Structural Analysis II
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Lecture – 20

Good morning. In the last few lectures, we have been looking at the displacement method and in the last lecture I told you that I have done as much as I can do to explain the different nuances in the application of the displacement method. In this particular lecture, I am going to end my discourse on the displacement method by actually taking you through the last few points that I would like to make in terms of the lecture. That essentially boils down to how to treat…. Support settlement and support flexibility are not issues in the displacement method because when you have support displacement and support flexibility, all it does is that it introduces another degree of freedom. Remember what we had talked was the various things: one was member loading, other was support settlement, support flexibility and then, we had lack of fit – lack of fit was specifically in terms of trusses and since I have not yet dealt with truss, I shall do that in this particular lecture; the only other thing, if you remember, was temperature-related effects.

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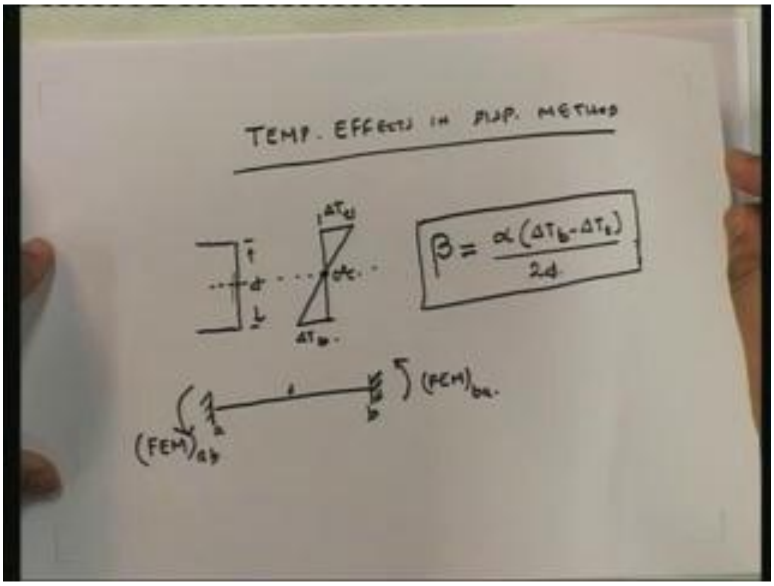

Today, I am going to start off by talking about temperature effects in the displacement method. If you are looking at a beam frame, we are only considering flexure; then we only consider the differential temperature. If you have a cross section in which this is the top and this is the bottom, this is your depth and this is your neutral axis. Then, essentially, what you are interested in is this variation. At the neutral axis, temperature is 0, this is the temperature difference – delta T; it is not as if at the center you have freezing, it is just that there is no rise of temperature at the neutral axis. Here, you have delta T at the top, you have delta T at the bottom, obviously for it to be 0…; this is only the differential part; in reality, you might have two parts to it; I am only looking at the differential part because that is the one that causes flexure.

If you remember, I had actually computed earlier that the curvature in the cross section due to delta T is given by (alpha into (delta T_{bottom} minus delta T_{top})) upon 2 d, where d is the depth of the cross section at that particular point – this is the curvature. Now, remember that temperature is a member effect brought in through the fixed end moments. The point to note here is that if you look at the member force end, member deformation, M_{ab} is equal to 4EI upon L into theta$_{ab}$ (this theta$_{ab}$ is due to displacements) plus 2EI upon L into theta$_{ba}$ (again theta$_{ba}$ is due to displacements) plus fixed end moment at ab. The member load effect is only incorporated in the displacement method by introducing the concept of the fixed end moment; the fixed end moment is one in which theta$_{ab}$ and theta$_{ba}$ are put equal to 0 and you find out the member end moments due to the load only. Here, there is no load, there is a temperature. However, it is again a member effect.

Remember I talked about that? What are member effects? Member effect is lack of fit, temperature, member load – all these are effects at the member level; in the displacement method, member-level effects are only incorporated in finding out the fixed end forces. In this particular case, since we are looking at flexure, the fixed end forces are the fixed end moments. Ultimately, if we can find out the fixed end moment due to a particular temperature profile, we have solved the problem. I am going to solve it for the simple case. This is the curvature induced due to the fixed end moments (Refer Slide Time: 07:46). How do I find out the fixed end moments? The way we found out the fixed end moments was by going back to first principles.

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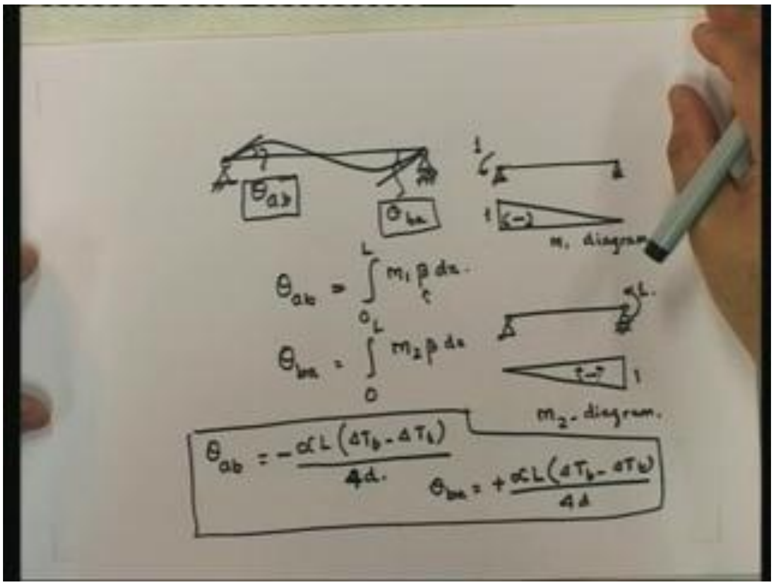

