Structural Analysis – II
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Lecture – 27

Good morning. We have started discussing about the matrix approach for structural analysis and in the last lecture, we looked at the basic concept behind the flexibility approach for a statically determinate truss. Today, we are going to continue that. Please understand that in the last lecture, the problem that I looked at – I was just looking at it; I did not solve the entire thing purely because I was just explaining the concept; later on, we will have enough opportunities to look at problems. Right now, the way I look at all my lectures is that I first look at all the concepts, use problems only as illustrative purposes to explain the concepts and then, once the concepts are described, I move on to solving problems in their entirety so that you know how to use the method in applying it to problems. Today, we are going to be looking at the flexibility method for beams and frames.

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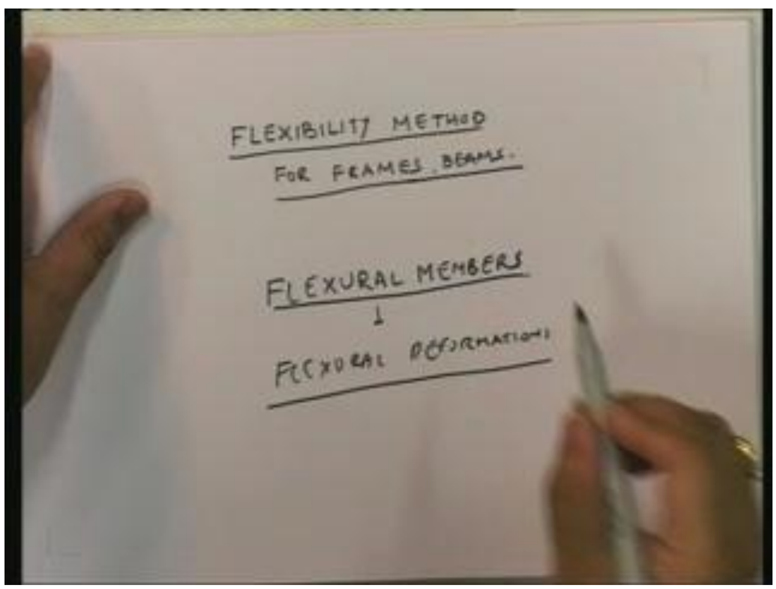

The essential difference between what we did for truss last time and what we are doing for beams and frames is that they are flexural members as opposed to axial members – in other words, they are flexural deformations. Just like we saw that in a truss, you started off by looking at the truss structure and then looking at the member forces in terms of the external loads and from there, we moved on to solving the structural problem.

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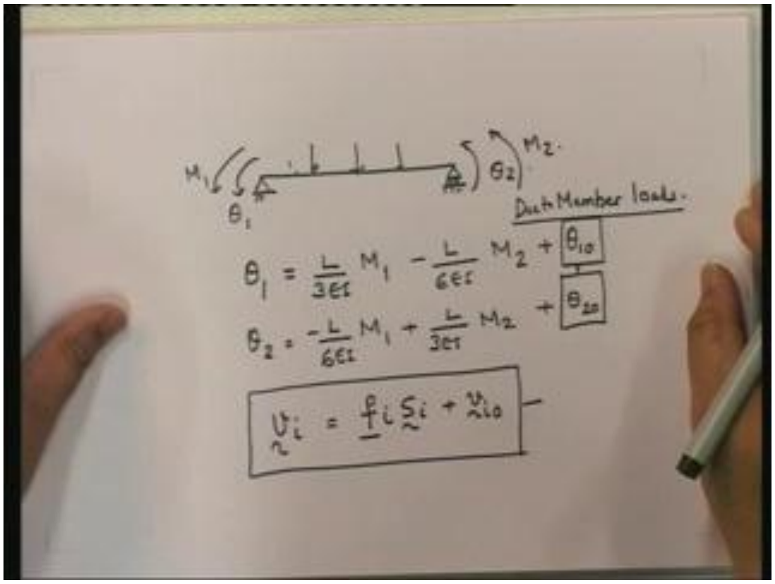

Now, let us look at a member – a member that has flexural deformations. Now, what are the member end deformations? We have already done this – I do not want to go into this. This is going to be theta$_1$ and this is going to be theta$_2$; these thetas are essentially due to moments M_1 and moment M_2. In addition to this, there may be thetas due to loads on the member; so if we write down this, we will see that – we have already written this down, I am just explicitly writing it down – theta$_1$ is equal to L upon 3EI M_1… I am actually going back to the old… I am just rewriting it in this form, and then plus theta$_{10}$. What is the theta$_{10}$ due to? Understand that theta$_1$, the rotation here would be related to M_1, M_2 as well as theta$_{10}$, which is the rotation at the left end due to the member loads; so this aspect is due to member loads. Similarly, theta$_2$ is equal to minus… both of these are due to member loads. If there are no member loads, then these two are 0 – this we have already seen. Now I am going to rewrite in this format. This is the format that I am going to write it down in. Let us see what these terms are.

