case, the internal virtual work is going to be equal to…. What is the virtual force in each member? This is A (Refer Slide Time: 21:58), this is B, this is C. In member AC, it is going to be delta F and what is the real deformation? $delta_{AC}$. This becomes $delta_{AC}$, which we have computed earlier, multiplied by delta F (that is the work done; also note that by definition, this is tensile (Refer Slide Time: 22:20), so this $delta_{AC}$ is elongation and elongation multiplied by tensile is always given in this fashion) plus the work done by $delta_{BC}$ multiplied by the virtual force in it, which is 0, plus $delta_{AB}$ multiplied by the virtual force, which happens to be 0. What is the internal virtual work? It is $delta_{AC}$ into delta F – that is the internal virtual work. What is the external virtual work? If I equate the two, what do I get? You will see that r_1 is equal to $delta_{AC}$ – this is what you get and this is not surprising. Let us go back here. This point is not moving anywhere. How much does this move? It moves by whatever is the axial deformation in member AC, so that is what you have got. The method of virtual work has given me a geometry by actually solving two equilibrium problems.

The first one was the real equilibrium problem (Refer Slide Time: 23:37) where you subjected it to the real loads; the second one is the real structure subjected to the virtual load. This in essence is how to use the virtual force method to get the geometry of a structure. I am just illustrating these points. I am not going to be solving anything right now but I am just looking at it purely from the point of view of the principle.

Once you understand the principle, we can then use it very easily; we will be using it for the rest of the semester; in all the lectures that I give you, you will be using these methods. We will continuously be using these methods so that you will be absolutely certain of what to do when you come to it. Now, let us look at the virtual method of virtual displacement.

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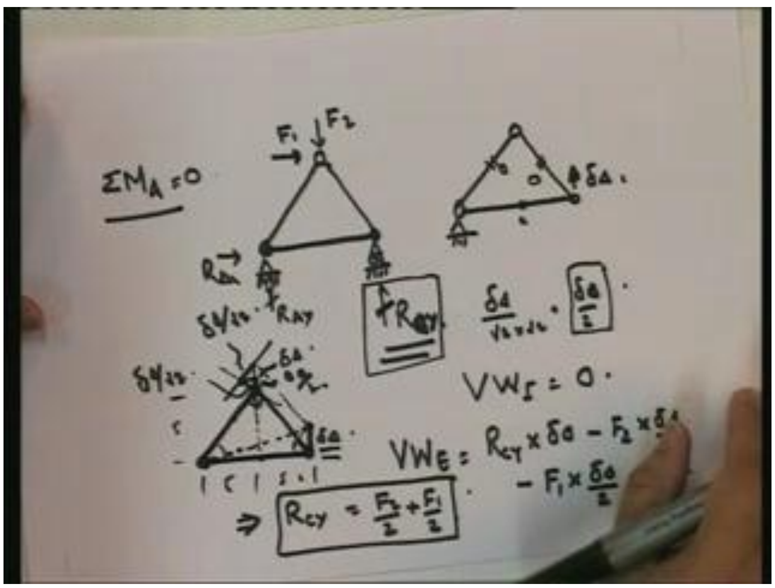

What is the virtual method of virtual displacement? Let me look at this: it is the same structure. The whole point here is that I am going to get you so involved in this structure that…. It is the same truss structure and I have a set of forces, again the same set of forces and I am going to use the method of virtual displacement. What is the method of virtual displacement used for? It is used to solve equilibrium problems. What does that mean? In this particular case, due to these set of forces, I need to find out what this is, what this reaction is (Refer Slide Time: 25:34).

Since this is a statically determinate structure, you can find it easily using equilibrium; but I want to illustrate the method of virtual displacement to you, so we are going to assume that you do not know how to solve equilibrium problems – let us assume that; without solving the equilibrium problems, can I get this reaction (Refer Slide Time: 26:03) given these set of loads? That is the question here and I have to use the method virtual displacement. How do I do that?

