Structural Analysis II

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Lecture – 26

Good morning. Just to review what we have done till now – we have looked at the force method of analyzing statically indeterminate structures, then we have looked at the displacement method for analyzing structures and finally, we looked at the moment distribution approach, which was a iterative approach to solving structural analysis. Now, I am going to change gears and move into what we call as the matrix methods of structural analysis. You will see that the matrix methods of structural analysis are nothing but the force method and the displacement method written in a slightly different format. Today, I am going to be discussing how to use flexibility approach. There are two approaches in matrix methods: one is known as the flexibility approach and one is known as the stiffness approach. Today, I am going to be discussing the flexibility approach to structural analysis – this is the matrix method.

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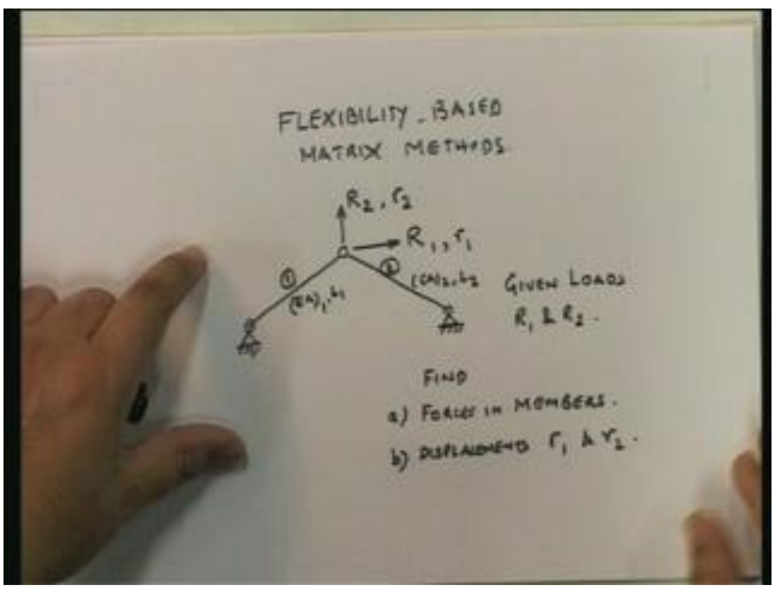

Flexibility-based matrix methods – what exactly is the flexibility approach? Let us now look at a statically determinate structure. This is a structure and you can see that there are six equations of motion, four unknown reactions and two forces – so it is a statically determinate structure. This is subjected to loads R_1 and R_2. Note that in a statically determinate structure, you do not need any method to compute the forces in the members, because the forces in the members can always be computed using equilibrium.

However, I am going to state this problem a little bit differently – I am going to say find out the forces in all the members and find out the displacements corresponding to the degrees of freedom. Here also, you will see…. How many degrees of freedom does this structure have? You will see that there are three joints and you have two degrees of freedom per joint in a truss, remember? Two degrees of freedom: one translation in the horizontal direction and one in the vertical direction, so two degrees of freedom per joint. You have four restraints, two hinges, so the number of degrees of freedom are two; and these are the displacements: horizontal displacement and the vertical displacement.

Let us say that I have this as $(EA)_1$ and this as $(EA)_2$; this is $length_1$, this is $length_2$ Anything else? You have got the complete information. The problem statement here is that given loads R_1 and R_2, find a) – this is member 1 and this is member 2 – find forces in members and b) displacements R_1 and R_2 – these are the problem statements. How would you do this in the normal case? You would find out the forces directly in terms of R_1 and R_2 – you can always find out the forces because all you need to do actually in this particular case is just take equilibrium of joint b. These are the two unknown forces and take sigma F_x is equal to 0, sigma F_y is equal to 0 and you can find out the two forces – member forces are very simple; so finding out the forces and members is just equilibrium.

How would you find out displacement R_1 and R_2? Corresponding to R_1, if I am applying the principle of virtual work, I would apply a virtual force equal to 1 corresponding to R_1, then find out the virtual forces and then take internal virtual work and external virtual work – you equate it and you can find out R_1. These are the steps that you would follow in a normal kind of a situation. Let me just write that down in a particular manner.