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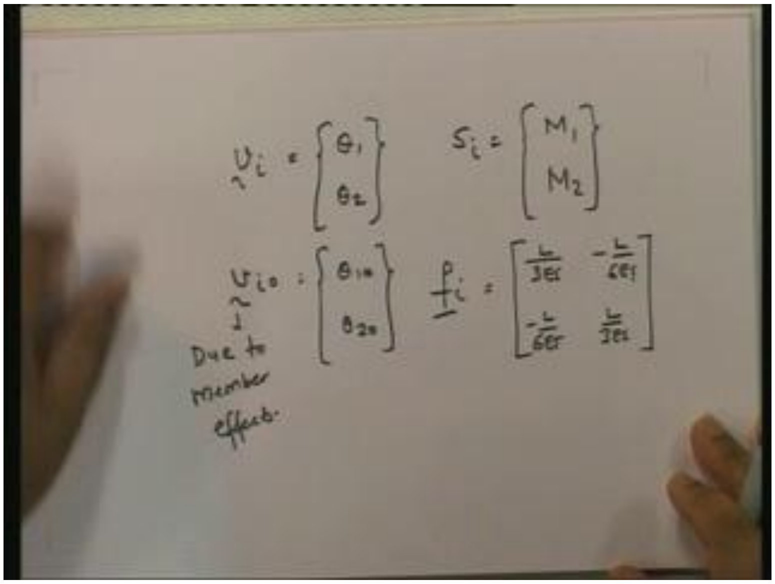

If you look at these terms, v_i is equal to $theta_1$ and $theta_2$, so in other words, the member deformations are the rotations at the two ends, these are the member deformations. I am using the same notation that we are going to be consistently using in this approach. What is S_i? S_i is equal to M_1, M_2. We know that v_{i0} is $theta_{10}$, $theta_{20}$ and v_{i0} is due to what I will call in a broader sense member effects because we have already seen that rotations can happen not just because of member loads but they can also happen due to temperature etc., etc. – we have already discussed that, so I am not going to go into the details. Finally, what is f_i? If you look at it, it is L upon 3EI, minus L upon 6EI, minus L upon 6EI, L upon 3EI. If these are defined in this fashion, then this is this is valid because this we know is valid from earlier. This is what we are going to be going with – this is one relationship. The other relationship is….

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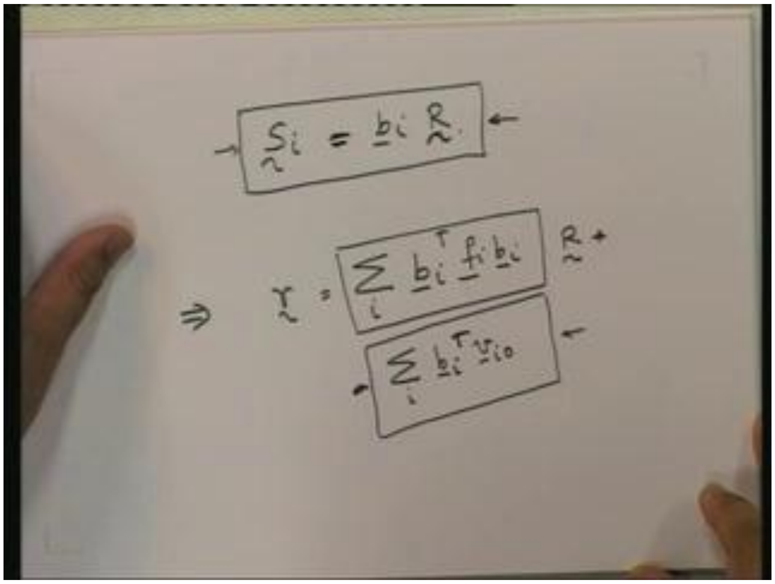

What is this relationship? If you look at this, all it says is how we get the member end moments corresponding to the loads apply at joints. Remember that loads applying on members are already considered in the effect that we considered in v_{i0}, so this is just member end loads and so if you really substitute it in here, we will see that ultimately, we get displacements in exactly the same way. I am not going to go into the details but once everything is set up, you will see that this is equal to the same terms the way I wrote them. This is exactly valid because once you use this, substitute the value of S_i into v_i and all those things, you will see here that this is one thing and then there is plus one additional term sigma over i b_i transpose v_{i0} – this is a very important term. These are member end rotations due to the loads applied on the members themselves. Therefore, by using virtual displacement, you will see that r can be related in this form.

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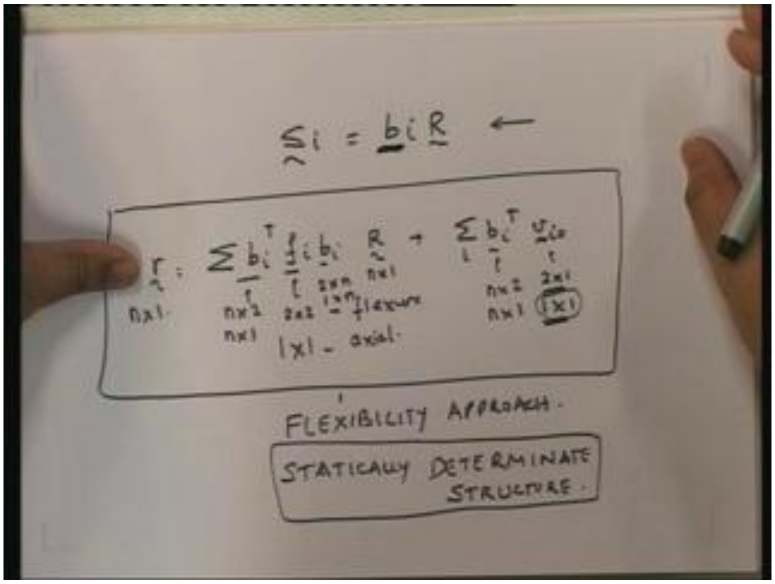

This in essence is the final form of r: r is equal to summation…. Here, for a flexural member, it is 2 by 2 – for flexure, it is 1 by 1 for axial. If this is n by 1, then by definition, this is going to be 2 by n or 1 by n depending on whether in a flexural member you are finding out the member end moments or in an axial member you are trying to find out the actual force in terms of this. This correspondingly is going to be either n by 2 or n by 1, this is going to be equal to a 2 by 1 or a 1 by 1. Now, I have this question, I did not introduce this in the truss but I am introducing this here: what is this effect? For a flexural member, we know what this is; this is due to member effects, for example, member loads, temperature etc., etc. In a truss, where would this come from? This would again come from member effects. What are the member effects that we have already considered? These member effects may be temperature or elongation or contraction due to temperature change or lack of fit – remember that.