We found out the rotations, we found out the member end rotations due to the member effect? What is the member effect? Member effect is temperature. How do I compute these? Using the virtual force, theta$_{ab}$ is equal to 0 to L M_1 into curvature dx; theta$_{ba}$ is equal to 0 to L M_2 into curvature dx. What is M_1? M_1 is the bending moment in the member due to a unit virtual force corresponding to theta$_{ab}$ – that gives me the virtual displacement; this is my M_1 diagram. This is M_2 corresponding to theta$_{ba}$ (Refer Slide Time: 09:48) and this turns out to be my M_2 diagram. For the temperature effect, we can find out theta$_{ab}$ and theta$_{ba}$. I have already solved this earlier, but I am just going through the steps again just to refresh it.

Now, I am going to assume the situation that everywhere along the length of the member, the top fiber sees delta T_{top} and the bottom everywhere sees delta T_{bottom}. In other words, the temperature gradient is uniform along the entire length. I am actually solving for a specific case where I have uniform temperature gradient across the cross section over the entire length; in this particular case, beta turns out to a constant given by this; this is a constant, so I just need to integrate M_1 dx. Therefore, I get theta$_{ab}$ is equal to area under the curve – L by 2, so it is going to be (alpha into L into (delta T_{bottom} minus delta T_{top})) upon 4d; M_1 is just the opposite of this because this is the curvature to get positive curvature, so this is negative, so I am going to get theta$_{ab}$ is negative. Theta$_{ba}$ is plus alpha L into (delta T_{bottom} minus delta T_{top}) upon 4d, because this is the same as that. Therefore, these are my theta$_{ab}$ and theta$_{ba}$. How do I get fixed end moments from these particular equations? We had to apply the moment which would give me just the opposite of these theta$_{ab}$ so that the sum total of them gave me 0, so that I could get my fixed end moments. Without much ado, I am going to go into computation of the fixed end moments.

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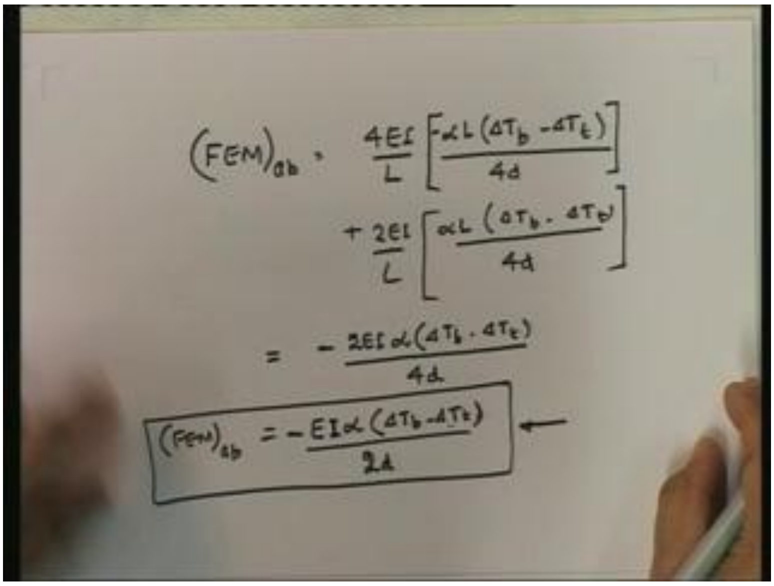

What are they? (Fixed end moment)$_{ab}$ is going to be equal to 4EI by L into ((minus alpha L into (delta T_{bottom} minus delta T_{top})) upon 4d) plus 2EI by L into ((alpha L into (delta T_{bottom} minus delta T_{top})) upon 4d) – this is equal to minus 4EI alpha plus 2EI alpha (Refer Slide Time: 14:18), this is going to be equal to minus (2EI alpha into (delta T_{bottom} minus delta T_{top})) upon 4d. This will be equal to minus EI alpha delta T_b minus delta T_t upon 2d – that is my fixed end moment at ab.

Let me check if I am dimensionally consistent. What are the units of alpha? alpha into delta T is going to be meter by meter, which is essentially dimensionless. What are the units of E? Newton per meter squared, this is meter fourth, so Newton per meter squared into meter fourth is going to be equal to Newton meter squared; Newton meter squared divided by d, which is meter, is going to become Newton meter and that is the fixed end moments; the units are consistent. This is the fixed end moment at ab (FEM)$_{ab}$.

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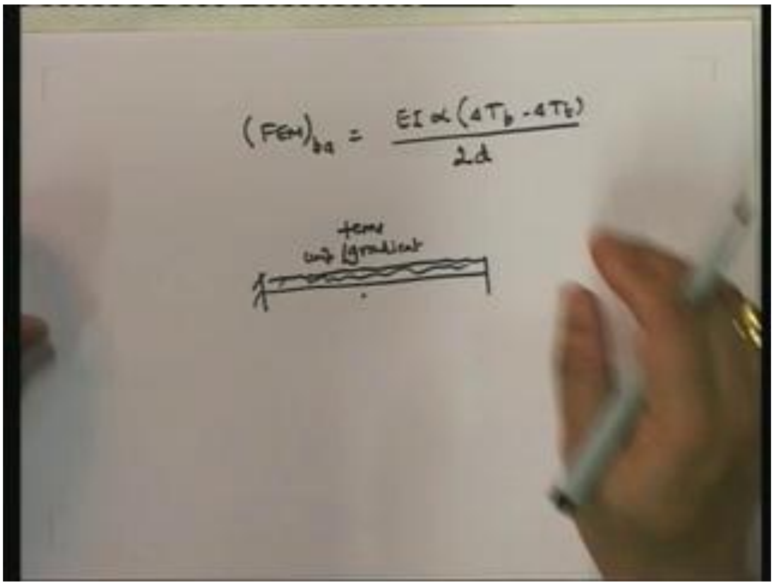

