Structural Analysis II
Prof. P. Banerjee
Department of Civil Engineering
Indian Institute of Technology, Bombay
Lecture – 16

Today, we are going to continue looking at the displacement method, which we have been spending the last few lectures looking at. Today, I am going to introduce a new concept that will lead to a simplification of applying the displacement method, essentially because it eliminates a few degrees of freedom which you may need to consider otherwise. Remember that everything is essentially member force deformation relationships and that is what I am going to be looking at right now.

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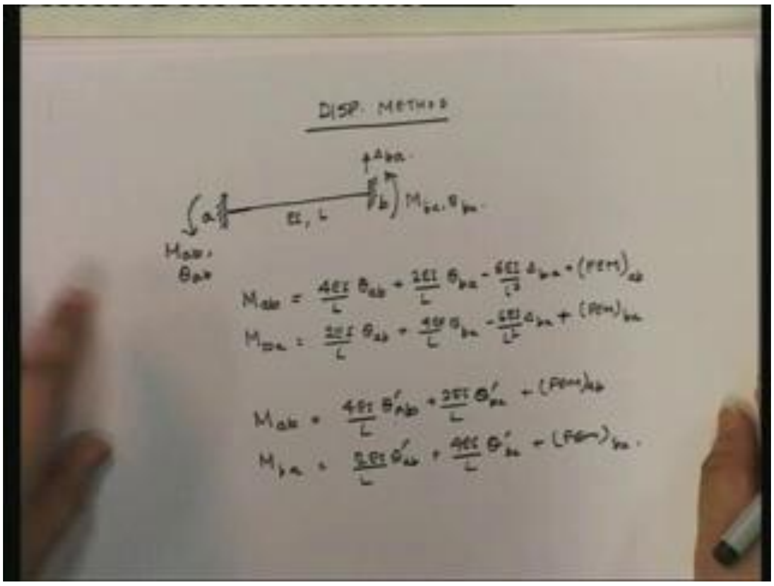

So if you look at the displacement method again, just to lay the foundation and in the displacement method for the member force deformation relationship, this is our structure and we say that this is M_{ab}, theta$_{ab}$ and then, we have M_{ba}, theta$_{ba}$ and there is also a displacement delta$_{ba}$. Remember I am just writing down the original slope deflection equations from last time and they come in this fashion. M_{ab} is equal to 4EI by L theta$_{ab}$ plus 2EI by L theta$_{ba}$ minus (6EI by L squared delta$_{ba}$) plus $(FEM)_{ab}$, where EI is the constant flexural rigidity and L is the length; M_{ba} is equal to 2EI by L theta$_{ab}$ plus 4EI by L theta$_{ba}$ minus (6EI by L squared delta$_{ba}$) plus $(FEM)_{ba}$. theta$_{ab}$ and theta$_{ba}$ are the rotation of the tangent from the original tangent.

Remember that from the original undisplaced tangent to the displaced tangent are the definitions and delta$_{ba}$ is the deflection of b. This is a, this is b, deflection of b is relative to a, upwards of b relative to a; theta$_{ab}$ and theta$_{ba}$ are positive anticlockwise and the fixed end moments and moments are positive anticlockwise. This is the original definition and in the last two lectures, I introduced you to the concept of M_{ab} is equal to 4EI upon L theta$_{ab}$ plus 2EI upon L theta$_{ba}$ plus $(FEM)_{ab}$; M_{ba} is equal to 2EI upon L theta$_{ab}$ plus 4EI by L theta$_{ba}$ plus fixed end moment at ba.

Here, what are the definitions of $theta_{ab}$ and $theta_{ba}$? These here (Refer Slide Time: 06:03) are defined from the chord to the displaced tangent; this chord is the straight line joining a and b (that is the chord) from the chord to the displaced tangent – that is $theta_{ba}$. Here, maybe I will call it theta this way just to distinguish from these (Refer Slide Time: 06:26), where these are from the chord. The fixed end moment here and here remain the same. These are the two definitions that I have looked at where it is essentially the fixed, where you have both M_{ab} and M_{ba}. Today, I am going to introduce to you a different member. This is based on fixed fixed member and this is the force deformation force or member rotation relationship for a fixed fixed beam. Today, I am going to introduce you to a different, I mean, a beam with different end conditions.

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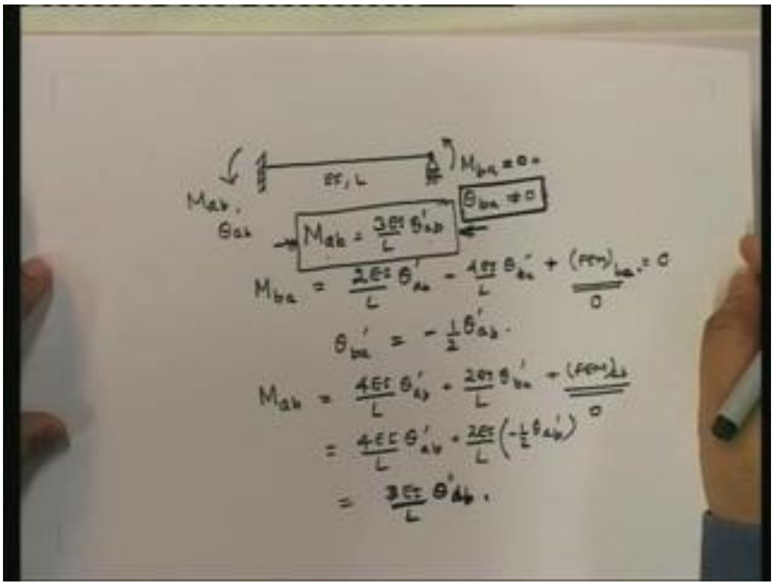

This is the beam that I am going to introduce you to. This is EI upon L (Refer Slide Time: 07:32) and note over here that the difference between the original fixed fixed beam and this fixed hinge beam (this is a hinge here) is the fact that you do not have.… You know that M_{ba} is equal to 0. Note that $theta_{ba}$ is not equal to 0. Rotation can occur, but M_{ba} cannot be non-zero, because at a hinge, this is the n hinge and at the n hinge, you know that M_{ba} is equal to 0. Today, I am going to introduce a different concept. This is M_{ab}, $theta_{ab}$. Note that over here, what I have to understand is what happens, what is the additional M_{ba} has to be equal to 0?