This is member effect and this one again is either n by 2 or an n by 1 and ultimately this is an n by 1. This, in essence, defines the flexibility approach for structural analysis. How do we find out the member forces? Member forces in terms of the external loads are this (Refer Slide Time: 14:14), so these are the member forces. Remember that all of this is for a statically determinate structure. This in essence defines… these are the member forces related to the structure loads and these are the structure displacements corresponding to the loads. This, in essence, is the definition of your structural analysis: find out the member forces and find out the deflections. Now, this again is for a statically determinate structure. Once we have this, let us see how we can apply it to a member and for a beam, it does not make sense, so I am going to take a frame.

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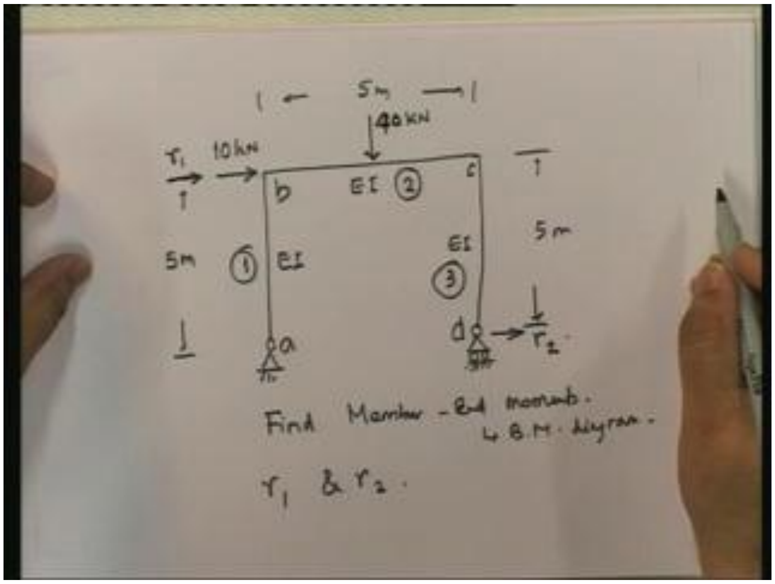

Let us take a frame problem and see how we deal with this; of course, the frame has to be statically determinate. Again, I am not going to go through the entire problem – I am just going to list out how to get what we want to. Now, let us put some loads on this structure, let me put some numbers here. This is EI, this is EI, this is EI. Let us say that this is 5 meters this is 5 meters, so this is – 5 meters – all of them are 5 meters. This load is let us say 40 Kilonewton. Here, what we want to find out is… There are lots of things to find out but let me just tell you that I want to find out the member end moments, because once I know the member end moments, I can draw the bending moment diagram, so I want to find out the member end moments and in this particular case, I want to find out how much… let us put some other number, let me put something else, let me put 10 Kilonewton here and I want to find out what is the displacement of this joint.

In other words, the lateral… note that since all members are axially rigid, automatically the horizontal displacement of this will also give me the horizontal displacement of this (Refer Slide Time: 17:55). In addition to that, let me find out how much this goes by. These are my two questions: how much does the top displace by and how much does this – this is my overall statement and I also want…. Therefore, the question is find member end moments. Once we find out member end moments, you can actually draw the bending moment diagram. What else? The values of r_1 and r_2. This is my problem statement. Now, I have to understand how I am going to go about this. Look at this and you will see that this is essentially a statically determinate structure, so let us see how we are going to go about it. Now, in the first step in this problem, what do we find out? For each member, we find out its member end displacement. Let me qualify this; I will call this member 1, this member 2, this member 3 and this is a, b, c, d; ab is member 1, bc is member 2, cd is member 3.

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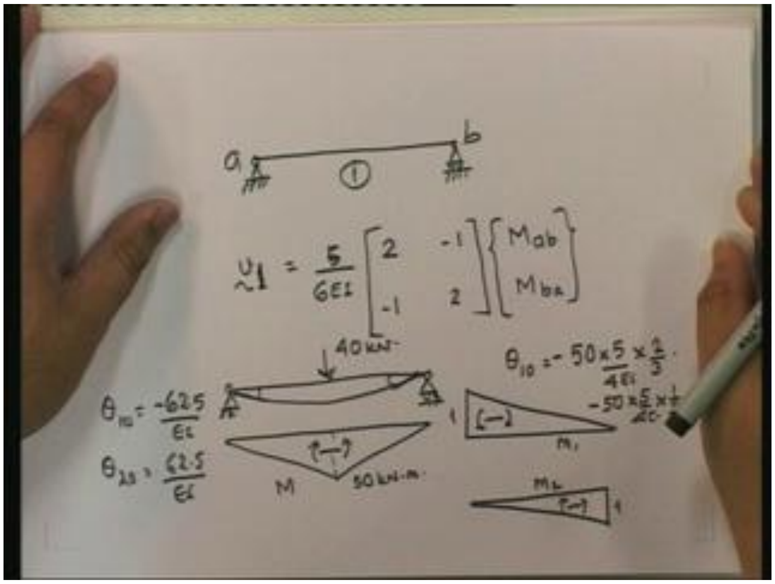

The first step: for each one, we write down the member force deformation relationship – for ab, what is this thing? There is no load and therefore, v_i, which are the two rotations at the two ends, are equal to – I am going to write this in this format – L upon 6EI and L in this particular case is 5 and note… see, 2 L upon 6EI is upon 3I, which is what you have, multiplied by S_i, which are M_{ab} and M_{ba}. Since there is no member load, for ab (which is member 1), v_1 is this. For member 2, this has a load of 40 Kilonewton acting here, so the first thing that we have to do… understand that the first part is this, but there is an additional v_{i0}, v_{20} that we have to write down.