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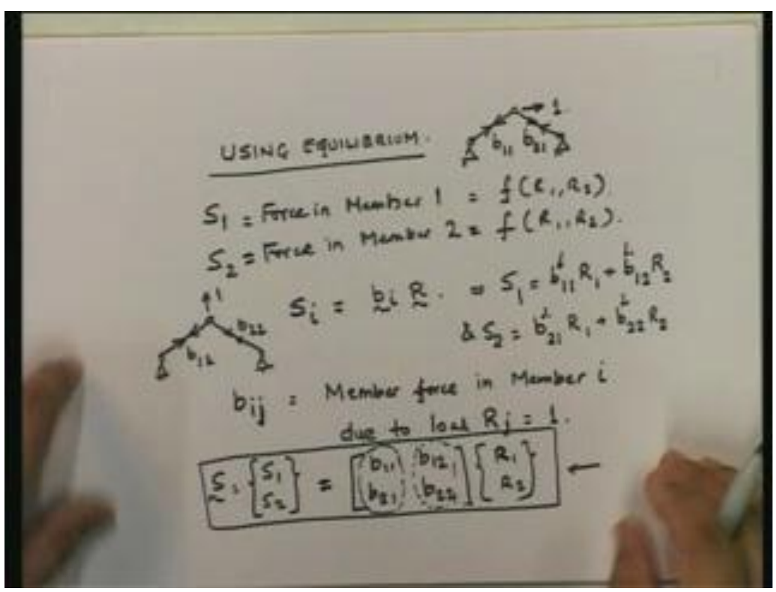

Using equilibrium, you can actually find out the force in member 1, it is a function of R_1 and R_2. Similarly, the force in member 2 is also a function of R_1 and R_2. These two you can find out using basic equilibrium. Let me say I am going to use start using notations – I am going to stick through these notations all the way through. This is slightly different: I am going to say this is equal to S_1 and this is equal to S_2. Up till now, I have been using force in member 1, force in

member 2. I am just defining a notation and I am saying that let me call the force in member 1 as S_1 and member 2 as S_2. Then, I can say that S_i can be given in terms of b_i into R. What does this mean?
This essentially means that S_1 is equal to b_{11} R_1 plus b_{12} R_2 and S_2 is equal to b_{21} R_1 plus b_{22} R_2. What are these coefficients? b_{11} or I will put it as b_{ij} is member force in member i due to a load R_j is equal to 1. In other words, the member bij is defined as the force in member i due to a load R_j is equal to 1 this is the definition of b_{ij}. In other words, I can write this in this format: I am using a matrix. If I write S as a row vector containing S_1 and S_2, I can actually write it as – think about it – b_{11}, b_{12}, b_{21}, b_{22} into R_1, R_2. This is the vector of member forces, this is the vector of loads on the structure and this is the matrix which actually is a force relationship that can be evaluated using equilibrium. Think about how will I evaluate this vector? Think about it. How will I find out this vector? I could find out this vector by this structure applied to a unit load and the force… so let me put it this way: this is equal to b_{11} and this is equal to b_{21}.

Therefore, in this matrix, each vector, each column vector can actually be found out by solving through equilibrium: putting a load equal to one corresponding to the load R_1 and unit load and finding out the member forces by taking equilibrium of joint b. You can find out the member forces – these member forces are denoted as b_{11} and b_{21}, so you can find out this. Then, how would you find this out? You could find this out by taking the same structure, applying a unit load and finding out the member forces. This would be b_{12}, b_{22}. What we are doing here is really saying that I do not care what load there is on the structure. In this particular structure, what are the possible applied loads? You will see that the number of applied loads that you will have is always equal to the degrees of freedom and each load is applied corresponding to a degree of freedom.

Look at it – how many of the loads can you apply on this structure other than these two, corresponding to the degrees of freedom? Whether you have these loads or not is not relevant. You might have a situation where only R_1 is not equal to 0 or R_2 is not equal to 0 and R_1 equal to 0, or you may have both R_1 and R_2. You see, the procedure that I am developing does not depend on whether a load exists on the structure. These values you can find out, but this is independent of what these values take up because these are…. What this is? These are the member forces due to R_1 is equal to 1, these are member forces due to R_2 is equal to 1 and therefore, this is very simple – you can find these out; normally, you know these displacements, so you can always find out the member forces. Therefore, all I am saying is that I am using equilibrium; however, even though I am using equilibrium, I am saying that I am writing down the member forces in terms of the loads and therefore I can always write it in this fashion.

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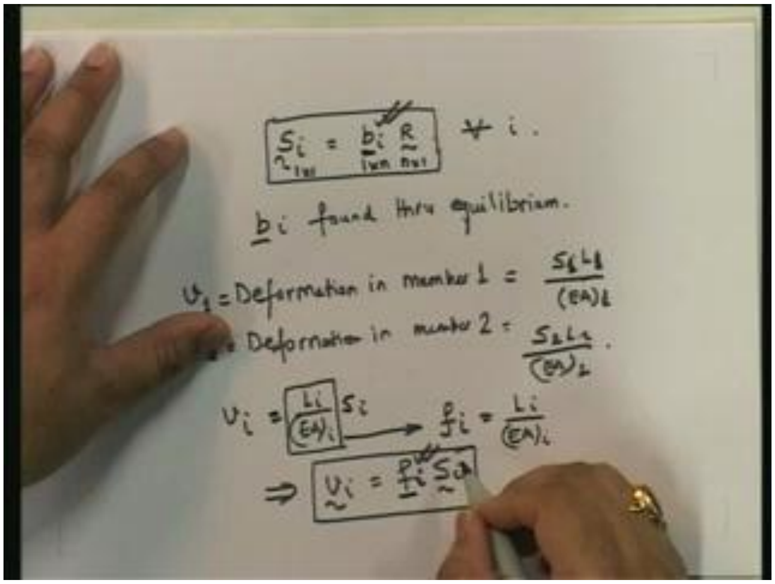

I am going to continue in this fashion; I am now going to put it together; I have just put it together. This is obtained for all I, so for every member, you can actually compute this; in this particular matrix, since this is a 1 by 1 in a truss and this is n by 1 depending on how many degrees of freedom there are and you will see that this is equal to 1 by n; if this is a column vector, this is a row vector, row vector into column vector gives you a scalar.