Here, I say that under these set of forces, the structure is in equilibrium; so what I am going to do is, I am going to apply a virtual displacement (note that I want to find out this reaction (Refer Slide Time: 26:38)). What I do is this: corresponding to the reaction that I want to find out, I release the restraint and I give it some displacement delta delta. Let us be very clear about it – all

I am interested in is to ensure that this structure…. Now, note that once I have removed the restraint, this is a mechanism; since this is a mechanism and is a single degree of freedom mechanism, I can give it a displacement and I can find out how every other point moves. Is that clear? I can do that.

Now what? Note also another thing: since it is a mechanism, this displacement is not going to lead to any deformation in any of these members; in other words, I can move this body as a rigid body, because it is a one degree of freedom mechanism. I give it a displacement and given this displacement, I need to find out the displacements of all other points – this is a geometry problem; let us see how I can do that. Let me again go back to the particular problem that we were looking at. I need to know the geometry to be able to give you the geometry. I give this a displacement d delta. Will this move in this direction? Note that this joint is here (Refer Slide Time: 28:50), this point cannot go anywhere and I need to move this way. It does not mean that it only moves in this direction; you have to ensure that every point moves the way it does in a rigid body; in other words, you cannot have deformation on any of the members. Remember that the displacement that I give, although I am showing it to be very large, is actually very small. Small displacements still hold true; the virtual displacement that I give has to be a small displacement; I might show it as very large, but I am just essentially exploding it to show you; it is not as if it is going to be moving in that direction.

Note that for this member not to axially deform, it is free to move in this direction tangentially. If it moves in any other direction, you have an axial deformation, which you cannot and so that means this point can only move up here. I am going to show this, it may look longer, but remember this displacement is very small compared to this length. Although it looks longer, it really is not; it is just that I have exploded the view. This one goes this way (Refer Slide Time: 30:15).

Now, I need to know where this point goes. How do I do that? That is the geometry problem. What I need to do is understand that this member (Refer Slide Time: 30:33) is connected over here and since it cannot deform axially, the only way that it can move is along this direction. How do I get where it has gone? Has it come here or has it gone there? That is what I need to find out – that is the geometry. For the geometry, if you notice, this point has come here (Refer Slide Time: 31:04).

Let us suppose that this joint was not constrained by this member – let us say that it is not constrained here, it is free to move. How would it move? If this joint moved here, to ensure that this member remains undeformed, this would have to go up because that would ensure that this was undeformed.

Therefore, this point would move here (Refer Slide Time: 31:38) if it was not restrained, but it is restrained. Obviously, since this joint has moved up this way, it cannot move up this way, so what does it have to do? This joint has to move in this direction, but note that this member is this way. This point has come here now. Now, for this member not to axially deform, how will it have to go? It will have to go perpendicular to this member; the point where this and this connect is where this joint actually is, to ensure that neither this member nor this member has axially deformed.

Therefore, this is the point where this member is. Where is that point? If you note the way I have drawn it with this as 5, this as 5 and this as 5, this is 90 degree, this is 45 and 45 degree – that is how I have managed the whole thing. Now if this is delta, when this has to move perpendicular to this, what does that mean? Perpendicular to this is this, so that means it moves parallel to this member. This part is d delta (Refer Slide Time: 33:20). How much has it moved horizontally? Let us see how much it has moved horizontally. You have to understand how much this motion is. Think about it. When you are going this way, how much is this motion? This is also 45, 45, so it is going to be d delta by root 2. Similarly, you will see that this is equal to d delta upon root 2. If you look at the total horizontal motion, it is going to be d delta by (root 2 into root 2), which is equal to d delta by 2. This point has moved vertically by d delta and horizontally by d delta by 2. Now, I can write down my virtual work equations.

The real forces undergoing the virtual displacements…. Note that internal virtual work is 0, because the internal deformations (the virtual deformations are 0) because I have ensured that you do not have them; so internal virtual work… even though under this you have real forces, the virtual deformations are 0; internal virtual work is 0. What is the external virtual work? First is this: R_{CY} multiplied by d delta. Note that this is up and this is up, so the work done is positive. Next is this force F_2. How much is that undergoing? d delta, but F_2 is downwards, displacement is upwards, so it is negative work and so, it is minus F_2 into δ delta by 2 (actually, this is the point, so this is the work done and that is d delta by 2). Then comes the work done by F_1.