How do we find out V_{20}? Under this, it is going to deflect like this and we need to find out this (Refer Slide Time: 21:55) and this. How do you find that out? I think we have done it enough, so I am just going to write down how to find it out. This is M, this is m_1 and this is M_2. Once we go through the entire process, we see that $theta_{10}$ is going to be equal to in this case 50 into 5 by 4EI; by the way, minus because these are opposite to each other, so 50 into 5 by 4EI is this part and for that, it is going to be equal to 2 by 3, then you will have minus 50 into 5 by 4EI into 1 by 3, so essentially, $theta_{10}$ is equal to minus 62.5 by EI and $theta_{20}$ is equal to 62.5 by EI. Actually if you look at it, it will work out to p L squared upon 48EI but that does not matter, we have computed this.

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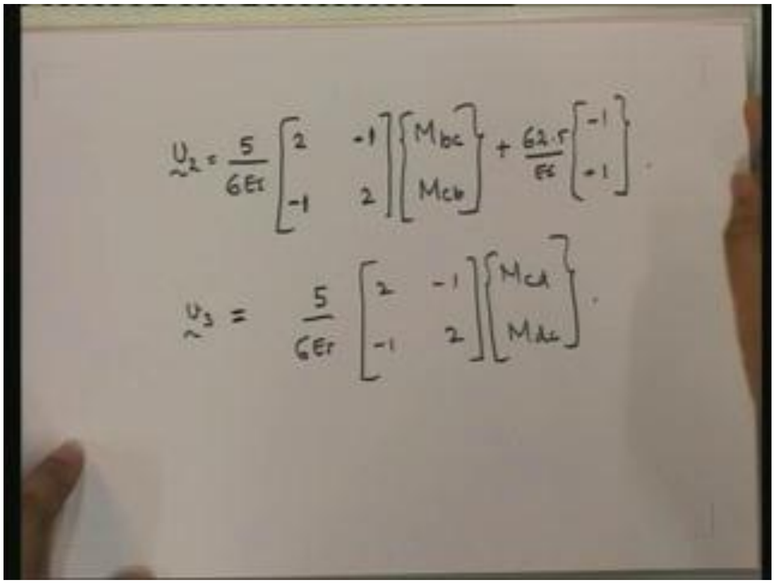

For b_2, it is going to be equal to 5 upon 6EI into 2, minus 1, minus 1, 2 into M_{bc}, M_{cb} plus 62.5 upon EI into minus 1, plus 1 and since there is no load on it, v_3 is going to be equal to 5 upon 6EI into 2, minus 1, minus 1, 2 into M_{cd}, M_{dc}. We have written down the force or deformation force relationship for each member. This part is the f into S_i and this part is the v_{i0}; for 1 and 3, there is no v_{i0} but for 2, there is a v_{i0}, which we have computed.

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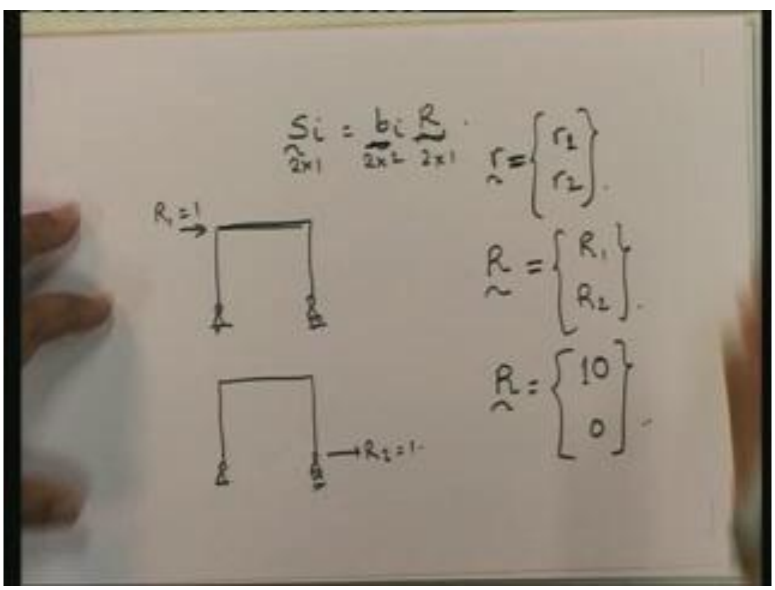

Now, the question comes: what next? Next is evaluating S_i – this is very very important. How do we find out this S_i into b_i R? What is R? In this particular case, what we are finding out? We are finding out small r is equal to r_1 and r_2 – those are the displacements at the top and at the hinge.

This is this way. Therefore, capital R will be R_1 and R_2 corresponding to the forces. If you look at this particular concept… in this particular case, what is R equal to? It is irrelevant what we are getting. R you will see will be equal to 10 and 0 in this particular case. How do we find out b_i? What we do actually is this. This is 2 by 1 and this is 2 by 1, this by force has to be 2 by 2. Now, this 2 by 2 has two columns. For each column, the way we find out the column vector is this: put R_1 is equal to 1 and find out the member end moments; put R_2 is equal to 1 and find out the member end moments.

