Structural Analysis - II

Prof. P. Banerjee
Department of Civil Engineering
Indian Institute of Technology, Bombay

Lecture - 13

Good morning. In the last lecture, we looked at how to solve a beam problem using the displacement method and in the end, I gave you a problem which included a support settlement; we are going to look at that problem in this lecture. Let me just reiterate that particular problem.

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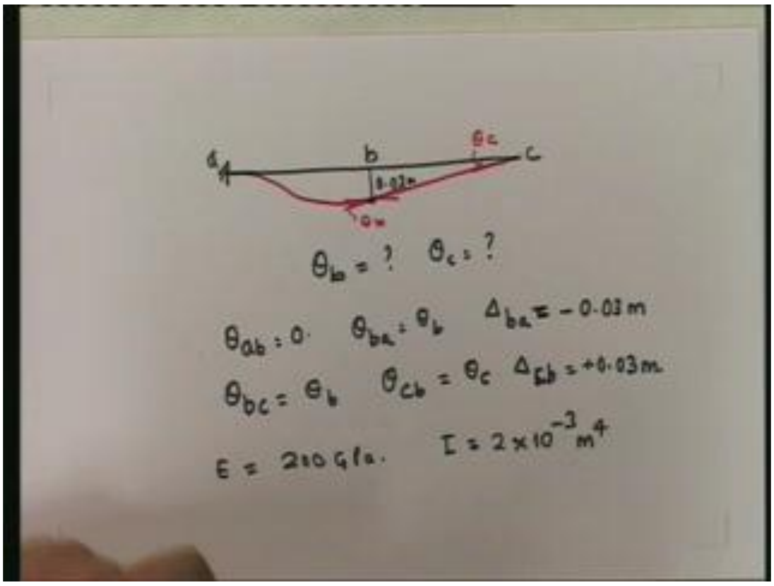

What you have is this, the support has settled and the support settlement is 0.03 meters. So, what will the deflected pattern look like? This is theta$_c$ (Refer Slide Time: 02:15) and this is theta$_b$ – the two unknown rotations, the two displacements corresponding to the degrees of freedom; so, theta$_b$ is unknown, theta$_c$ is unknown.

Going back, what do we have? By compatibility condition, theta$_{ab}$ is equal to 0 and theta$_{ba}$ is equal to theta$_{b;}$ delta$_{ba}$, now, this is interesting. What is delta$_{ba}$? delta$_{ba}$ in the previous case was 0, but here, you see b, a, c; b has gone down relative to a, so that means delta$_{ba}$ is not 0. Remember delta$_{ba}$ was considered to be positive if b went up relative to a; since it is going down, it is negative and the value is 0.03 meters. Let us look at member bc; theta$_{bc}$ is equal to theta$_b$, theta$_{cb}$ is equal to theta$_c$ and what is delta$_{cb}$ equal to? b has gone down relative to c, so we can say that actually c has gone up relative to b, so it is plus 0.03 meters. Note that we are given that E is equal to 200 GPa and I is equal to 2 into 10 to the power of minus 3 meter fourth – these are what are given; we can substitute them into equations. By the way, what are the fixed end moments?

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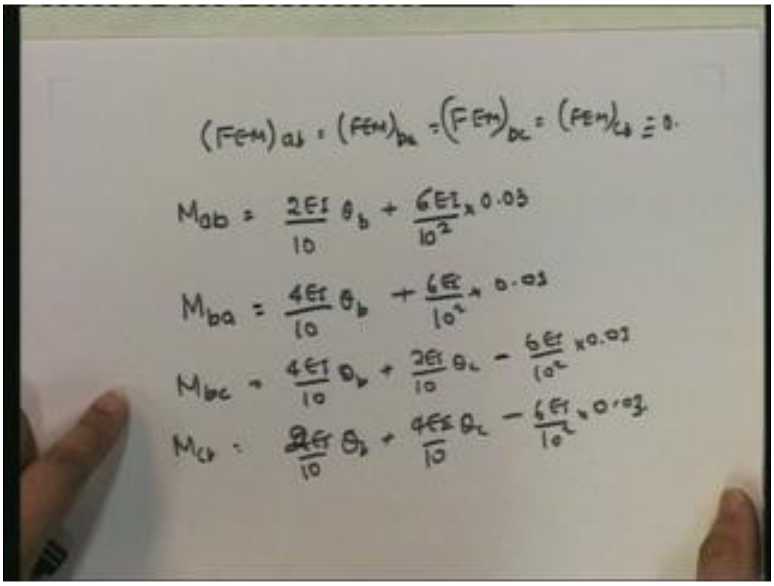