We can evaluate this. This can be found out – b_i can be found through equilibrium. This is the actual load, these are the actual member forces (Refer Slide Time: 17:48)**.** Now let me find out what the member deformations are. If I look at deformation in member 1, what is member deformation? If S_i is the load, then you will see that it is S_i into L_i upon $(EA)_i$. What is S_i upon A_i? S_i upon A_i is the uniform stress in the member, then S_i upon $(EA)_i$ is the uniform strain in the member and uniform strain integrated over the whole length… essentially since you have a uniform frame, the strain integrated over the whole length is going to be just strain into L_i and this is the deformation in the member.

I have been using till now delta; however, I am going to use consistent formulation, so I am calling this v_1 – v_1 is the axial deformation in the member 1 due to the load S_1. This is the deformation in the member due to the loads. What is v_2? Deformation in member 2 is equal to S_2 L_2 upon $(EA)_2$. Note that we are just saying the lengths could be different and EAs could be different; whether they are different or not actually depends on a particular structure. Here, note that I am not making any assumption excepting for the fact that the structure is a statically determinate one, so that I can relate the member forces to the only assumption till now, which is that it is statically determinate and of course that the members are uniform. These are the only two assumptions: statically determinate structure and uniform members – those are the only two assumptions that I have made. If I can do that, I can find this out and I can find out deformation in the member.

I am going to call this as v_i, which is equal to L_i upon $(EA)_i$ into S_i – I have just rewritten this particular thing. Now, think about it: what does this relate to? This actually is a relationship between the member force – this is the force deformation relationship or if you want to call it you can call it deformation force relationship. What does this particular quantity signify? This quantity (Refer Slide Time: 21:35) is the deformation in a member due to a unit member force – deformation in member i due to unit S_i; this by definition is called as the flexibility of member i.

What is flexibility of member i? It is equal to L_i upon $(EA)_i$. What is the flexibility? What is the physical definition of the flexibility? The physical definition of the flexibility is the deformation in member due to a unit member force. Once we have that, this implies that v_i is equal to f_i S_i. In this particular case, v_i happens to be 1 by 1, S_i happens to be 1 by 1 and therefore, f_i is also 1 by 1, but I can actually relate it in this fashion: if I have more than one member force, then the deformation corresponding to that force is going to be related through the flexibility matrix or in this particular case, the flexibility coefficient – I am just going to keep it in this fashion. Now since this is true and I can find this out and note that I know this because I know this and this loading in normally given, I can find this out. If I substitute this into this, what do I get?

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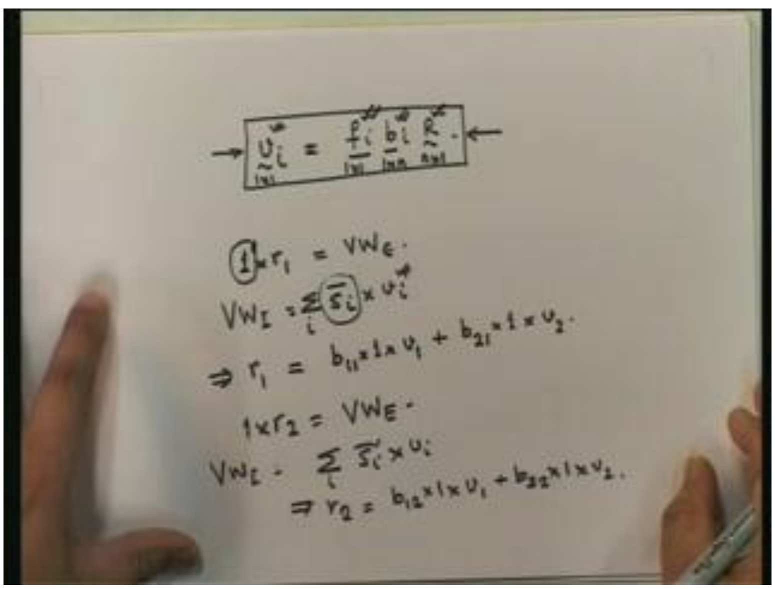

You see that I get v_i is equal to f_i into b_i into R – I have just substituted S_i equal to b_i into R into b_i equal to f_i S_i; that is all I have done. What does this relate actually? In this particular case, since I am dealing with a truss member, this is a 1 by 1, this is a 1 by 1, this is 1 by n and this is n by 1. Here, the major point is…. What does this give me? This gives me the member deformation in terms of the member loads, where this is given by the force deformation relationship of a member and this is given from equilibrium.