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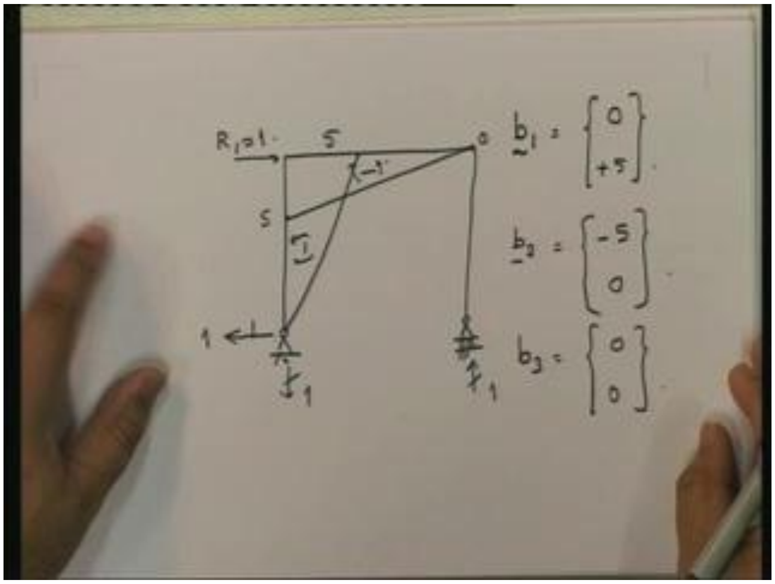

If we look at that, this becomes 1. Then, how would we do it? If you look at this, you will see that this gives rise to 1. The reactions: sigma F_x is equal to 0, sigma M is equal to 0, take moments about this point and then sigma F_y is equal to 0 gives me all of these. Once I have all of these, my member end moments become this. How I find out is… The first column of b_1, I am finding out the first column for b_1 – what would that be? Let us look at it. If I look at this, I can draw the bending moment diagram. How would the bending moment diagram look? It will look like this (Refer Slide Time: 30:11). This would be 5 here, this would be going in this fashion and on bc, it is going to be like this, 5 and 0 here, this is going to be… and over here, it has to be 0 because this 1 does not take any moment. If you look at b_1…. At this M_{ab}, what is the moment? 0, so this is 0; at this point, what is the moment? Anticlockwise 5, so anticlockwise 5 is plus 5.

Let us find out b_{20}. It is clockwise at this end – it is clockwise 5, so this is minus 5, here, it is 0, so it is going to be 0. What about b_3? You will see it is equal to 0, 0 because there is no moment at this end. This is the first part for R_1 is equal to 1.

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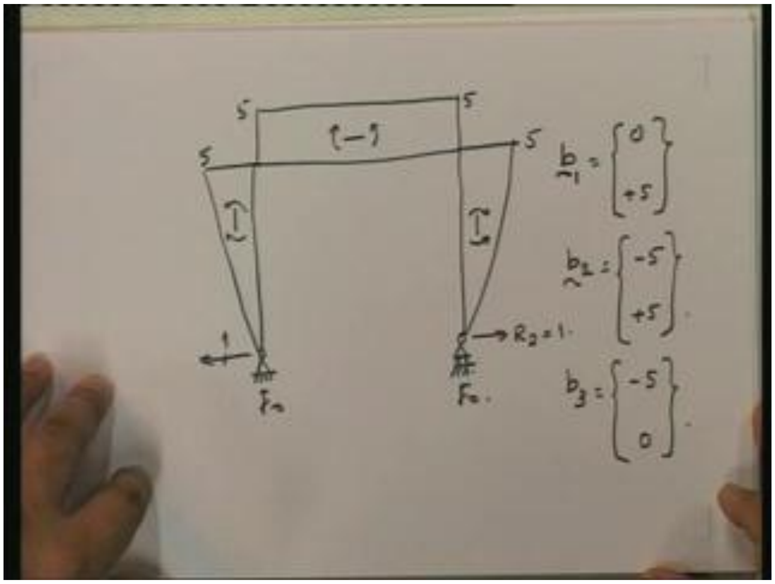

Then, we are going to put R_2 is equal to 1; let us put R_2 is equal to 1; I am now putting R_2 is equal to 1. If you look at this, this will give rise to this and these two are going to be equal to 0. How will the…? This is 5, 5, 5. This is the bending moment diagram and therefore, if you look at b_1 due to R_2 is equal to 1, this is going to be equal to 0 and this is again anticlockwise, so it is going to be plus 5; b_2 is going to be… this is clockwise, so it is minus 5, here it is anticlockwise, so it is plus 5; b_3: here, it is clockwise, so it is minus 5, here, it is 0.

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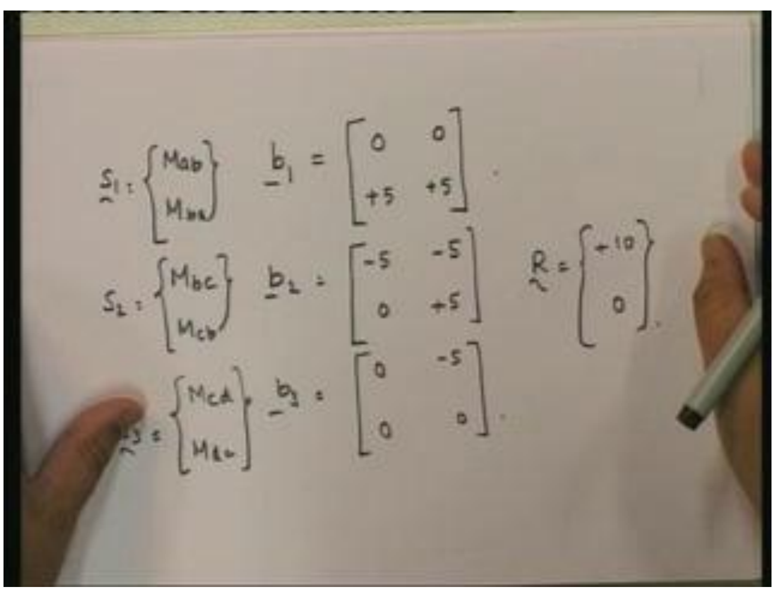