In other words, what does that mean? It means that M_{ba} is equal to 2EI by L $theta_{ab}$ (I am still using the definition of from the chord) plus 4EI by L $theta_{ba}$ plus fixed end moment at ba is equal to 0. I will come to the fixed end moment later on but for now, let us just say that this is 0 for now, let us say it is 0 (it is normally not 0, but for now, let us call it as 0). Then, what do you have? You have 2EI upon L $theta_{ab}$ plus 4EI upon L $theta_{ba}$ is equal to 0. What does that mean? It means that $theta_{ba}$ prime is equal to minus (2EI upon L divided by 4EI upon L, so that is half $theta_{ab}$ prime). Do you acknowledge that? This is just by putting it equal to zero; I am assuming this fixed end moment equal to zero. $theta_{ba}$ is equal to minus of … and now, I substitute…. Therefore, if M_{ba} has to be equal to 0, $theta_{ba}$ has to be equal to minus half of $theta_{ab}$.

Let us see what happens to M_{ab}. M_{ab} which is equal to 4EI upon L theta$_{ab}$ prime plus 2EI upon L theta$_{ba}$ prime plus $(FEM)_{ab}$, which for now I am considering…. In other words, I am considering the situation where there is no member load. What is this equal to? Here, I substitute this here (Refer Slide Time: 11:12), so this becomes 4EI upon L theta$_{ab}$ prime plus 2EI upon L and this is equal to this. If you look at this, this becomes equal to 3EI upon L theta$_{ab}$ prime.

In other words, what happens is, if you look at it, M_{ba} is equal to zero and M_{ab} is equal to 3EI upon L theta$_{ab}$ prime. In other words, the only relationship that we are left with in this particular situation is the force deformation relationship; it is M_{ab} is equal to 3EI upon L theta$_{ab}$ prime because M_{ba} is equal to 0. What is theta$_{ba}$ equal to? It does not really matter in this particular case, because remember the only reason why we are finding out rotations and displacements is essentially to find out the moments; and what happens here is, since you know that the moment over here (Refer Slide Time: 12:41) is 0, you really do not need to find out this at all; you do not need to find it out. However, if you are really interested in finding it out, since this is equal to 0, you can always find out what theta$_{ab}$ is equal to from this formulation. For a fixed hinged member, this is the force deformation relationship; this angle (Refer Slide Time: 13:23) is from the chord to the….

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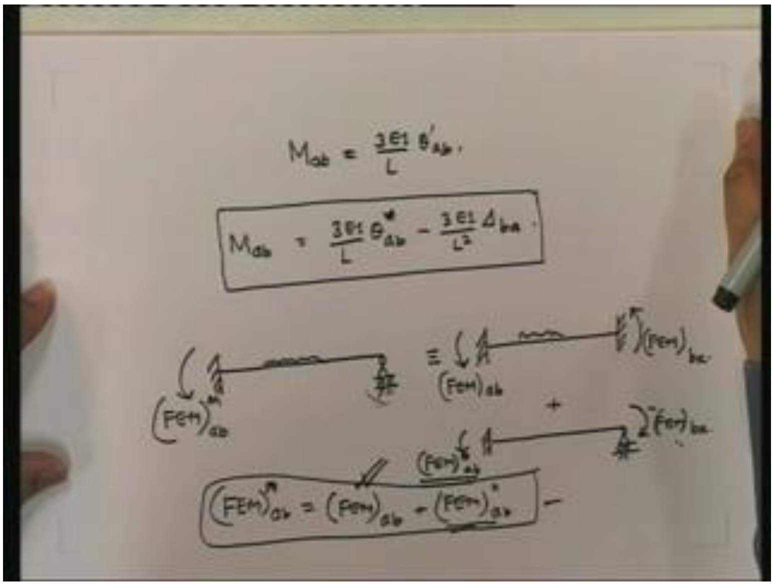

If theta is taken from the chord to the displaced tangent and if you want to define it in the original sense, then you can show…. I am not going into the details here but exactly in the same way, you can show that this is the original definition. This is from the original tangent to the displaced tangent (the rotation is the angle from the original tangent to the displaced tangent) and delta$_{ba}$ is the displacement upwards of b relative to a. This is it. Now, what about the fixed end moment? Up till now, we have not considered the fixed end moment. The advantage of this is that I can consider the fixed end moments separately and add it because it is after all superposition, so let us look at what happens. To consider the fixed end moment, it is very easy. This is equal to this (Refer Slide Time: 15:16), where…. Note that for this, we know the fixed end moments, so it is $(FEM)_{ab}$, given any loading.

We know the fixed end moment (Refer Slide Time: 15:32) and we know ba. You know how to compute these also, I have shown it to you. This is here; this is the modified fixed end moment, so we can call it modified fixed end moment; this is equal to this plus this. What is the reason? Look at this. I am saying that this is the original one which we know how to compute and this plus this (Refer Slide Time: 16:52)…. What I am doing is, I am actually applying the negative of the fixed end moment. Why am I applying the negative of the fixed end moment? Because this plus this will ensure that this becomes equal to 0, there is no fixed end moment because there is no fixed end; and all I have to do is find out this, because then I know that the modified fixed end moment at ab is equal to $(FEM)_{ab}$ plus $(FEM)_{ab}$ prime, because the principle of superposition is valid. So, all I have to do is find out.… I know this, this I know from before (this, all I have to do is find out what this is, given this load. Let us see. If I apply a load….