Since there is no loading, note that $(FEM)_{ab}$, $(FEM)_{ba}$, $(FEM)_{bc}$ and $(FEM)_{cb}$ are all 0. Now, substituting into the equations, what do we get? Let us see what we get. We get the equation M_{ab} minus 2EI by 10 $theta_b$ plus… note that it is actually minus 6EI L squared into delta but since delta is negative, this is going to be 6EI upon 100 into 0.03. Then, M_{ba} is equal to 4EI upon 10 $theta_b$ plus 6EI upon 100 into 0.03 and M_{bc} is equal to 4EI upon 10 $theta_b$ plus 2EI upon 10 $theta_c$ minus 6EI upon 10 squared into 0.03. Note that in bc, $delta_b$ is positive, so that is why this is this way and M_{cb} is equal to 2EI upon 10 $theta_b$ plus 4EI upon 10 $theta_c$ minus 6EI upon 10 squared into 0.03. If I plug in the values of EI, see what I get.

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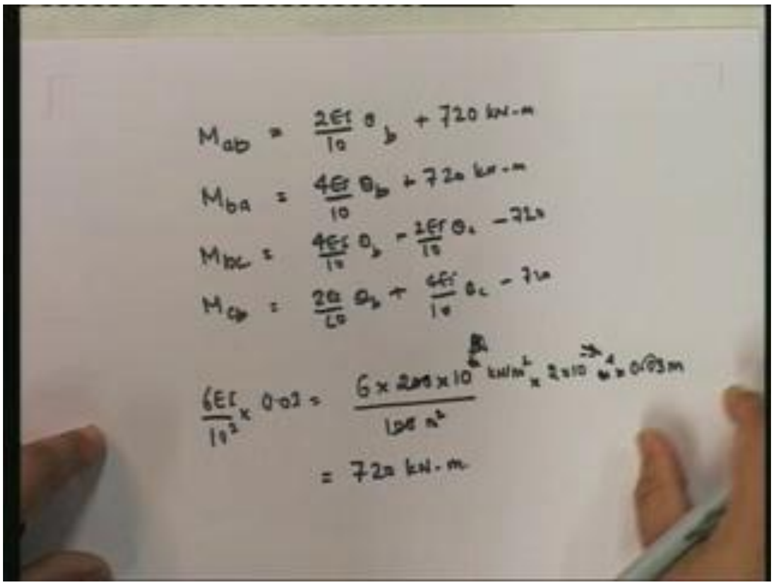

I get M_{ab} is equal to 2EI by 10 $theta_b$ plus …. You plug in the value of EI, note I am just going to go through a little bit; let me show it to you, since you might be a little bit confused. What I am doing is 6EI upon 10 squared into 0.03; EI is 200 Gpa, which if you write in Kilonewton (note that everything is in Kilonewton) becomes equal to 200 into10 to the power of 9 Newton per meter squared; this is going to be equal to 200 into 10 to the power of 6 Kilonewton per meter squared divided by 100 meter squared multiplied by I. What is I? 2 into 10 to the power of minus 3, that is 6EI upon L squared meter fourth and then we have 0.03 meters. Let us see what units I land up getting.

This is going to be equal to Kilonewton (Refer Slide Time: 09:20), this is going to be meter fourth at the bottom, meter fourth cancels here, so this is going to be Kilonewton meter, this goes into this, this goes into this three times and this 3, if I bring it here, this will become 1; this will become 6 into 6, 36, 36 into 2 is 72, into 10 is equal to 720; that is what I have over here, this is 720 Kilonewton meter. These are the units and then, 4EI upon 10 theta$_b$ plus 720 Kilonewton meter; M_{bc} is equal to 4EI upon 10 theta$_b$ plus 2EI upon 10 theta$_c$ minus 720, the same; and then, M_{cb} is equal to 2EI upon 10 theta$_b$ plus 4EI upon 10 theta$_c$ minus 720. Now, the equilibrium equations are the same.

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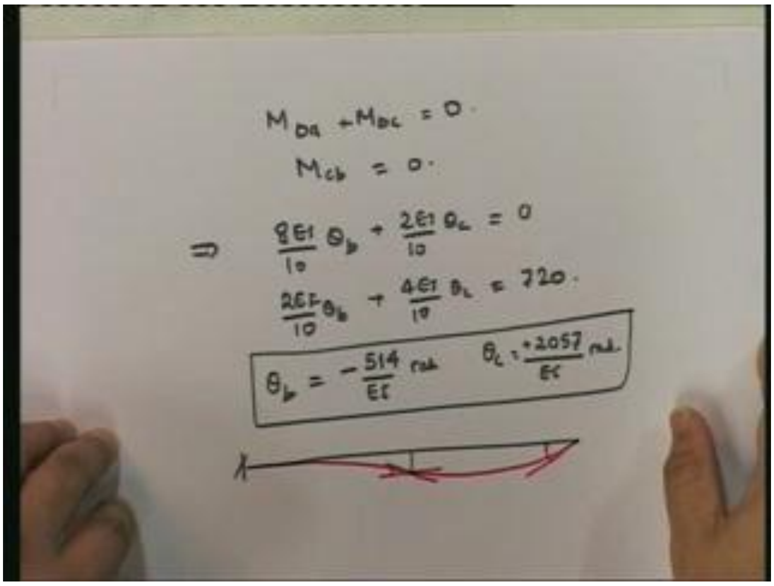