Ultimately, having done these, if we write down b_1, this is going to be equal to 0, plus 5 – this is coming from R_1 is equal to 1; the next one is coming from R_2 is equal to 1. Similarly, R_1 is equal to 1, R_2 is equal to 1 (Refer Slide Time: 34:36) in each column; b_3: R_1 is equal to 1 and R_2 is equal to 0. Now here, for b_1, what are the things? S_i is written in this way, M_{bc}, M_{cb} and this is equal to M_{cd}, M_{dc}. This is S_1, this is S_2, and this is S_3. If I have got these, now I can plug in. What is my R? I know, I have already given my R: R is equal to plus 10, 0, so I can find out my S_i very easily – that is the first step: find out the member end moments.

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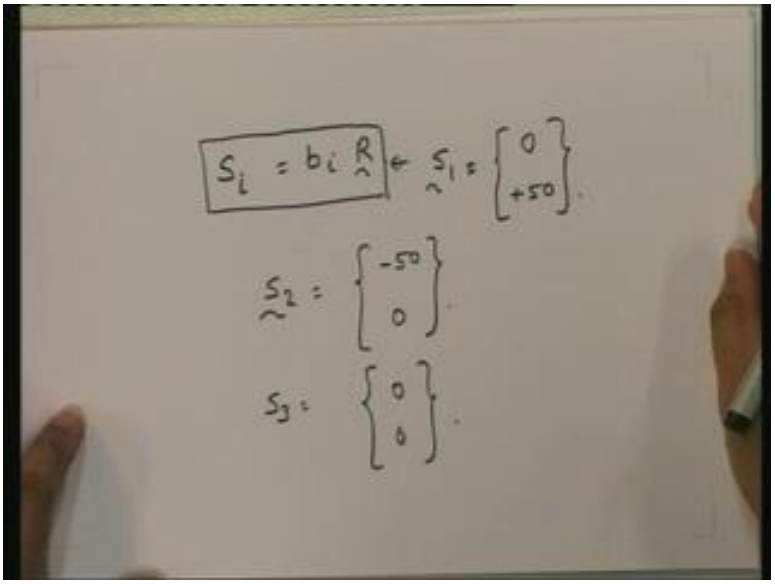

If you look at the member end moments, you will see that they are equal to… If you put in, you will get S_1 equal to 0 plus 50, S_2 is equal to minus 50 and 0, and S_3 is equal to 0, 0. Satisfy yourself that this is indeed the case for this particular problem. We will come to problems later where we will see that this is not a complete picture of my S_i – in fact, S_i actually has some additional terms that we will look at later; right now, I am not too concerned with this. For this particular problem, you will see that this is indeed how my S_1, S_2 are. Once I know my b_1, b_2, b_3 and I have also got this, what is my R going to be equal to?

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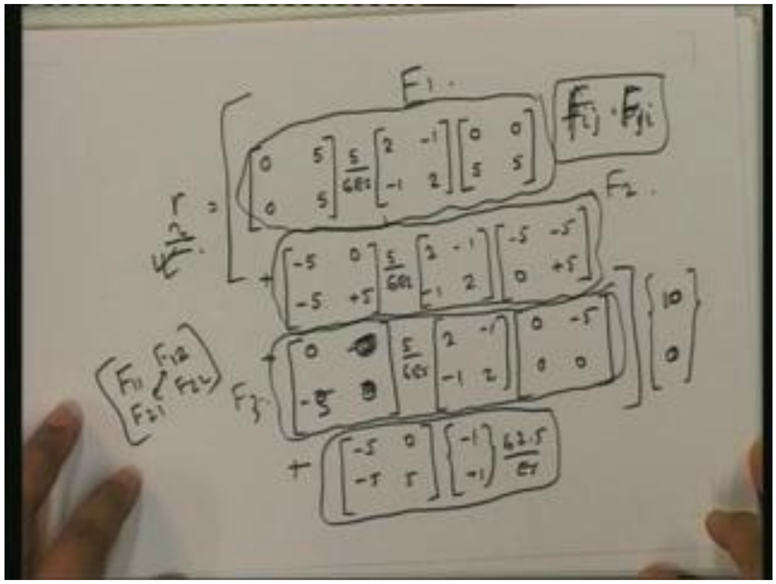

I am going to just put it down. My R is going to be equal to…. If I just put it down in…. You can actually do the matrix multiplication later. This is v_1 transpose into ((5 upon 6EI) into (2, minus 1, minus 1, 2) and then v_i, which is (0, 5, 0, 5) plus (minus 5, 0, minus 5, plus 5) into (5 upon 6EI) into (2, minus 1, minus 1, 2) into (minus 5, 0, minus 5, plus 5) plus (0, 0, minus 5, 0) into (5 upon 6EI) into (2, minus 1, minus 1, 2) – sorry, this would be (0, minus 5, 0, 0) – (0, minus 5, 0, 0)) and this entire thing multiplied by 10, 0 plus (minus 5, minus 5, 0, 5) multiplied by minus 1, plus 1 into 62.5 by EI. If you do all these computations, you will get your displacements R_1 and R_2. You have got your S_1 and S_2 and you have got your R and that in essence is….