Now, what is the next step? Note what my ultimate goal is: given these loads, what are R_1 and R_2? Note that I have already found out the member forces – the first thing that I did was find out the member forces, because they are related in terms of b_i and you can always obtain b_i by actually solving the equilibrium equations for this statically determinate structure. The next step is I am trying to find out R_1 and R_2. How would I find out R_1 and R_2? Well, we know that due to this load, we have real deformations in the members. If I use the concept that I have these and I want to find out R_1, what would I do? I would apply a unit virtual force corresponding to the degree of freedom R_1, so let us do that: unit multiplied by R_1. Note that since I have only applied a unit virtual force corresponding to R_1, the virtual force corresponding to R_2 is 0 and therefore, that will not do any work, so this is equal to the external virtual work.

What is internal virtual work? Internal virtual work is going to be equal to the S_i – these are the virtual forces due to the unit applied load corresponding to R_1 multiplied by v_i (I am continuing the fact that they are scalars). What does this give me? Virtual member force times the deformation – this is the work done in each member; this summed up over all the members is going to give me the internal virtual work. What do I have to now compute? Now, I know this, because I know this, this, this and I know this (Refer Slide Time: 27:07), so I know this. Now the only thing I need to find out is the member forces due to a unit virtual force corresponding to R_1. Let us go back. Which was R_1? R_1 was this horizontal degree of freedom.

What is my force? Virtual force is this. What are the virtual member forces? The member forces are these (Refer Slide Time: 27:41) – these are the same. Whether this force is real or virtual, it makes no difference – the member forces are going to be identical. I know what those are, so I just need to substitute those. What are those? These are equal to in this particular case b_{11} into 1 into v_1. This is the virtual member force due to the unit moment times the deformation plus b_{21} into 1 into v_2.

Think about this. What is this? This essentially means that this r_1 is equal to b v_1 into this thing. The next equation; I need to find out is r_2. I apply a unit virtual force corresponding to r_2 and that is my external virtual work. My internal virtual work is again going to be S_i into v_i, but this S_i is due to this degree – let us see what that is. Look at this: R_2 is this degree of freedom, so if I apply unit load, it is going to be b_1; whether this is virtual or real, it makes no difference, it is going to be this; only thing is if this is virtual, these are virtual forces. If you look at this, this implies that r_2 is equal to b_{12} into 1 into v_1 plus b_{22} into 1 into v_2. If I write this, do you notice something? I can write these two equations in matrix form. What would that become?

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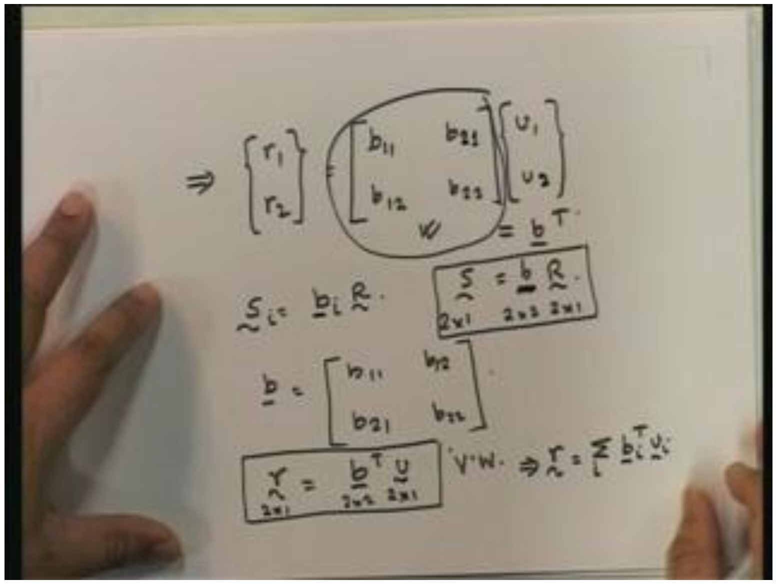

You will see that this implies r_1, r_2 – I am just writing in a matrix form – b_{11}, b_{21}, b_{12}, b_{22} into v_1, v_2. Note that I have all I have done is written these two equations in a simultaneous in a matrix form – that is all I have done; but note something: what does this look like? Remember that I had S_i is equal to b_i R? You can look at this. This you will see is going to be equal to… and if I take S_i equal to… or I can write it as S equal to b R where S… since I have two…, if you look at it, this is what I have written – this one.