Similarly, you can actually get that the fixed end moment at ba is going to be equal to (EI alpha into (delta T_{bottom} minus delta T_{top})) upon 2d. Therefore, if I have a uniform temperature gradient with the bottom being hotter than the top, it expands in this way so that you get it to be equal to the opposite. I would just like to make a point here: remember that theta$_{ab}$ is equal to minus and theta$_{ba}$ is equal to plus.

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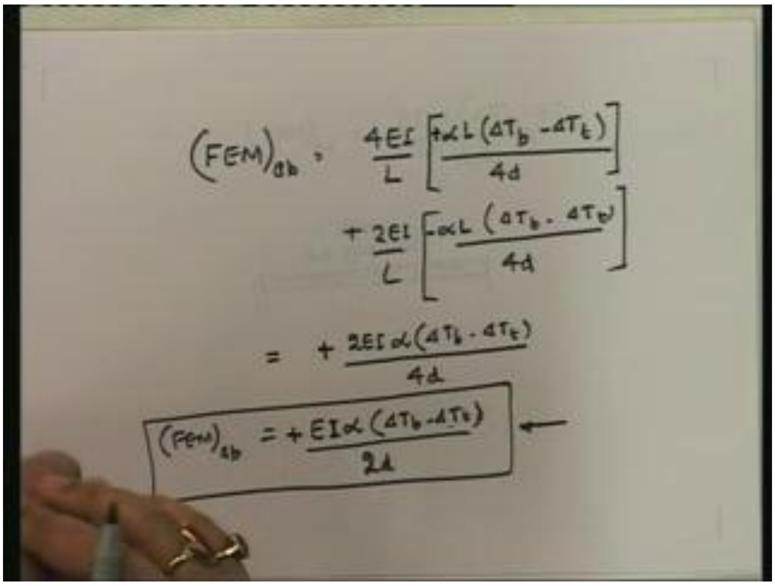

Therefore, when we substitute in here, this has to be plus and this has to be minus, therefore this becomes plus and this becomes plus. Remember that the rotation that we get is this plus this rotation has to be equal to 0. Therefore, when this is minus, this (Refer Slide Time: 18:24) plus this is going to give you 0; then this plus this is going to give you 0. Therefore, the fixed end moment here has to be positive.

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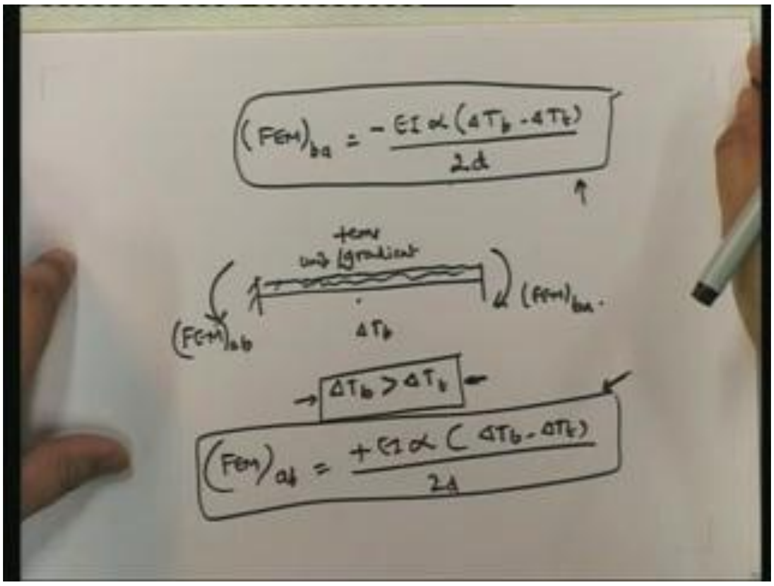

Similarly, the fixed end moment at this has to be negative. If we show it for uniform temperature gradient where delta T_b (the temperature differential at the bottom) is greater than delta T_t (the temperature differential at the top) and it is uniform across the entire length, then we get $(FEM)_{ab}$ and fixed end moment at ba equal to…. I will just write it down, I have got $(FEM)_{ba}$ over there. $(FEM)_{ab}$ is equal to this (Refer Slide Time: 19:35). Note that these two fixed end moments that I have written down over here are essentially for the case where the increase in temperature at the bottom is more than the increase of temperature at the top. Mostly, it would be increase of temperature in the bottom and a decrease of temperature at the top because you have to have neutral axis 0 for pure flexure.

Note also the fact that it is uniform over the entire length; in other words, the top fiber across the length sees delta T_{top} – that is a negative temperature; delta T_{bottom} is uniform across the length – that is a positive temperature to get the directions that you have. Of course, if delta T_b and delta T_t turn out to be different, you will get them to be minus. This gives you the expression. Now, if there is a different variations of temperature across the lengths, then of course you will not get these two, you will get some other expression, but the fact is that you should, by now, be totally certain about how to evaluate this effect.