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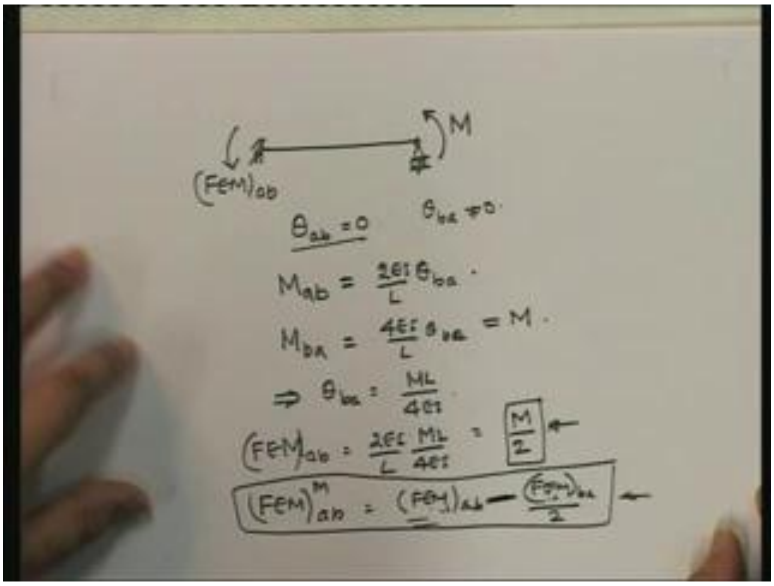

Let us go back to the original equation. Let me say that I am applying a moment here. We need to find out what is the fixed end moment at this. If I can do that, I have solved my problem. Let us see what happens. If you look at this particular situation, what do you have? You have theta$_{ab}$ is equal to 0 and theta$_{ba}$ is not equal to 0 – this is the condition. Here, what I am doing is, I am finding out M_{ab}. This is the only loading on the system, so M_{ab} is equal to 4EI into theta$_{ab}$, which is 0, so it going to be just 2EI upon L theta$_{ba}$. Do you agree to that? We do not know what theta$_{ab}$ is, but do you agree to this? Since there is no load, there cannot be any fixed end moment. What is M_{ba} equal to? M_{ba} is equal to 2EI by L into theta$_{ab}$, which is 0 into… plus 4EI upon L theta$_{ba}$ and since there are no moments applied, there is no fixed end moment. What is M_{ba} equal to? By definition, we know M_{ba} is equal to M, which implies that theta$_{ba}$ is equal to M L upon 4EI.

All I need to do is substitute that into this and what do I get? 2EI upon L into M by 4EI, which if you notice is equal to M by 2. In other words, what is this M_{ab}? M_{ab} is the fixed end moment since theta$_{ab}$ is equal to 0; that means if I apply a load M here (Refer Slide Time: 20:43), the fixed end moment is M by2. I have derived it using the equations and since fixed end moment is equal to a, since I am applying minus fixed end moment at ba, what is this fixed end moment equal to? It is going to be equal to minus fixed end moment at ba upon 2.

What does that become? Ultimately, the fixed end moment modified at ab is equal to fixed end moment at ab due to the fixed fixed condition minus ($(FEM)_{ba}$ upon 2), where these are obtained using the using the fixed fixed. We know how to obtain this; let us apply it to a particular equation and see whether we get it from first principles also.

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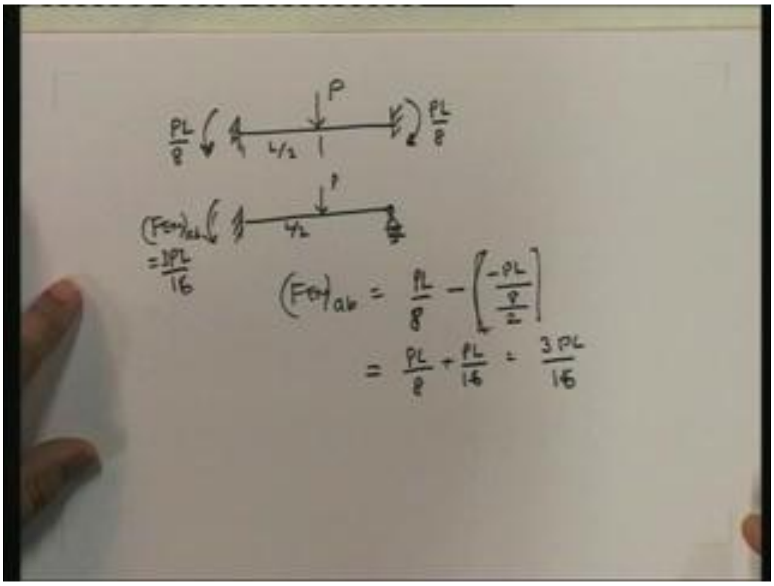

I am taking an example of load P being applied at L by 2 (Refer Slide Time: 22:24) and what are the fixed end moments? We can derive them but we already know what the fixed end moments are. These are equal to PL upon 8, PL upon 8 this way. If I were to find out for this situation, load applied at the center, what is this fixed end moment equal to for the modified beam? The fixed end moment is equal to PL by 8 (the fixed end moment at ab) minus PL upon 8 divided by 2 (note that the fixed end moment here is negative because it is clockwise). This becomes PL upon 8 plus PL by 16 and this becomes 3 PL upon 16; so, the fixed end moment here is equal to 3 PL upon 16 (Refer Slide Time: 23:55) when this end is hinged. Now I can derive this from first principles. How will I derive it from first principles? Let us think back. I want to satisfy you that whatever we have done, we should be able to obtain it for this also from first principles. Let us see what happens when I apply first principles.