S is a 2 by 1 because I have two members. This is b into R; b is b_{11}, b_{12}, b_{21}, b_{22}; so in this particular case it is 2 by 1 (Refer Slide Time: 32:08), 2 by 1, 2 by 2. What is b? It is equal to b_{11}, b_{21}, b_{12}, b_{22}. If you look at this, what is this equal to? You will see that this is nothing but equal to b transpose. This is very interesting: S is equal to b R, but if I use virtual work I get that r is equal to b transpose v, where v is a 2 by 1 – each member in the column vector corresponds to the displacement and this is a 2 by 1, this is a 2 by 2. Virtual work gives this, this is equilibrium, this is from virtual work. You notice something very interesting and now, what I am going to do is I am going to now split it up. I am going to split it up and you will see that this can be written in this format: r is equal to summation over all (b_i transpose v_i). What is this now (Refer Slide Time: 33:54)? This is the deformation in each member. Let me see what this implies.

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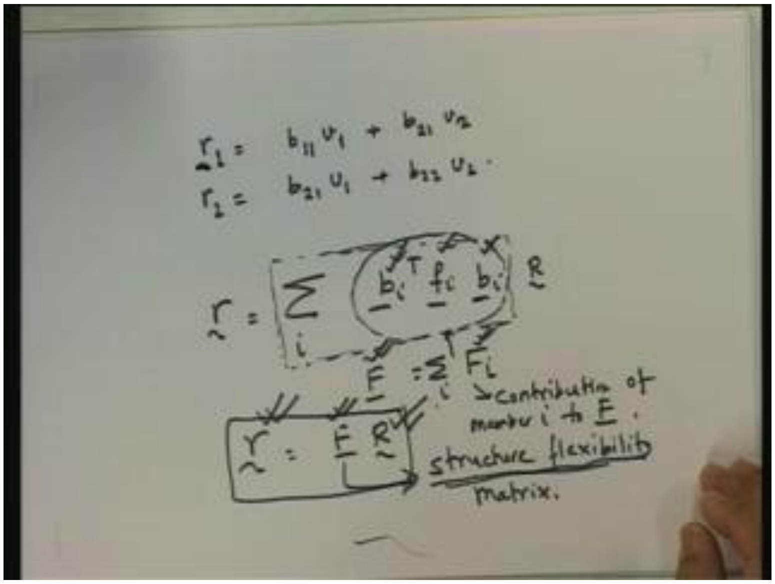

Here, it means that r_1 is equal to b_{11} b_i transpose; b_i transpose would be equal to…. What is b_i? b_i is b_{11}, b_{12}, so this is going to be equal to…. r_{11} is going to be equal to b_{11} v_1 plus b_{21} v_2 and r_2 is equal to b_{21} v_1 plus b_{22} v_2 – this is a summation (Refer Slide Time: 35:21). If I put this, let us see what happens here. What is v_i equal to? It is equal to f_i b_i into R (Refer Slide Time: 35:37). I am now going to plug that in and this r will then be equal to summation over all the members (b_i transpose f_i b_i) into R.

Note that R is outside, so actually, the thing that you have is this (Refer Slide Time: 36:09). Very interesting, very interesting. This part comes from equilibrium, this is S_i equal to b_i R, then f_i into S_i is v_i and then, from virtual work, we know that r is equal to b_i transpose v_i; therefore, ultimately, when you put it all together you get this. This means that this particular term is summed up over all i's. What is this term actually? What is this term totally? It relates the displacement to load. This represents something like that – if these were one, what would these be? These would be the displacements given unit load, so this matrix I can call it as F_i and summed up over all i is equal to F and I can write it in this fashion, where this is now the structure flexibility matrix and this is the contribution of member i to F.

Now, I can find out the displacements because I know this (Refer Slide Time: 38:15), I know this, I know this so I can find out this and summing it up over this, I can find this. Once I have this, note one thing: nowhere do I need to find out anything. Once I have found this out, given the load, I can always find out the displacements of the structure – this is the beauty of the matrix method. Once you go through the steps… the steps are identical to what we have done earlier using the virtual work principle, we have already discussed the virtual work principle; the only thing that I have done is explicitly write down the virtual work, first equilibrium… I have just related it in a general format and ultimately I have come up with an equation that directly relates the displacements and the loads through the structure flexibility matrix. This is the reason why this is known as the flexibility approach matrix method because ultimately you really are finding out the structural flexibility. And if you look, along the way you are actually finding out the member flexibility term.

This entire procedure is set up very very easily. Of course, what is the assumption? The assumption is that I have a statically determinate structure and of course, it is a truss-type structure because the only forces in the members that I have written till now are truss because axial force is only axial forces and axial deformations. Let me just illustrate this and then, I will show you next time that exactly this same thing can be written down even for a statically indeterminate structure. Let us see how we can find out the member forces by applying it to an actual example structure.