Once we have the fixed end moments, there is no difference between the member loads or member temperature, excepting that once you have got the member end moments in temperature, you do not have additional bending moment in the member due to the temperature effect as you do for the load. In other words, the simply supported beam subject to temperature has 0 moments, whereas a simply supported beam subjected to a load will always have a bending moment diagram associated with it, which has to be superposed on; in the temperature, you do not have the superposition. Otherwise, once you have calculated the fixed end moments, you use the same principle that we have already; there is no difference when you have to consider the temperature effect.

We are now done with considering the temperature effect and how to incorporate it into the displacement method. I am now going to move on to look at the displacement method for a truss. How do we apply the displacement method? The reason why I am looking at this is that the displacement method is exactly the same excepting for the fact that in a flexural member, the member end force deformation relationship essentially related the moments with the rotations. In this particular case, the force deformation relationship in a truss, it only has axial forces and axial deformations, so the member end force deformation relationship should essentially relate the axial force and the deformation in the member. Let us look at how to tackle that.

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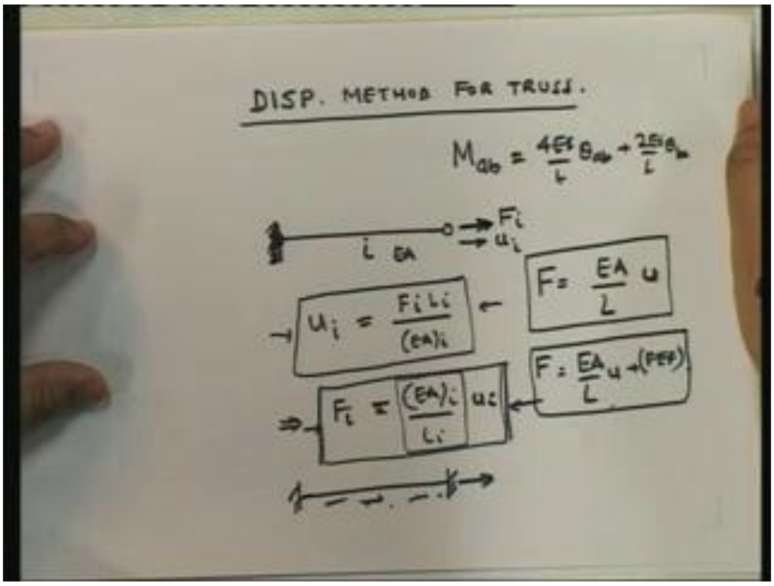

This is my truss, I am going to consider it to be fixed over here (Refer Slide Time: 23:43). Just like I considered a simply supported … which was the simplest statically determinate flexural member, I am going to consider the simplest statically determinate axial member, truss member. Here, I have load F and due to this, I have a displacement which is u. Since this is member i, I will call this force i and member i. How do I determine this relationship? I know that u_i is equal to F_i L_i upon $(EA)_i$ – this is assuming that EA is a constant and only the load is applied at the member end; I know this, I have already evaluated this earlier.

That means my force deformation relationship will be F_i is equal to $(EA)_i$ upon L_i into u_i – this is my member end force, this is my member end displacement and this is (Refer Slide Time: 25:13)…. Look at the difference between the member end for flexure: in flexure, you have two rotations and two moments to define the member end forces; for the truss, you only have a single force and a single displacement; and this is the force deformation relationship for a member. By and large, in trusses, you do not have forces acting in the center; you normally always have member end moments and that is the reason why the force deformation relationship is essentially this for a member; so I can write it as F is equal to EA by L into u.

What was it for a flexure? It was M_{ab} is equal to 4EI upon L theta$_{ab}$ plus 2EI by L theta$_{ba}$. Of course, in a flexural member, you may have member loads and that is why you have the fixed end moments etc. In this case, you normally do not have that and therefore you do not have this aspect. Of course, you might have a temperature problem and therefore, this can be written generally as F is equal to EA by L u plus fixed end force. Which is the fixed end force? This is the force developed here due to a member effect – we will see that later.

Let us look at this moment (Refer Slide Time: 27:20). EI by L is Newton meter, so the unit here is Newton meter – it is moment per unit rotation, so this unit is Newton meter per radian and it is like a torsional coefficient, torsional stiffness constant. Here, EA by L is Newton per meter – something like a stiffness constant or spring constant, so this is similar to that, it is force per unit displacement. Once we have this, how do we tackle a particular problem? Let me take a specific problem here and see how to solve it. I will take a simple problem here because you can actually take it for any kind of thing. I just want to reduce the number of degrees of freedom and that is why I am considering this effect.

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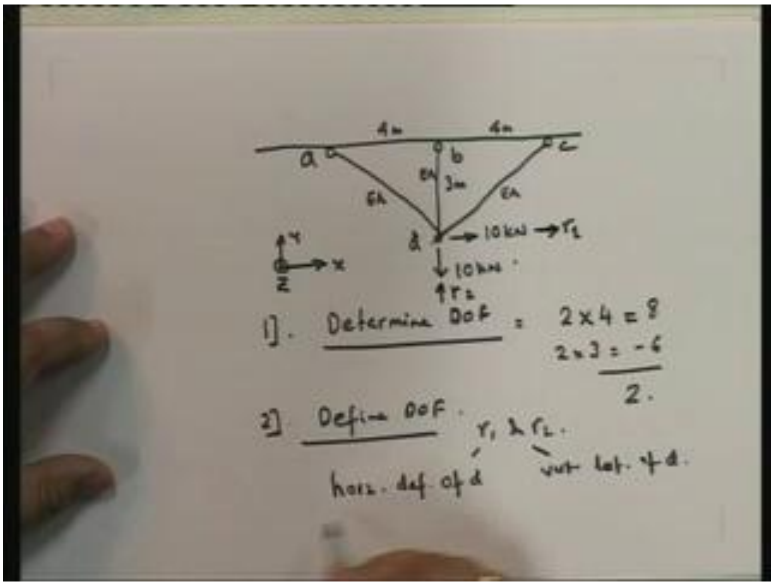

The reason behind why I have chosen this is in the displacement method, what do you start off with? You start off with defining the degrees of freedom, so this is essentially all truss members; although this does not look like a truss that you would have normally, the reason why I am taking this is because it is simple; I will take up another thing also in which we will see how it works.