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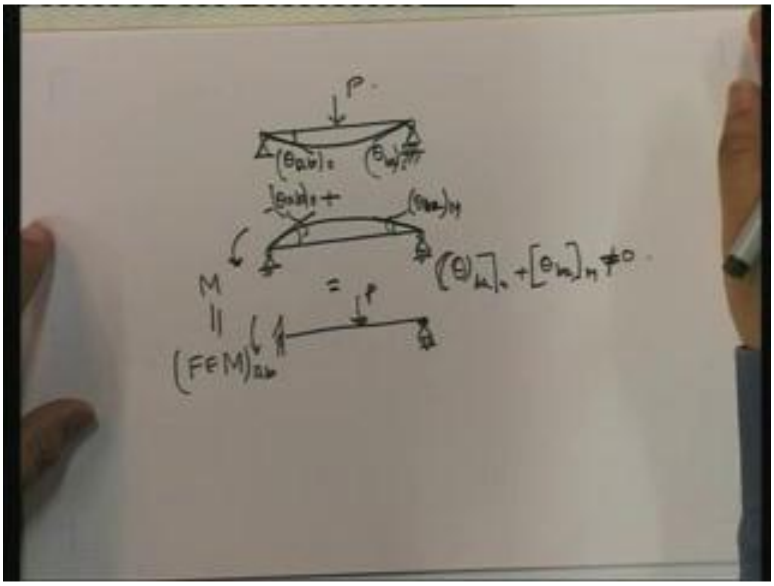

How do I go from first principles? I consider this, apply the load here and under this load, what is going to happen is that it is going to go like this. What I am saying is that this plus… a moment applied here such that if this is $(\text{theta}_{ab})_0$, I will get minus $(\text{theta}_{ab})_0$. It is going to be equal to this. If this is the case, then this (Refer Slide Time: 25:46) is equal to $(\text{FEM})_{ab}$. We have already looked at this; the only difference between the original and this one is that since this is the only one fixed, we do not need to satisfy this $(\text{theta}_{ba})_0$, and this can be theta_{ba} due to M.

Nowhere does $(\text{theta}_{ba})_0$ plus theta_{ba} due to the moment have to be equal to 0, no. Why? Because this is a hinge, the rotation can go anything. The only thing is we cannot apply a moment here and I am not applying a moment anywhere. Note that I am only applying the moment here (Refer Slide Time: 26:34); since this is a hinge, its moment is 0 and this is a hinge, its moment is 0. Since I am not applying a moment, this plus this is not going to give me a moment here and this plus this is not equal to 0, because there can be rotation; if rotation is allowed, it cannot take a moment; so you see the difference between the original one and this one. Let us see what happens here, let us go through the steps.

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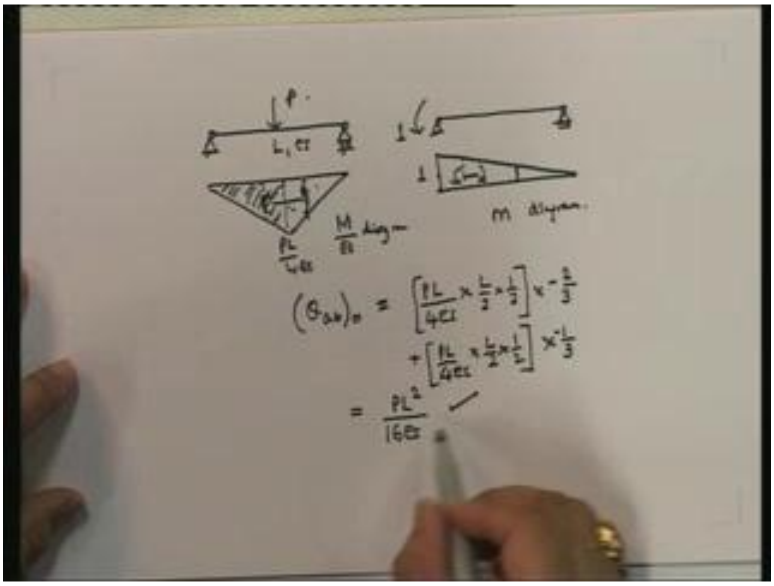

Let me apply the load; due to the load, the bending moment is PL by 4 and it is sagging. Now, I need to find out $theta_{ab}$ due to the loading. How do I do that? L is this. Virtual force I need to find out this, so I apply a unit virtual force and I find out the moment diagram. The moment diagram looks like this (Refer Slide Time: 28:05), this is my small m diagram, this is my M diagram. Since EI is a constant, this gives me directly the M by EI diagram and this gives me the small m diagram. What is $(theta_{ab})_0$ equal to? $(theta_{ab})_0$ is equal to the area under this curve – half; because there are two areas, I have to consider PL upon 4EI into L by 2 into 1 by 2, which is half length into base (that is the area of the triangle). Since this is this way, I am taking that as positive, this is going to be negative.

The value at this point (the centroid) is equal to minus 2 by 3; the centroid of this part is at two-thirds of L by 2, which is one-third L, so we look at one-third L from this side. Thus, you will see the value is equal to 2 by 3 and negative because if I consider this positive, then this is negative plus…. I have to consider this part, so it is going to be PL upon 4EI into L by 2 into 1 by 2 multiplied by the centroid value at this point. Look at the centroid; again, two-thirds L by 2, that is at one-third, so the value at this point is equal to one-third and again negative, because if this is positive, this is negative. This is equal to P L squared upon 16EI. This is identical…. Remember we had computed this earlier also. This P L squared by 16EI still remains the same. Now, what I need to do is I need to find out a moment that will give me minus $theta_{ab}$.

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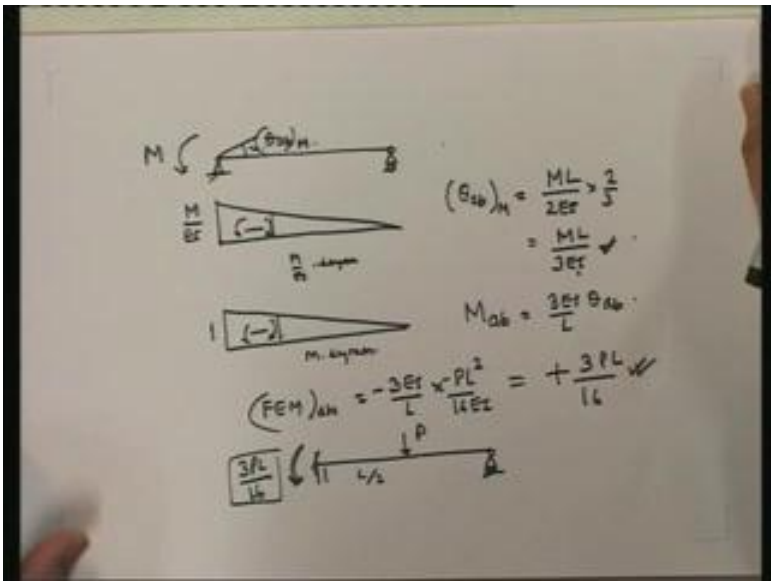