The equilibrium equations are M_{ba} plus M_{bc} is equal to 0 and M_{cb} is equal to 0. This gives us that M_{ba} plus M_{bc} is equal to 8 EI upon 10 theta$_b$ plus 2EI upon 10 theta$_c$. If you have plus 720, minus 720, that cancels out and this becomes 8 EI upon 10 theta$_b$ plus 2EI upon L theta$_c$ is equal to 0. The second equation M_{cb} is equal to 0 gives us 2EI upon 10 theta$_b$ plus 4EI upon 10 theta$_c$ is equal to 720. Solving for this gives us that theta$_b$ turns out to be equal to (minus 514 by EI) radians and theta$_c$ is equal to (plus 2057 by EI) radians. What does that mean? It means that if I were to draw the deflected shape over here due to this settlement (Refer Slide Time: 12:58), the system becomes this way; this is positive and it is clockwise because this is negative and this is anticlockwise (Refer Slide Time: 13:27) because it is positive. This is the deflected shape and now, once we get theta$_b$ and theta$_c$, we can substitute it back into the equations for M_{ab}.

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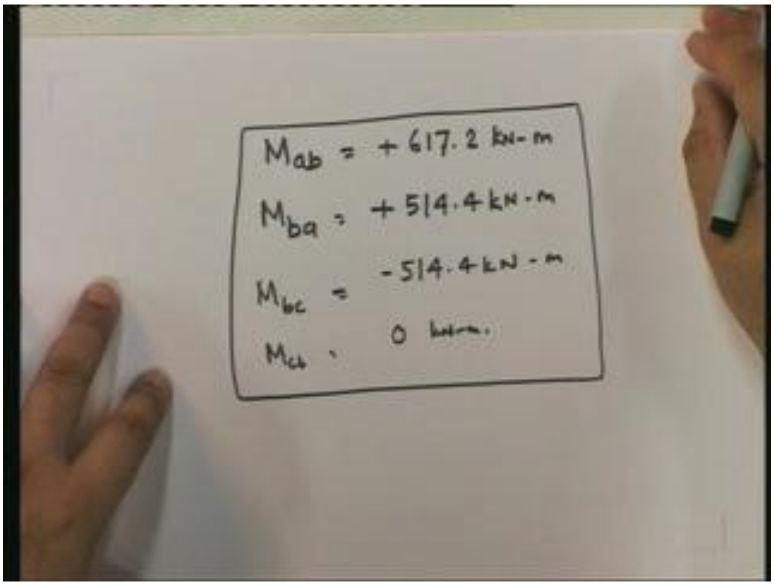

In other words, once I have found out $theta_b$ and $theta_c$ I can substitute $theta_b$ and $theta_c$ into these equations, plug in the values of $theta_b$ and my final M_{ab} is equal to plus 617.2 Kilonewton meter, M_{ba} is equal to plus 514.4 Kilonewton meter, M_{bc} is equal to minus 514.4 Kilonewton meter and M_{cb} is equal to 0. Once you get these moments, then you can draw the bending moment diagram just we as we had done last time; it is not very different from what we had done, excepting that there is no loading.

The interesting point to note here is how much did the moments come out to be? Think about the loading; you had 120 Kilonewton loading and then we had 500 Kilonewton on 10 meters and what kind of forces did we get? We got much less moments than we got due to a settlement of only 3 centimeters. In other words, settlement problems can actually give rise to significantly higher stresses in structures than loads can and that is the reason why support beams are always very very… support settlement can have very differential settlement, can have extremely problematic consequences for your structures unless you design them for these kinds of loads. So much for this particular analysis; I think I have done enough number of problems for beams.

What I would like to now do is, I would like to go on to solving a particular problem for a frame, so that it gives you an idea of how the whole concept of displacement method can be used for frames. You will see another very interesting thing and that is, in a beam, we only had $theta_{ab}$ and $theta_{ba}$; in frames, you will see that $delta_{ba}$ will also come in and it will be an unknown. In a beam, the only problem that we did where $delta_{ba}$ came in was when we had support settlement; we knew the value of delta. In frames, you will see that that will also be an unknown and therefore, you have some interesting equations of equilibrium that you have to develop when you go through it. Let me take a very simple problem and go through it for you.