This is a statically indeterminate truss. You cannot find out the forces in each of the members given this load. How do you use the displacement method? Let us go through the steps. First, determine the degrees of freedom. I have a, b, c and d. Remember that when you have a truss member, you do not have the rotation, so you only have two displacements and so the unconstrained degrees of freedom are 2 into 4, so 8. There are three constraints, each one has two constraints, so constraints are equal to six, so there are two degrees of freedom (Refer Slide Time: 30:03). What are those two degrees of freedom? I will define them as r_1 and r_2. Note that I always define my displacements along the positive. This is my coordinate system for the structure, so my degrees of freedom are positive in the positive direction; so r_1 and r_2 – these are my two degrees of freedom. Two: Define the degrees of freedom – r_1 and r_2; r_1 is the horizontal deflection of d and r_2 is the vertical deflection of d; we have defined that. The third step is defining the member end force deformation relationship.

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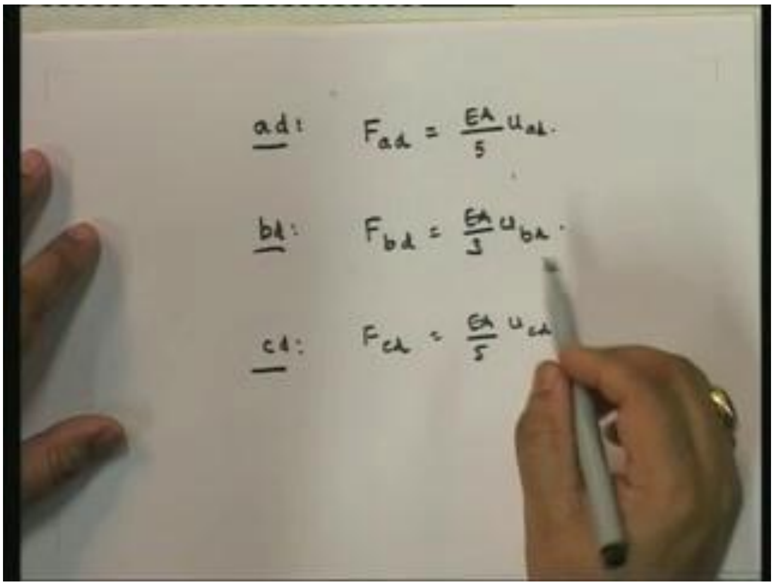

How many members do I have? I have ad, I have bd and I have cd. For ad, what is the member end deformation relationship? F_{ad} is equal to EA upon L; L is equal to 5, so EA upon 5 into u_{ad}; then F_{bd} is equal to EA by 3 into u_{bd}; F_{cd} is equal to EA by 5 u_{cd} – simple relationships but we have written the member end force deformation relationships. I am actually going through the displacement method as I have defined it. The next step is the kinematics. What do I need to do? In terms of r_1 and r_2, I need to find out u_{ab}, u_{bd}, u_{cd}. What I need to do is put….

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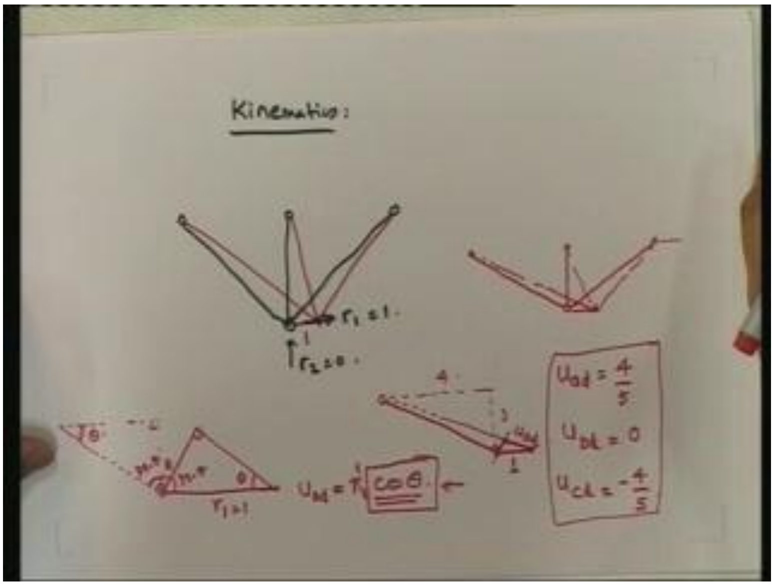

Next is the kinematics. How did we do it? Put r_1 is equal to 1 and r_2 is equal to 0 and look at the displaced shape. When I do this, what happens? This becomes 1**.** Note that since they are hinged, they go like this (Refer Slide Time: 33:25). Now, I need to find out elongation. How do I find out elongation? I drop a vertical along this direction. I will show this a little bit better because it gets a little bit complicated, so let me draw it properly; otherwise, we will have a problem. Since this point has come here and these points remain where they are, your new lengths are these; I need to find out what the increase in length is. Let me take ad. It is given in this form where this is 3 and this is 4; this point has gone here (Refer Slide Time: 34:40) and I am going to expand it by 1. This is my new shape and I need to find out the length. To find out the length, what do I do? I drop a 90 degree here because in small displacements if it goes perpendicular, then there is no change in length; so all I need to do is now find out, if I have dropped a perpendicular here, how much this is and this is going to be my u_{ad} (Refer Slide Time: 35:18). How would I find that out? Let us draw it.