What I need to find out is due to a moment here (Refer Slide Time: 31:02), what is the rotation here? How do I find that out? You will see that it comes out from the original equation that we have – the force deformation relationship, but I will do it from first principles. Due to this moment, what is the moment diagram? The moment diagram looks like this and this is equal to M and since EI is a constant, I am drawing the M by EI diagram. I want to find out this $theta_{ab}$, so I apply a moment here and this is unit moment here and this is going to be the small m diagram. I am not writing them big because you know them already.

Here, what is the $theta_{ab}$ due to the moment? You will put this and what you get is the area under this curve, which is ML upon 2EI, M upon 2EI into half of length. Where is this? This is at two-thirds the distance at that value and note since this is hogging, this has the same direction , so it is positive and this becomes equal to ML upon 3EI. You could have found this out easily because M_{ab} is equal to 3EI upon L $theta_{ab}$ – you know this and you see that you will get exactly the same relationship from here too. Since we know the $theta_{ab}$ and this has to be equal to…. If M has to be the fixed end moment, then this (Refer Slide Time: 33:21) plus this has to be equal to 0. Therefore, what we have is fixed end moment at ab becomes equal to 3EI upon L. Note that this was minus, so you see what we are getting is that $theta_{ab}$ is equal to minus (P L squared upon 6EI); minus is obvious because this is clockwise, minus means clockwise. This plus this (Refer Slide Time: 34:01)… is equal to 0 because that is what you have here.

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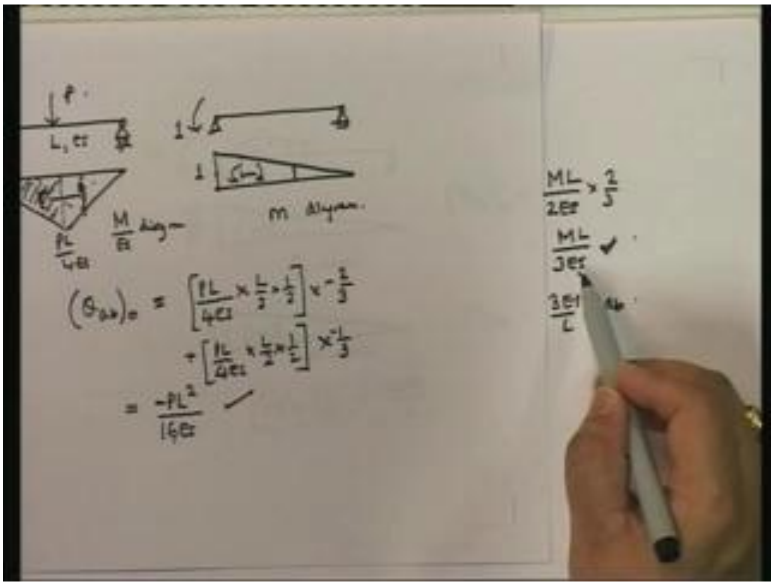

It has to be minus of $(\text{theta}_{ab})_0$ (Refer Slide Time: 34:15). Therefore, if we put that in, what do we get? We get it equal to minus (3EI upon L) – because this goes on the other side – multiplied by minus of (P L squared by 16EI); this becomes plus 3 PL upon 16. What is the fixed end moment? For a load applied at L by 2, the fixed end moment over here is positive 3 PL by 16, which means anticlockwise – you have already derived that earlier. Therefore, essentially what happens is that this I have derived from first principles and we have got exactly the same thing from here too and since we had this from the same, therefore I can say for certainty that I can write down the fixed end moment for this modified beam by just taking the fixed fixed and just subtracting ba by 2; I will always do this. This is the uniform equation that I am going to use because I know the fixed end moment for the fixed fixed case. Now, what I am going to do is to see how this changes.

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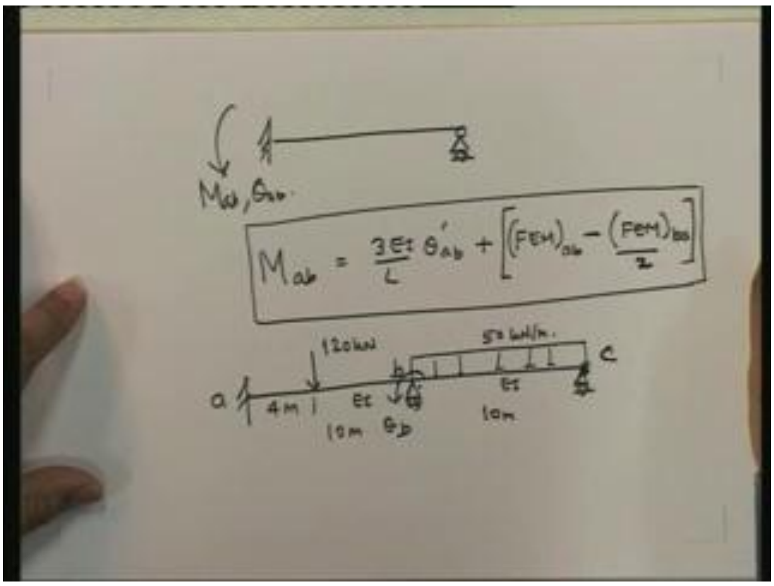

Ultimately, if you look at the equation for this, it turns out to be M_{ab} is equal to 3EI upon L into theta (this is where you take it from the chord plus and here, I am going to write down the expression for the fixed fixed case – these are the fixed end moments from the fixed fixed case; and this is algebraic – if the sign of ba is different from ab, it becomes positive. This becomes my new equation; I am going to use this to solve a problem that we had solved earlier.