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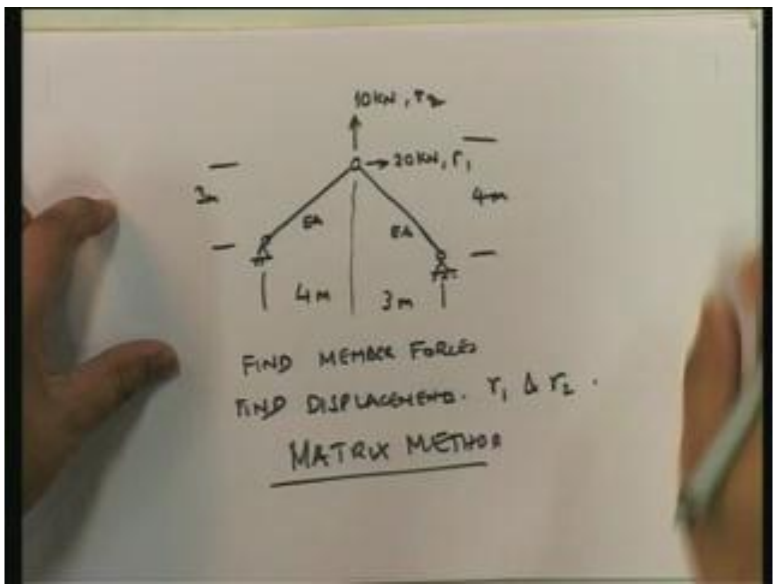

Let us have a situation. This here is 4 meters, this is 3 meters, this is 3 meters, this is 4 meters. Both of them have the same EA. Let me just put some loads here: 10 Kilonewton, let me apply 20 Kilonewton. The problem statement is to find the member forces and find displacements r_1 and r_2 –this is what the problem statement is. This is a statically determinate structure, so I can actually find out the member forces directly, so let me find out the member forces. The only difference is I am not going to find out the member forces directly for the given load – I am going to be using the matrix method.

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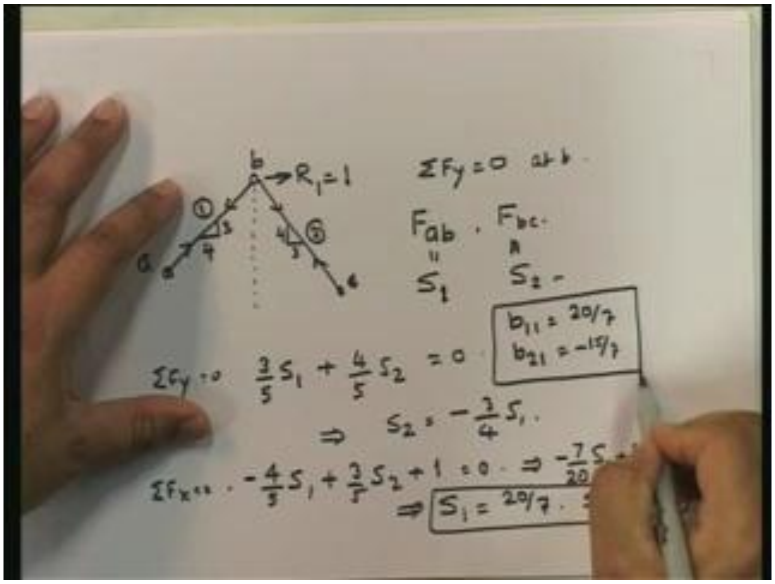

If I use the matrix method, then I do not solve the structure for the loads directly. What I do is I put R_1 is equal to 1 and find out the member forces. Let us see what I get. This is a 4, 3 slope and this is a 4, 3 slope this way. Let me call this a, b, c. If I take sigma F_y is equal to 0 at joint b, you will see that F_{ab} which is my member 1 and bc is member 2 F_{ab} and F_{bc}… I will call this in my notation as S_1 and this as S_2. These are the forces in the members and note that member forces are positive when you have tensile forces – that is the assumption that we make and we are going to continue making that (positive tensile forces – negative…).

Once I do that, let us see what happens. The vertical component of S_1 is going to be 3 by 5 S_1. Since this is positive, this is positive, you will see that S_1 is downwards – the vertical component plus 4 by 5 S_2. Since there is no vertical force it has to be equal to 0. What does this give me? This gives me that S_2 is equal to minus 3 by 4 S_1. This is sigma F_y is equal to 0; sigma F_x is equal to 0 gives me that minus 4 by 5 S_1 (that is the horizontal component pulling in this direction) plus this direction 3 by 5 S_2 plus 1 is equal to 0, but note that S_2 is equal to 3 by 4, so this becomes 9 by 20 S_1, 9 by 20 minus 16 by 20 is going to be minus; this implies minus 7 by 20 S_1 plus 1 is equal to 0; this implies S_1 is equal to 20 by 7 and S_2 and this minus 3 by 4, this is going to be minus 15 by 7. Satisfy yourself that this indeed does satisfy: take 3 by 5 of 20 by 7 – it is going to be equal to 12 by 7; take this minus 12 by 7 is equal to 0; and here (Refer Slide Time: 47:55), take minus 4 by 5 – this is going to be equal to minus 16 by 7 plus 9 by 7 and that is going to be equal to minus 7 by 7 so that is minus 1 plus 1 is equal to 0. But note that this is S_1 due to R_1, so what is this? This I have found out that b_{11} is equal to 20 by 7 and b_{21} is equal to minus 15 by 7; these are what I have found out.