What are these terms? This term is the contribution to the flexibility matrix of member 1, this is F_2, this is F_3. Since I had F_1, F_2, F_3 that gives me the complete flexibility matrix multiplied by 10 into 0. This gives me the structure flexibility matrix for this particular structure and you will also notice something else. In a structure flexibility matrix, do you remember what the member flexibility matrix was? It was 2, minus 1, minus 1, 2 into L upon 6EI. Did you notice something? It was symmetric. When you do these computations, you will see that the structure flexibility matrix is also symmetric. What does that mean? It means that b_{ij} is equal to b_{ji} or I am F_{ij} is equal to F_{ji}.

What is this flexibility? What is f_{11}? In this, you will get F_{11}, F_{12}, F_{21}, F_{22} – you can evaluate each one. What do these mean? Let us look at this. What does F_{11} mean? It is the displacement at 1 due to a unit R_1, capital R_1; F_{22} is the displacement at 2 due to a unit load corresponding to R_2. What is F_{21}? It is the displacement at 2 due to the unit load corresponding to R_1. What is F_{12}? The displacement at 1 due to a unit R_2. Now, think about this. Displacement at 1 due to a unit load at 2 and displacement at 2 due to a unit load at 1 – should they not be the same? Which law is that? The Maxwell–Betti's Reciprocal Theorem. If you look at that, then automatically these two are equal to each other and you will see that this is also a symmetric matrix; a symmetric matrix is one in which the transpose is equal to itself – that is what a symmetric matrix is. This additional term essentially comes from the effect of the local load. We have considered the load effect of the structure loads and the effects of the member loads and obviously, this is going to be a superposition of the two – this, in essence, is structural analysis the way it is. Remember I said that S_i is equal to b_i R in this particular case. When do we need to consider some additional terms?

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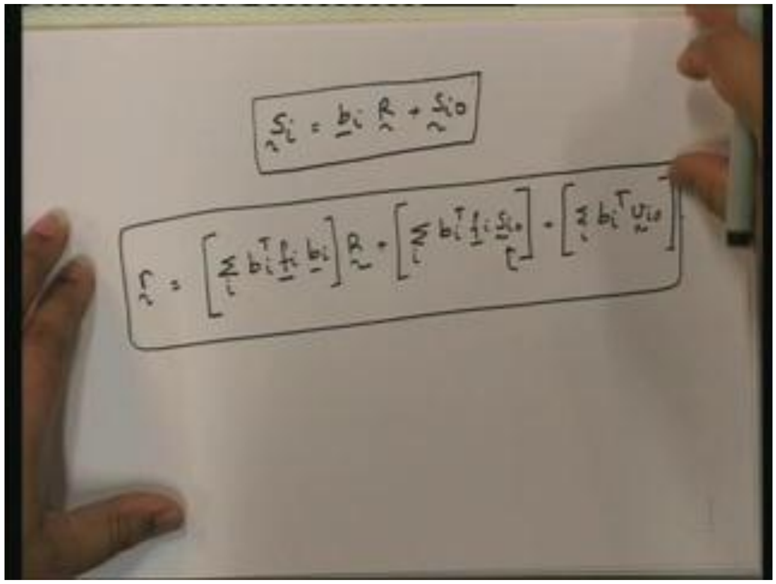

We will see that S_i will not only be just b_i R, there will also be an S_{i0}. Where does this S_{i0} come from? In other words, when you plug this in, you will see that ultimately, r will have three terms: one term will be (summation over i of b_i transpose f_i b_i) into R plus (summation over i of b_i transpose f_i S_{i0}) plus (summation of i b_i transpose v_{i0} – this, in essence, is the real thing. Where does this S_{i0} come from? Let us take a particular problem that I am going to look at. Do not be worried that I am not solving all the problems – I am illustrating all the equations that I am talking about. Let us take the same problem that I have.

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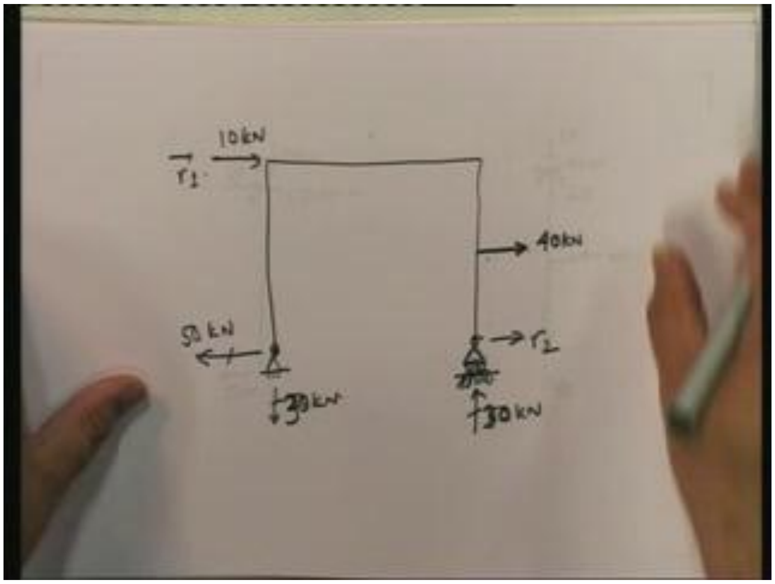

Instead of applying the load here (Refer Slide Time: 45:41)…. I still have this 10 – let it be and I have a 40 Kilonewton load acting at the center. Note that there will be two aspects to the problem: one is… this member load. We are still finding out r_1 and r_2 and we are finding out the S – the member end moments. If you look at this particular case, there will be two aspects to this. In this particular structure, one aspect of this load is going to be the effect of the local v_i and there will also be another if you look at the member end moments. Let us do the member end moments through the traditional approach.