If you look at F_1, the work done by F_1 is negative because this is facing in this direction and the force is in this direction, so minus F_1 into d delta by 2 and that is equal to 0. You get that R_{CY} is equal to F_2 by 2 plus F_1 by 2 – this is what the principle of virtual displacement gives you. Let us see whether that is true by actually looking at equilibrium; because this is a statically determinate structure, I can find out from equilibrium. Let me take moments about this point (Refer Slide Time: 36:34). Note that this is R_{AY} and this is R_{AX}, let me take moments about this point. If I take moments about A, what do I have? Let me look back at that particular problem; let me go back.

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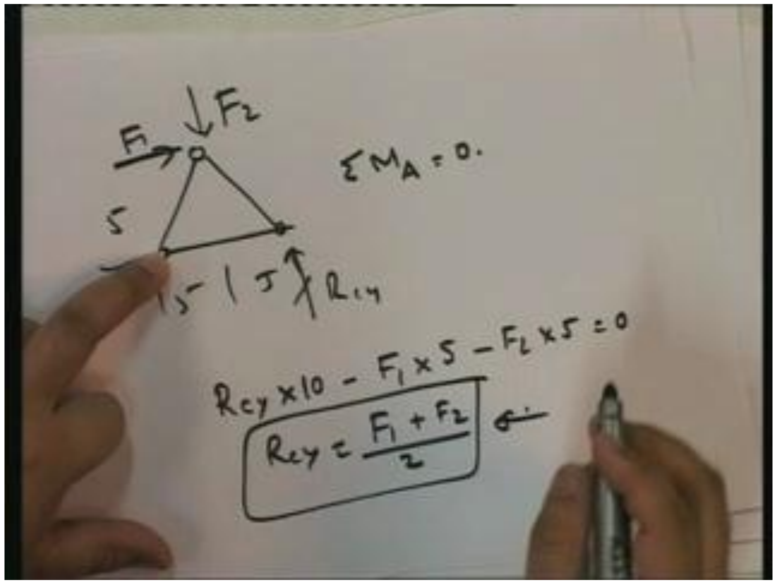

This is 5, this is 5 (Refer Slide Time: 37:09) and this is 5, this is F_2 and this is F_1 and I need to find out R_{CY}. Take moments about A is equal to 0. What do I get? R_{CY} into 10, this is a counterclockwise movement and both of these give rise to clockwise movement, so the clockwise movements are F_1 into 5 minus F_2 into 5 is equal to 0, and R_{CY} is equal to (F_1 plus F_2) by 2.

Both the methods of virtual displacement and equilibrium give me the same value of R_{CY}. This in essence is the method of virtual displacement. What I have done is that I have reviewed with you the method of virtual work, which actually has two different virtual work methods. One is the method of virtual force, which is used to find out displacements. Remember? Let us review that. What did I use the virtual force method for? I used the virtual force method to get the displacement; this is the virtual force. The virtual force method is used to find out displacements in the structure; the method of virtual displacement is used to find out the forces in the structure using geometry. That is the point that I was trying to make right at the beginning: the virtual force method is used to define geometry using equilibrium and the virtual displacement method is used to define forces using geometry.

We are going to continuously use these methods over and over again during the entire course that we have over here and you will see how useful this method is. Note that some people tend to use the method of virtual work and some people use the method of stationary potential energy or minimum potential energy – all these methods are used to analyze a structure, these are just methods, all of them really give you the same. You might have used the method of minimum potential energy to analyze the displacements of a structure; you have used the Castigliano principle to get the displacements of a structure. What is so good about the method of virtual work? The reason why I am going to use the method of virtual work while I teach you structural analysis of statically determinate structures is because all the other methods are essentially mathematical principles where you need to differentiate something, you need to integrate something – it is all mathematical, whereas I like the method of virtual work essentially because it is very very physical.