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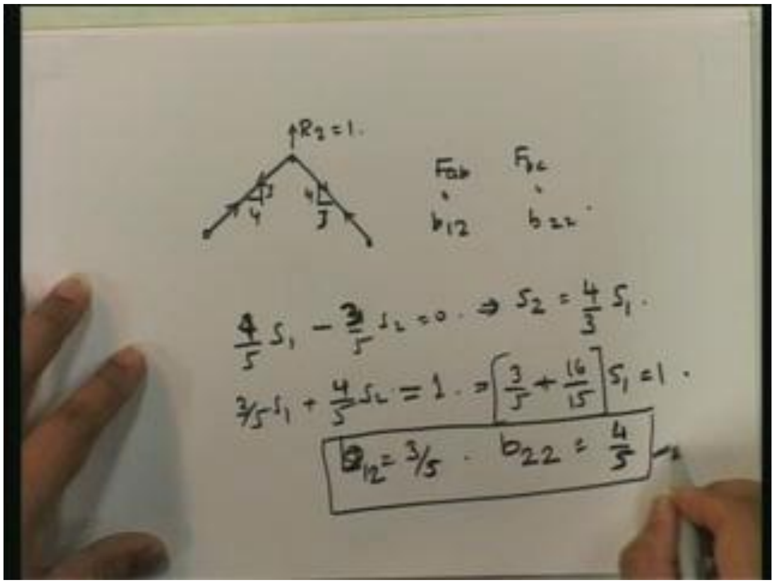

Let us move on to the other one, which is R_2 is equal to 1 and let us find out this. I am going to directly put that F_{ab} if I put R_2 is equal to 1 is actually b_{12} and F_{bc} is going to be equal to b_{22}. If I write it in this fashion, we will see that this is 3, 4 and this is 4, 3. sigma F_x is equal to 0 gives me that 4 by 5 S_1 (this is a horizontal component) minus 3 by 5 S_2 equal to 0. This implies that S_2 is equal to 4 by 3 S_1 – this is sigma F_x is equal to 0; sigma F_y is equal to 0 gives me 3 by 5 S_1 plus 4 by 5 S_2 is equal to 1. If I plug this in, I get… this is 4 by 3, so this becomes (3 upon 5 plus 16 upon 15) into S_1 is equal to 1. Ultimately if you look at it, if you put these through, 9 and 25, 25 upon 15 is 5 upon 3, so that means S_1 is equal to 3 by 5, that means b_{12} is equal to 3 by 5 and b_{22} is equal to… plug it in and you will see that this is equal to 4 by 5. The horizontal vertical component of this is going to be 12 by 5 and the horizontal component is 12 by 5 – they cancel each other out; the vertical component of this is 9 by 25, this is going to be 16 by 25, 9 plus 16 by 25 is 25 by 25 is 1, so this is okay. Ultimately, I have found out the b_1.

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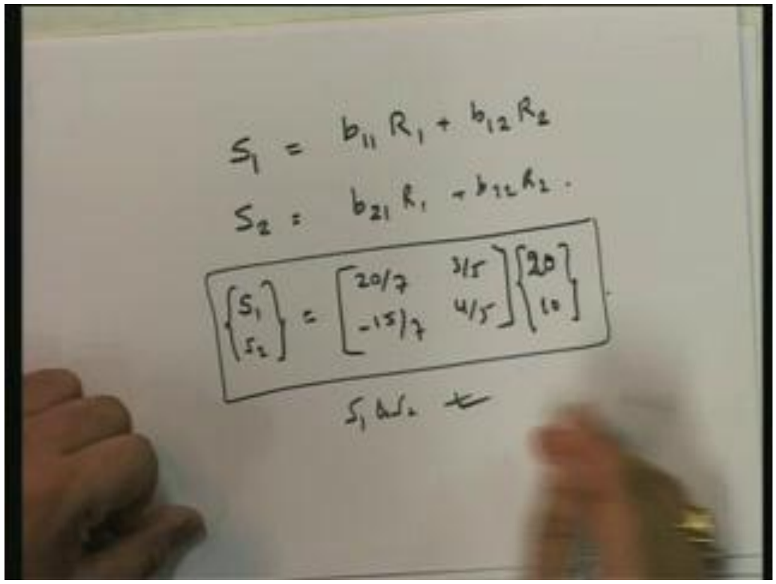

Therefore, now I can say that S_1 is going to be equal to b_{11} R_1 plus b_{12} R_2 and S_2 is going to be b_{21} R_1 plus b_{22} R_2. I know what R_1 and R_2 are, I am given R_1 and R_2 – R_1 is equal to 20 and R_2 is equal to 20. If I put in it, I can get my S_1 and S_2 as equal to 20 by 7, minus 15 by 7, 3 by 5, 4 by 5; this is 20 and this is 10 (Refer Slide Time: 53:08). From that, you can find out S_1 and S_2. The first part of the problem is done and the second part of the problem – I can actually find out; both the members are 5 meters long.