If you look at this and look at this (Refer Slide Time: 35:50), I say this makes an angle theta, where this is 90 degrees; since this is 90 degrees, since this is theta and this angle is 90 minus theta, what happens is that since this is 90 degrees, this angle becomes theta; and since this angle is theta, this angle is theta and this angle is 90 minus theta.

If you look at the change in length, the change in length is given by u_{ad}, which is given as r_1 into cosine of theta; if r_1 is equal to 1, this is 1 and this is cosine theta – u_{ad} is cosine theta. In this way, I can find out that whatever angle that it makes with the horizontal, the u_i of that is given by cosine theta. In this particular case, it is going to be that u_{ad} is equal to cosine of theta, which is 4 by 5 – positive 4 by 5. What about u_{bd}? What is this theta? It is 90 degrees. What is cosine of theta? 0. It makes sense, right? If it moves perpendicular to itself, the change of length is 0. What about u_{cd}? It is going to be equal to the angle made with the angle. It is going to be shortening and that is minus 4 by 5 these are the displacements given in terms of cosine theta. Similarly, you will see that corresponding to r_2 … in other words, the kinematics for trusses is actually very simple; it does not get very complicated.

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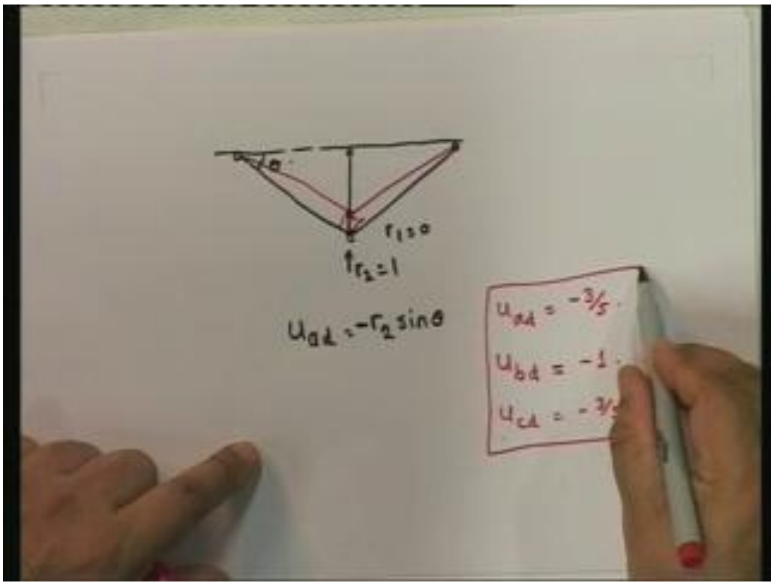

For trusses, you can actually write it down. In this particular case, this is my structure and I give r_2 is equal to 1 and $r_1$is equal to 0. If you put that, what happens? This will go this way and the displacement pattern will be this (Refer Slide Time: 39:53) and this; and all we need to do is find out how much of shortening you have. The shortening or lengthening will depend on… again, perpendicular from this. I leave that up to you: you will see that u_{ad} is going to be equal to minus r_2 sine theta, where theta is this angle and this happens to be minus because it is shortening. Same thing for all of them; in this particular case, you will see u_{ad} will be equal to minus 3 by 5 and here, when you do sine theta for the vertical member, sine 90 is 1, you will see that u_{bd} is equal to minus 1 (that is true because here, whatever is the displacement is directly the shortening of the vertical member); u_{cd} is going to be equal to minus 3 by 5. We are done with the kinematics now. If you plug this into the kinematics and into the equations, what do you get?

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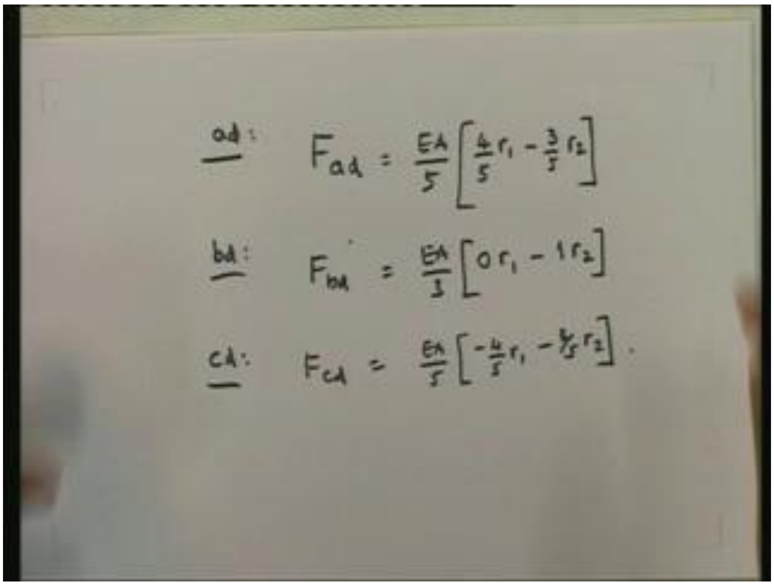

For ad, my F_{ad} becomes equal to EA by 5 and we have 4 by 5 r_1 minus 3 by 5 r_2 – all I have done is I have substituted u_{ad} in terms of r_1 and r_2; I have got the influence coefficients for r_1 and r_2 from the kinematics and that is all I do. Similarly, for bd, you have F_{bd} equal to EA by 3 into (0 r_1 minus 1 r_2) and for cd, F_{cd} is equal to EA by 5 into minus (4 by 5 r_1 minus 3 by 5 r_2). These three expressions give me the forces in the members ad, bd and cd in terms of the displacements corresponding to the degrees of freedom. Once I have those, what is the next step? I am following exactly the same steps as I did for the flexure; only thing that I have in this particular case is that I have instead of flexural equations, the truss equations.