What will be the member end moments coming out to be? In the traditional approach, we take the load and find out; this is a statically determinate structure and I can solve for it. Let us solve for it. What will we get? Here, due to these load you will have sigma F_x gives you that this is equal to 50. Take moments about this point (Refer Slide Time: 47:32). What is this into 10 plus 5 into 40 going to give you? It is going to give you 200, 250 and 250 divided by 5, so this is going to land up being equal to 50 and this one is going to be equal to 50. If we draw, if we find out the member end moments… let me draw the bending moment for this.

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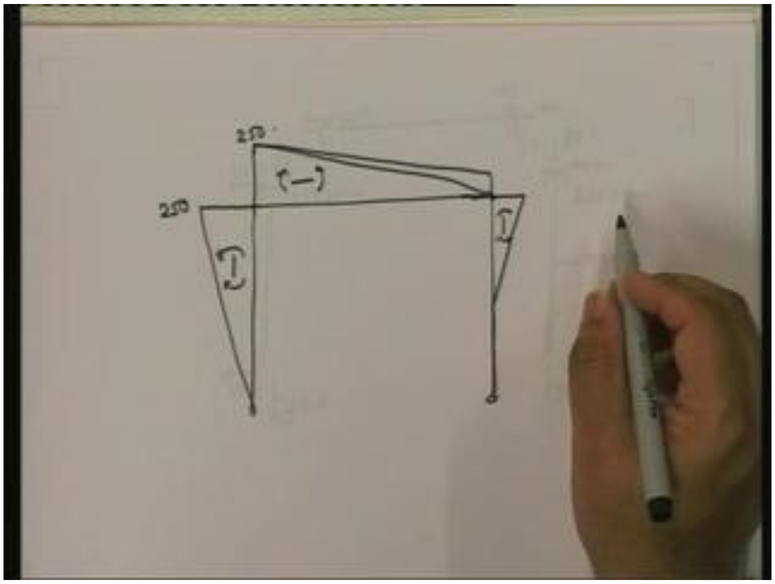

How will the bending moment look? It is 250 here. What kind of a moment is this is going to be? A moment like this. Here, this is also 250 obviously from equilibrium considerations and you have 50 over here, so we are going to be having 50 into 5, 250 and so this is going to be equal to… let us see. Over here, you have (Refer Slide Time: 49:19)…. If you look at this particular point, what moment do you have? You will see that you will have 0 moment here. Let us look at this. Here, what is the shear force? Here, you will see that the shear force is going to be 40, so if you draw the bending moment…. Let me just take you through this step by step.

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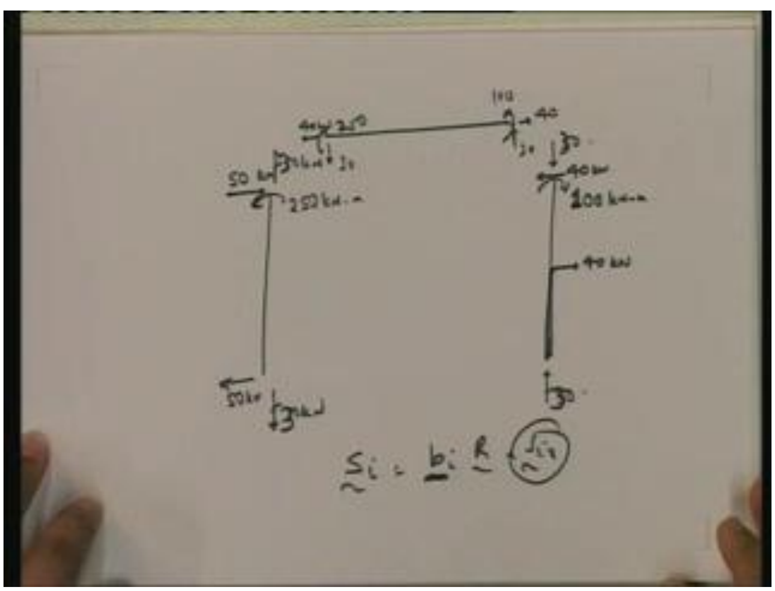

This is going to see this (Refer Slide Time: 50:07) and on this side, it is going to be in this direction – this is 50, this is 50, here this is 50, on this side, this is going to be 50, the moment over here is going to be in this direction and it is going to equal to 250 Kilonewton meter. Look at this: here, this is 0, from here to here, there is no moment and then here, what do we have? We have 40 here, 40 here and we are going to have 50 here (Refer Slide Time: 51:13). What is going to be the moment at this point? You will see that since this goes this way, this will go this way and it is going to be equal to….

The fundamental mistake: actually, this is where I made the mistake, this is 30 (Refer Slide Time: 51:39) and this is 30 because if you take moments about this, 40 into 2.5, so that is 100, 100 plus 150 divided by 5 will give me 30. This is going to be 30, 30, 30, 30. Over here, what do I have? I have 40 into 2.5, so that is 100 Kilonewton per meter. Let us look at this one: this one is going to have 40 this way, 40 this way, 30 this way, 30 this way and the moment over here is going to be 250 over here and so if you take moments about this point, this is going to be 30, this is going to be 100 over here (Refer Slide Time: 52:53). If you look at the bending moment diagram, it looks like this – that is going to be the bending moment. Now, the problem that you will have over here is…. This is the bending moment diagram as we have it. Now, I am going to show you…

I am going to end my lecture over here, but next time I am going to be showing you how we are going to get exactly the same bending moment diagram by using the concept of S_i is equal to b_i into R plus S_{i0} and we will see what S_{i0} actually stands for, in general. Thank you for today. We are going to continue looking at how to look at the flexibility approach for beams and frames; beams – we are doing simply supported, so simply supported statically determinate beam is not an interesting problem, so that is why we were looking at a statically determinate frame. We will continue with this particular lecture as we look at it next time.