What was the problem? I will make the problem statement for you; the problem statement is a problem that you have seen earlier and I am going to just put it down. I am going to use this procedure to solve this. This was the problem; this is a, this is b, this is c; this was the problem that we looked at earlier. In this particular case, if you look at member ab, at b, you have continuous, so there is going to be a moment here and so we cannot consider this in this, but if you look at bc, c is an end support and at an end support, we know that moment is equal to 0.

Remember we had applied that moment at c is equal to 0 as one of the equilibrium equations. In this particular case since I know that the moment is equal to 0 and I am not interested in knowing what the rotation over here is, what is the only degree of freedom that I have? Theta_b because note that I am not interested in theta_c because I know that the moment over here is equal to 0. So, I am going to use the modified… for the ab, I am going to use the original element and for bc, I am going to use a modified element; let us see what happens. I am going to define this problem in a different way. Earlier, what we had taken ab as fixed fixed, bc as fixed fixed, and we had defined theta_b and theta_c, both of which are unknown displacements. Here, I am only going to consider one unknown displacement and let us see how we can solve this problem. You see the advantage of this problem automatically – with only one degree of freedom, we only need one equation and things will become much simpler if you put that in. Let us go back and revisit that problem.

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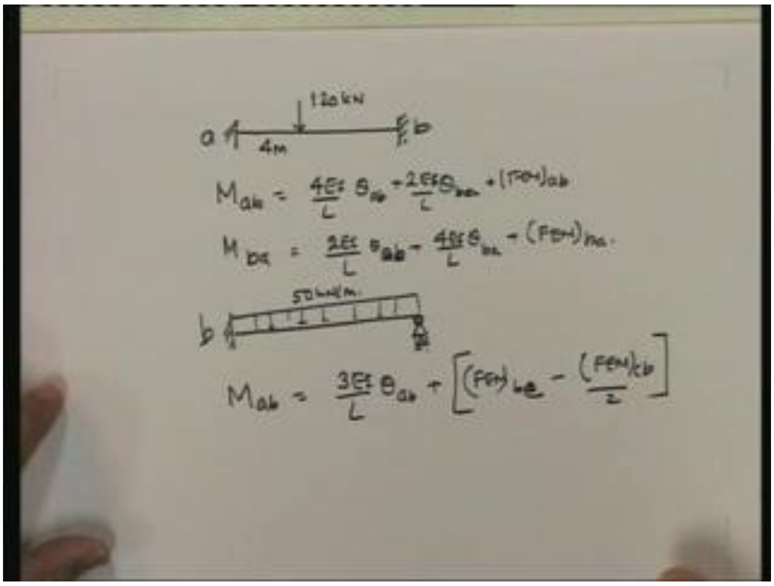

What we are going to do is, we going to be considering ab as a fixed fixed. As far as ab is considered, we are going to say M_{ab} is equal to 4EI upon L theta_{ab} plus 2EI by L into theta_{ba} (note that since there are no displacements, I am not even writing down the delta because the chord… or you can say take it from the chord, it does not matter) plus fixed end moment at ab; and M_{ba} is equal to 2EI upon L $theta_{ab}$ plus 4EI upon L $theta_{ba}$ plus $(FEM)_{ba}$. So, for ab, this is what I do and the fixed end moment comes from this 120 Kilonewton force acting at 4 meters. For bc, I am going to consider it this way: b is fixed but c is the end hinge. Whenever you have an end hinge, you know that the bending moment at that point is equal to 0 and so, I am going to use this and I have this loading, this is 50 Kilonewton per meter. I am going to put M_{ab} is equal to 3EI upon L $theta_{ab}$ plus (here, I am going to put down that these fixed end moments are the ones I had considered; they were fixed fixed) [$(FEM)_{bc}$ minus ($(FEM)_{cb}$ by 2)]. For ab, I consider this (Refer Slide Time: 42:30) and for bc, I consider this. Now, for $theta_b$ is equal to 1, let us go through the steps. For $theta_b$ is equal to 1, what does the thing look like?

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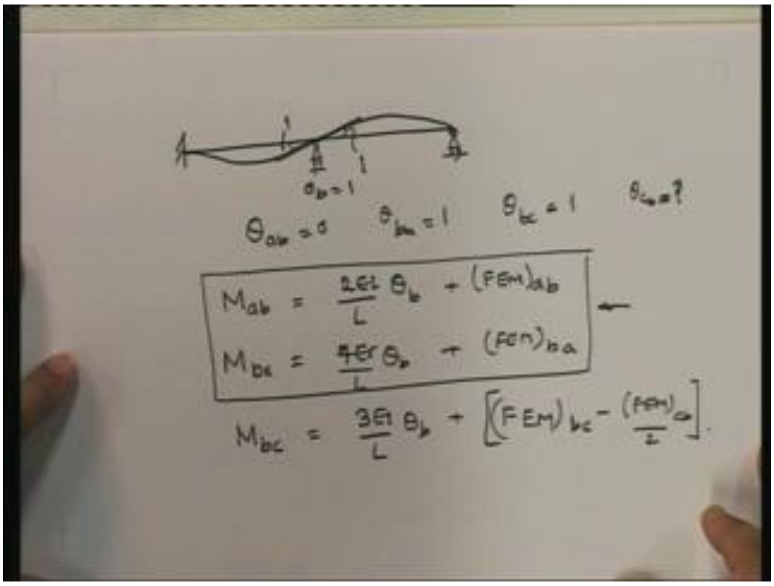

Note that that is the only degree of freedom so this is going to become 1. I am going to have a displacement pattern that looks like this and note that over here, it does not have to go to 0 because this is a hinge and it can take any value.