I apply a force, I find out those forces and then I use the virtual work principle which says that the total work done by displacements and forces in a structure under equilibrium is 0, the virtual work done is 0. In other words, when I use the method of virtual displacement, I actually displace the structure (physically displace the structure), find out all the displacements and then I multiply all the virtual displacements by the corresponding real forces acting at those points and I have got my work. It is a very physical way of getting what you have here.

Similarly, the method of virtual force to define geometry; you have so many different methods: the Castigliano principle, the conjugate beam, so many different ways. Why are we using the method of virtual force? We are using the method of virtual force to essentially use equilibrium. All the other methods that you see are either geometry or mathematics. In this one, you actually apply a virtual force, find out all the forces in the structure and once you have found out the forces in the structure, you multiply all the virtual forces by the real displacements and deformations and put that equal to 0 and you have the unknown displacement that you need. It is a very physical way of obtaining the forces and that is the reason why I am going to be using the method of virtual work for the rest of the semester to solve all problems and that is the reason why I have spent one lecture looking at how to use the method of virtual work. Of course, today, I just used it for a very very simple problem; we will use the same methods to solve very very complicated problems later. Today was just an illustration.

This brings me to the end of the review section of this particular course. I have reviewed how to get the static indeterminacy and I have reviewed how to get the kinematic indeterminacy of a structure. Therefore, given a structure, you should now be very comfortable in finding out the static indeterminacy and kinematic indeterminacy of a structure. I recommend that you go back and look at any book on structural analysis, pick up a few problems, look at them and try to get the static and kinematic indeterminacy for those structures. Once you gain confidence, you will see that it is really trivial. Please remember: use the algorithmic way of getting the static and kinematic indeterminacy, which I have discussed over the last two lectures.

Today, I reviewed the method of virtual work which all of you are familiar with and all of you have used – it is just one method that you have used to get the displacements and forces in the structure. I am using it here because I am going to be using it over and over again for the rest of the semester. I think I am going to be ending here today, because I would like to end the review module here. From next lecture onwards, we are going to be getting into the basic methods of analysis. Since I have a little bit of time today, I am going to look at the methods that I am going to be using.

First, I am going to be looking at the force method. Initially, I will be looking at the basic force method, which by now you should be familiar with, but I will still look at it because I am going to be using the method of virtual work to solve the force method problems – that will be the new way. Then, I am going to be introducing to you the matrix approach. Ultimately, the matrix approach has several advantages with the standard approach of the force method; so I am going to introduce you to the matrix approach of the force method. I will be using the force method to solve truss problems, to solve beam problems and to solve frame problems. We are going to restrict most of the problems to planar problems because ultimately, I will be using them for illustration purposes but all the things that I talk about are going to remain exactly valid even for space problems or space structures.

I am going to first look at the basic force method. Then, I am going to be looking at the matrix approach of the force method; in that, I am going to be using the method of virtual forces to develop the equations in the force method. After that, I am going to be looking at the displacement method for analyzing structures. In the displacement method, I am going to be using the method of virtual displacement to set up the equations that we are going to be solving in the displacement method. This is the reason why I have spent today on the method of virtual work: it is going to be used over and over again.

Again, in the displacement method, I am going to look at the basic formulation and then I am going into the matrix approach, which is essentially referred to as the stiffness method; the matrix approach for the displacement method is known as the stiffness method. We are going to be looking first at the basic force method, matrix analysis of the force method, then we are going to be looking at the basic displacement method, then we are going to be looking at the matrix approach for the displacement method (the stiffness method) and that in essence will give you the overview of the basic methods that can be coded in the computer to solve complicated structural analysis problems. After that, of course, I will be looking at a simpler approach known as moment distribution which is on how to solve complicated structures using hand calculations. Finally, as I said, we will be looking at influence lines, how to obtain the maximum loads, design loads and structures under moving loads.

Thank you very much.