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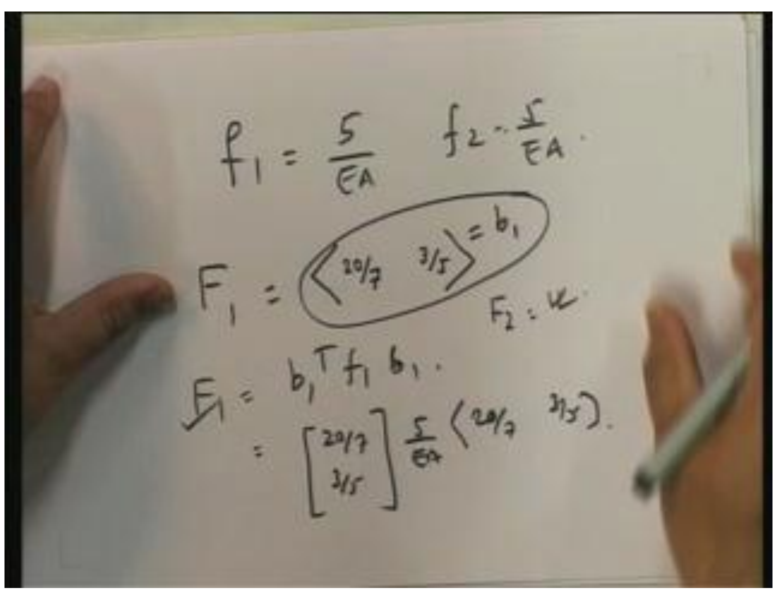

Therefore, f_1 is equal to 5 upon EA, f_2 is equal to 5 upon EA and therefore, the contribution of the first member is going to be equal to b_{i1} b_{12} so that is going to be equal to in this way 20 by 7, 3 by 5. What is this? This is equal to b_1 (Refer Slide Time: 54:26), so F_1 is equal to b_1 transpose f_1 b_1; this is going to be equal to 20 by 7, 3 by 5, 5 upon EA 20 by 7, 3 upon 5 and you can find out F_1. Similarly, you can find out f_2.

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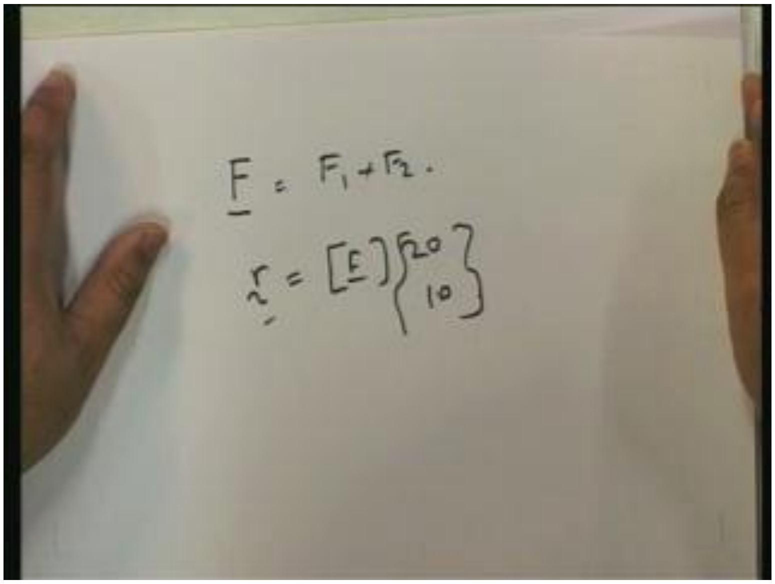

Once you find out F_1 and F_2, then you have F is equal to F_1 plus F_2 and finally r is equal to F into 20, 10 and you have got your displacement.

We will continue with this particular problem in the next lecture and then I am going to extend this to see how we can use the same procedure to solve statically indeterminate structures. We are going to be using the flexibility approach to solve statically indeterminate structures also. I hope I have been able to give you a brief background on the matrix method flexibility approach. You will see that there is nothing new in what I have talked; it is all exactly what I have talked before; it is just that I am using notation and putting it into a matrix format and then using matrix algebra. I suggest that if you are uncomfortable with matrix algebra, you go back to any standard matrix algebra book and study the concepts in the matrix algebra book. I am going to bring out a few points in the next lecture, which will show that for the kind of flexibility matrix that I have, there are specific forms of the flexibility matrix – more on that in the next lecture.
Thank You.