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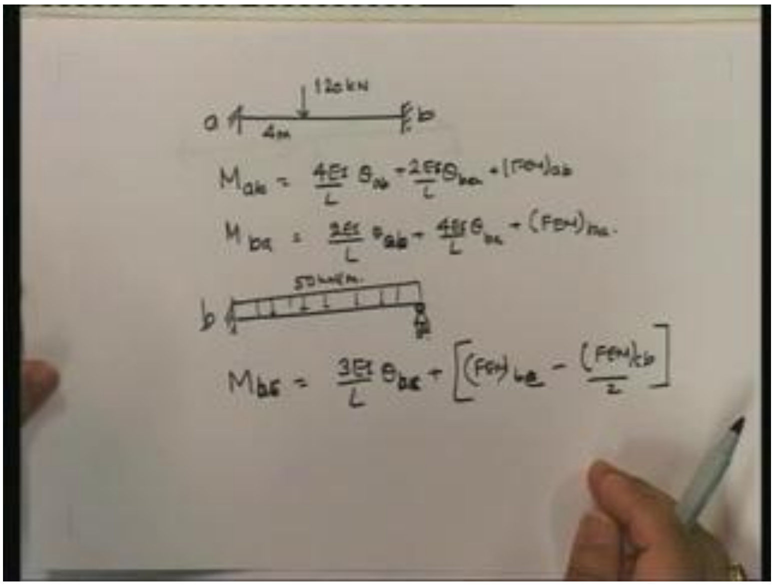

What is the value over there? Actually, if you notice, this one is M_{bc} (Refer Slide Time: 43:20) and it is equal to theta$_{bc}$ (Refer Slide Time: 43:22) and since in this equation there is no theta$_{cb}$, I do not really care what the rotation over here is. If I put this, I get that for theta$_{b}$ is equal to 1, theta$_{ab}$ is equal to 0, theta$_{ba}$ is equal to 1 and theta$_{bc}$ is equal to 1 and theta$_{cb}$ – I do not know and I do not care because theta$_{cb}$ is not going to get into my force deformation relationships anyway. Once I have that, it means now I can put down the expressions. I get M_{ab}; theta$_{ab}$ is equal to 0, so I get 2EI upon L and since theta$_{ba}$ is equal to theta$_{b}$, this becomes theta$_{b}$ plus $(FEM)_{ab}$. M_{ba} is equal to 4EI upon L theta$_{b}$ plus $(FEM)_{ba}$. These do not look any different from what we had originally; they look identical. The only difference here is in this equation – M_{bc} is equal to 3EI upon L into theta$_{b}$ (since theta$_{bc}$ is equal to theta$_{b}$) plus ($(FEM)_{bc}$ minus ($(FEM)_{cb}$ by 2)). I need to find out ab and bc and note that I have already done those earlier, so I am just going to show you that we have already evaluated these.

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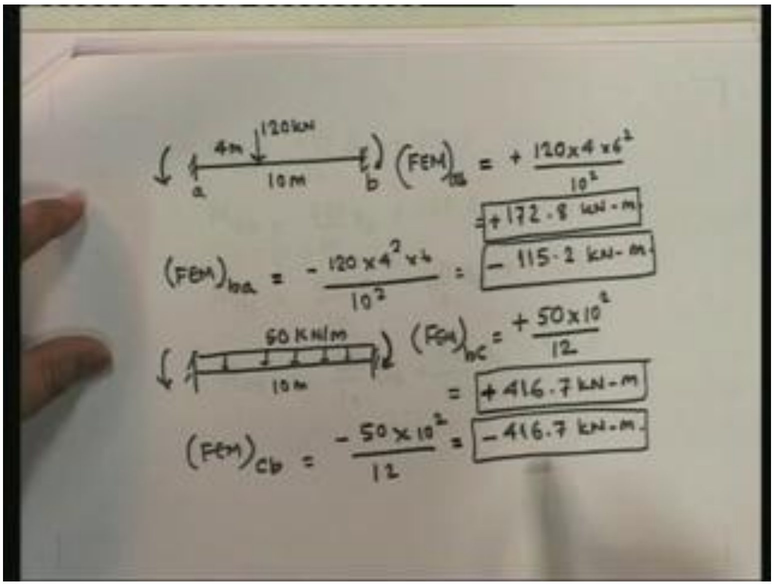

These were the fixed fixed, so ab is equal to plus 172.8, $(FEM)_{ba}$ is equal to minus 115.2 and $(FEM)_{bc}$ is equal to 416 and minus 416. Having done that, I can plug in those values directly. Therefore, what do I get?

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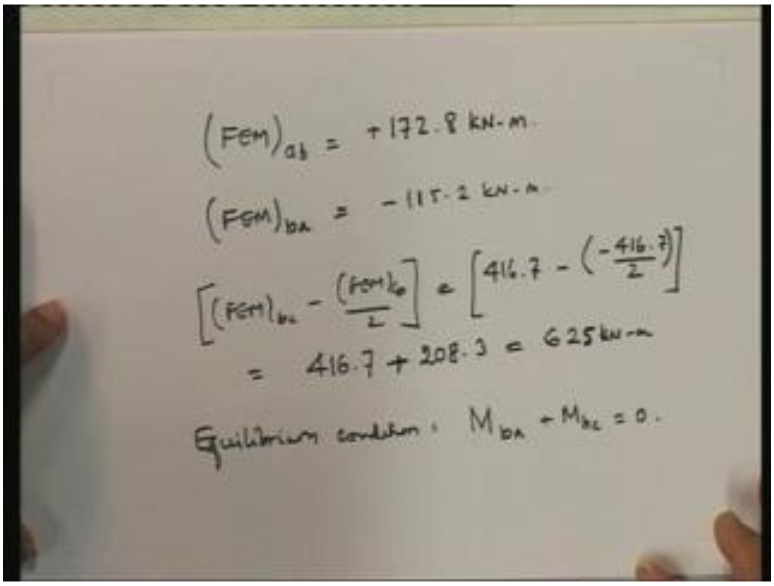

My $(FEM)_{ab}$ is equal to plus 172.8 Kilonewton meter, $(FEM)_{ba}$ is equal to minus 115.2 and I need to evaluate $(FEM)_{bc}$ minus ($(FEM)_{cb}$ by 2), which is going to be equal to 416.7 minus of (minus 416.7 by 2); this becomes 416.7 plus 208.3, which is equal to 625 Kilonewton meter. These are the fixed end moments that I have; I have put them in. Once I have all of these relationships, I need to write down the equilibrium condition. Since there is only one, the equilibrium condition, satisfy yourself, the equilibrium condition becomes M_{ba} plus M_{bc} is equal to 0. If I put that in, what do I get?