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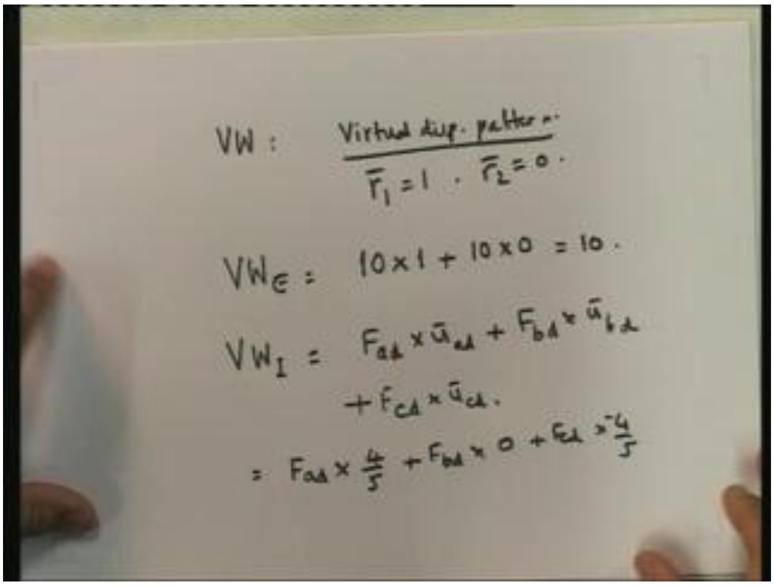

The next step is virtual work. Since I have two degrees of freedom, I have to write two independent equations. My first equation is my virtual displacement pattern: r_1 is equal to 1, r_2 is equal to 0 and I find out the external work done. What is the external work done? Let us look at what the force was. What were the forces? There was a horizontal force of 10 and a vertical downward force of 10, so external virtual work is going to be 10 into 1 plus 10 into 0, because r_2 is equal to 0; so this is going to be equal to 10. What about the internal virtual work? The internal virtual work is going to be F_{ad} into u_{ad} plus F_{bd} into u_{bd} plus F_{cd} into u_{cd} – how are these are related to r_1? I already have the influence coefficients, so this is going to give me F_{ad} into plus 4 by 5 plus F_{bd} into 0 plus F_{cd} into minus 4 by 5. What I get is the following. I am going to substitute F_{ad}, F_{bd} and F_{cd} in.

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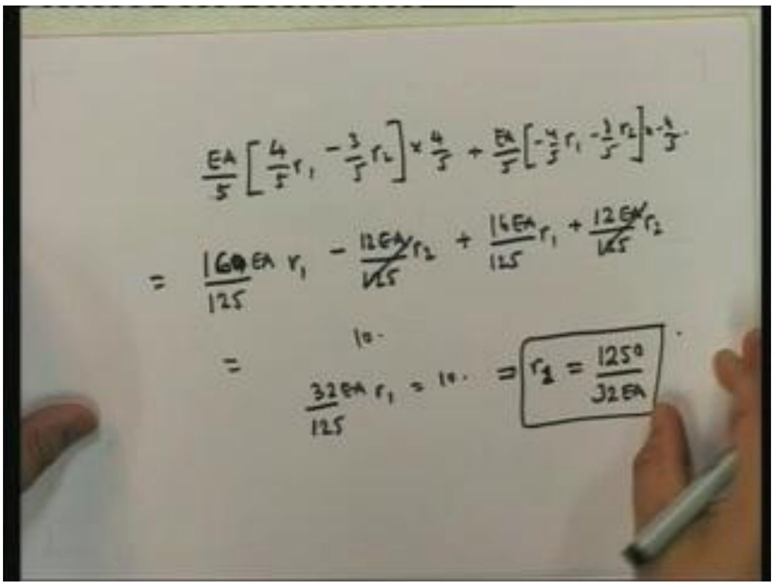

I am going to get (EA by 5 into (4 by 5 r_1 minus 3 by 5 r_2) multiplied by 4 by 5) plus (EA by 5 into (minus 4 by 5 r_1 minus 3 by 5 r_2) into minus 4 by 5). Taking 4 into 4, 5 into 5 into 5, this is going to be 16 125 upon EA into r_1 minus 12EA upon 125 r_2 plus 16EA upon 125 r_1 and this one (Refer Slide Time: 47:33), minus, minus gets plus, it becomes plus 12EA by 125 r_2, which is equal to … this cancels this (Refer Slide Time: 47:50), this becomes 32 upon 125EA r_1 is equal to 10. In this particular case, we actually get an uncoupling; normally, you would not get an uncoupling. r_1 is equal to this.