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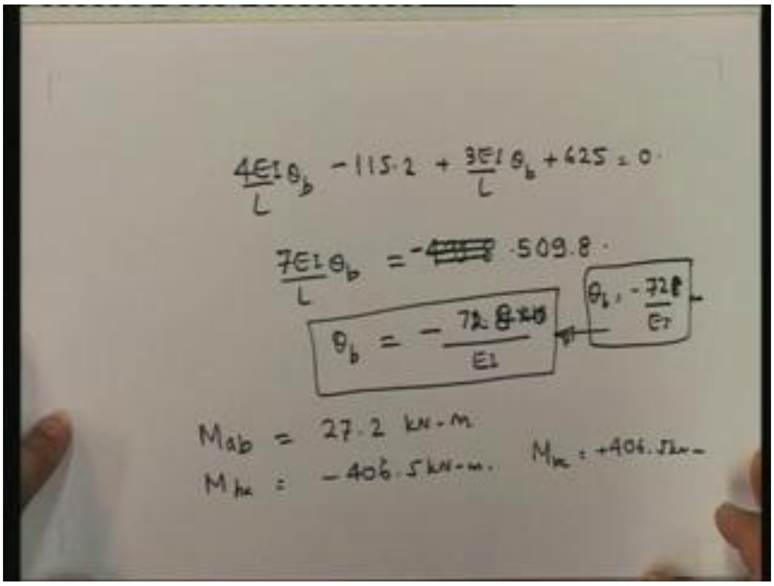
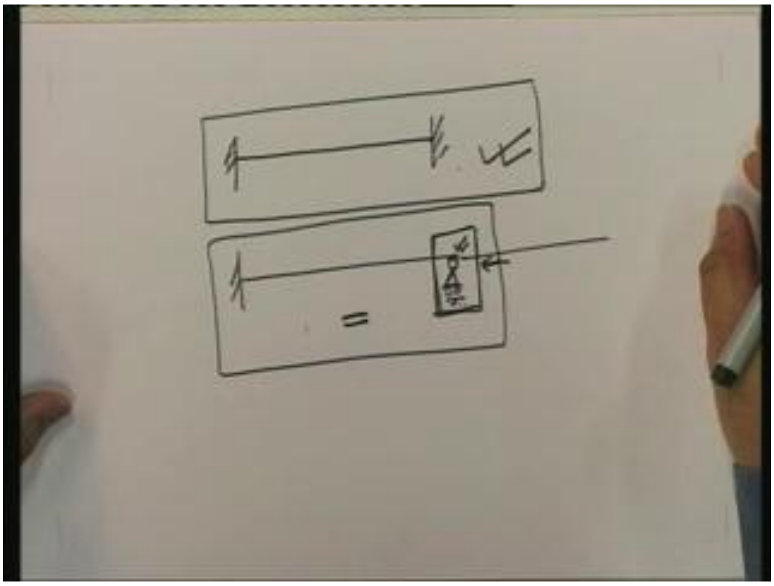


I get 4EI upon L into $theta_b$ minus 115.2 – this is M_{ba} and M_{bc} is equal to 3EI upon L $theta_b$ plus 625 (this is the fixed end moment) is equal to 0. This becomes 7EI upon L $theta_b$ is equal to 499.8 and $theta_b$ becomes equal to minus 71.4 L upon EI. Once I have my $theta_b$, I substitute it into my equations and I get M_{ab} is equal to 27.2 Kilonewton meter, M_{ba} will be equal to minus 406.5 Kilonewton meter and M_{bc} will be equal to plus406.5 Kilonewton meter. This is what you got the last time, so you will get those values just by substituting this. Now, since L is 10, this is multiplied by 10 (Refer Slide Time: 50:20).

The beauty of this essentially becomes this: that there is just one degree of freedom, so the equations become much easier. The major advantage is that I have already made two degrees of freedom into a single degree of freedom. You may say that there was not too much of a difference between two degrees of freedom and a single degree of freedom, but when you go into multiple degrees of freedom and you can eliminate a few of these degrees of freedom, you will see that it becomes a big issue.

This simplification does nothing to the solution process; it still remains exactly the same; the only thing that happens is that you land up getting a much better solutions base. That is all I have to say in this particular case. We will see that as we go along, we are going to get many many problems. By the way, let us go back to this; I have made one small mistake and I just want to correct that mistake because you may find yourself…. This becomes actually 509.8 and if you look at this, this becomes 72.8 and these give you this; this basically becomes $theta_{ba}$ is equal to minus 728 upon EI; you will see that 728 gives you this; I have just made a mistake in putting it down over here; satisfy yourself that this is going to happen. Now, I am going to just state the basic concept that you have to work with.

(Refer Slide Time: 52:48)



Right in the beginning, you have to decide whether you want to use this (Refer Slide Time: 52:49) or whether you want to use this. If either joint of the member is continuous, in other words, some other member is connected over here, then you always have to use this. You can use this only when this end happens to be the end support of a structure and so, this can only be used if there is an end support; you cannot use this if there is another member connected over here; you cannot. If there is another member, then the moment at this point is not equal to 0. Is that clear? You can use this only if you have an end support; otherwise, you have to always use this and use the member force deformation relationship for this; and for this, I have just shown you today how the force deformation relationship is different for this one.

I hope today's lecture has made it clear to you that even in the displacement method, you can actually bring in simplifications to enable you to reduce some of the degrees of freedom and thereby reduce the equilibrium conditions that you need to specify; ultimately, you reduce the number of algebraic equations that you need to solve for finding out the displacements and hence the member end forces. Thank you very much.