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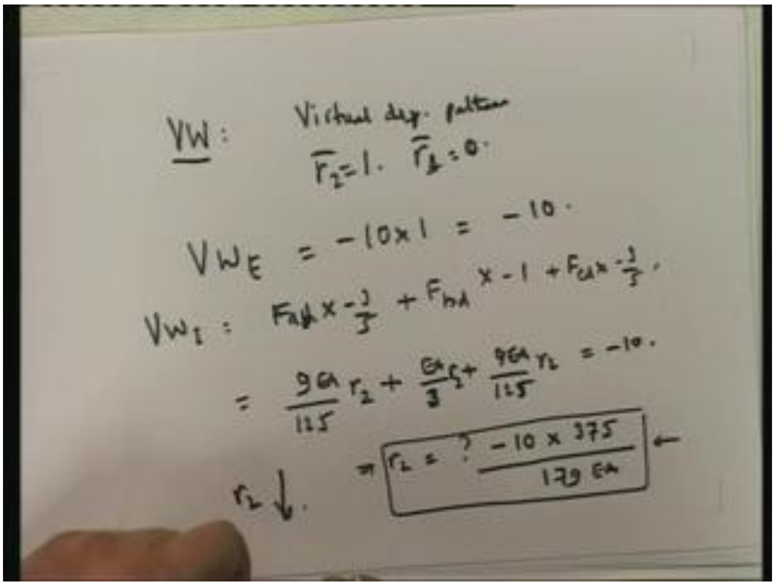

The other equation is going to give me…. The virtual displacement pattern is r_2 is equal to 1 and r_1 is equal to 0 independent. In this particular case, the external work done is minus 10 into 1, so this is going to be minus 10. Internal virtual work is going to be equal to F_{ad} multiplied by r_2, which is minus 3 by 5, plus F_{bd} multiplied by minus 1 plus F_{cd} multiplied by 3. We will see that the positive and the negative will cancel out; what we will be left with is 9 EA by 125 r_2 plus EA by 3 plus EA by 3 r_2 plus 9EA by 125 r_2 is equal to minus 10 and so this is going to give me r_2.

What is r_2 going to be equal to? Let us take stock of that. This is going to be equal to minus 10 multiplied by 375, this is going to be 125 into 3, 375, so this is going to become 27 plus 27, 54; 54 plus 125 is equal to 179EA, so 179EQ. This is my r_2 (Refer Slide Time: 50:58). This r_2 is negative. Why? Because r_2 is taken to be positive upwards and so all that means is r_1 is positive, it is to the right and r_2 is negative, which essentially means that r_2 is downwards – this is how it should be because the way we have defined r_1 and r_2 is positive to the right: r_1 is positive and r_2 is upwards positive, so obviously under the loading, you will see that r_2 will go down and r_1 will go up. Now, how do I find out these values?

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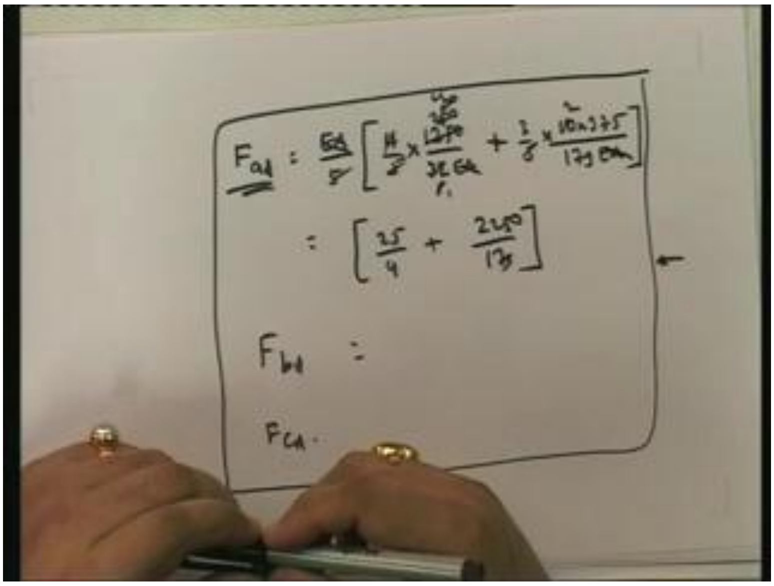

Once you know r_1 and r_2, you can substitute and you can get F_{ad} is equal to EA by 5 into (4 by 5 (which is r_1) into 1250 upon 32EA) plus (r_2 is minus, so this becomes plus) (3 by 5 into 10 into 375 upon 179EA). EA, EA cancels, EA cancels here; you get 5, 250, 50, 25 by 4, this becomes 25 by 4 plus goes 2, so this is 6 into 375, this is 1125 into 2 is 2250, 2250 upon 179. This is my F_{ad} and in exactly the same way, I can find out F_{ba} and F_{cd}. Once I know these, I know my member forces and I have analyzed the structure – I have found out the displacements and I have found out the member forces.

In trusses, it is relatively easier because the member force deformation relationship is simple, the kinematics is very simple – the kinematics essentially depends on sine theta, cosine theta, so kinematics is simple. Therefore, actually, the displacement method is very simple for trusses but the only thing is that for regular trusses, the number of degrees of freedom is so large that you cannot do a hand computation – that is the reason why I have taken a simple example with axially loaded members to illustrate the concept. You will have to use computers to be able to solve for larger trusses.

However, the procedure still remains the same: determine the number of degrees of freedom, define the degrees of freedom, define the member force deformation relationship, do the kinematics, do the virtual work and then solve for displacements, incorporate the displacements and get the member end forces, member forces – that is all there is to the displacement method. Thank you very much. I hope at the end of this lecture, series of lectures, you are now comfortable with now applying the force method and the displacement method for obtaining forces and displacements in both plane trusses and plane frames.
Thank you very much.