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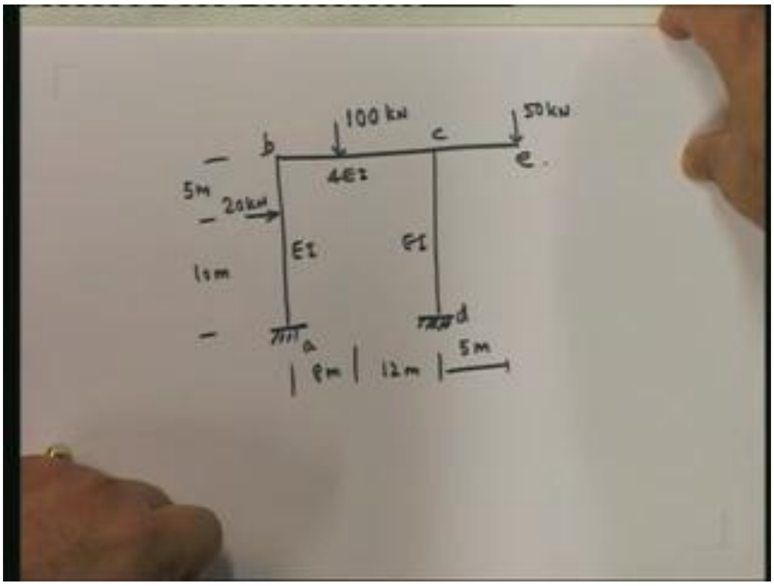

This is a frame that we have and if you look at this particular frame, it has fixed supports at a and d and it is a single-storey, single bay frame with an overhang. At the end of the overhang, you have a load, you have one load in member bc; member ab has a load, bc has a load, ca has a load and cd has a load. In this particular case, you will see that I am going to eliminate ce because I am not quite interested. If you look at ce, can I draw the bending moment diagram for ce? Sure, I can.

At this point, the bending moment is going to be 0 (Refer Slide Time: 20:09) and at this point, it is going to be 50 into 5, 250 Kilonewton meter; I know the bending moment in here, so I do not really need to consider this. So, what I am going to do is, I am going to consider that this load (Refer Slide Time: 20:31) is going to get transferred to c with the appropriate equivalent forces. With this problem, I can solve it as far… I need to only solve abcd with the given fact that the loading from ce comes out to be this.

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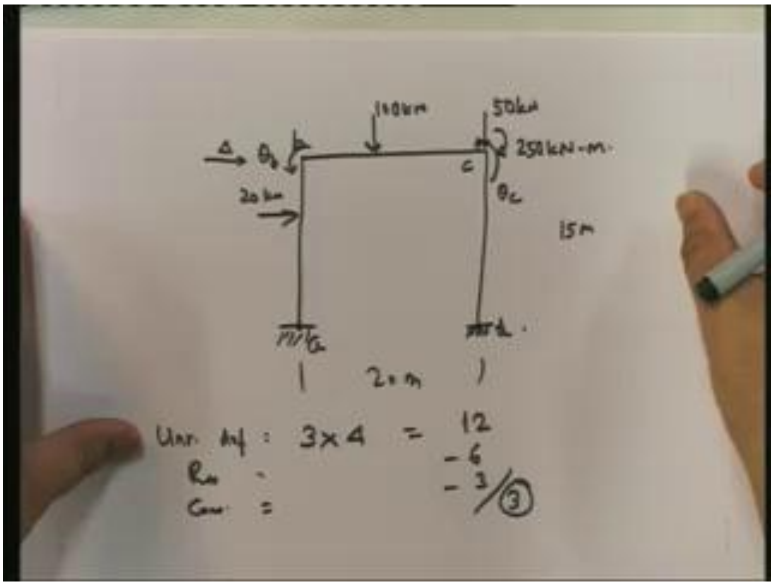

I can solve this problem. 15 meters, 20 meters, load here (Refer Slide Time: 21:12), load here and this load, I can transfer here as 50 Kilonewton, but just transferring the load is not good enough. What happens? Equivalent load system, I also have to transfer a moment and the moment is equal to 250 Kilonewton meter (Refer Slide Time: 21:29). Now, I am solving abcd. This is the problem that we have to solve using the displacement method.

What is the first step? The first step is to find out the degrees of freedom. How do you find out the degrees of freedom? Let me go through all the steps of the degrees of freedom. How many nodes? 1, 2, 3, 4 so four joints; three into four is the unrestrained degrees of freedom, twelve. How many restraints? Restraints are three here and three here (Refer Slide Time: 22:22) because you are restraining all three degrees of freedom at this point and you are restraining all three degrees of freedom, so six restraints. Constraints? I have three, axial deformation is neglected, three members, three constraints. So, how many degrees of freedom? Three. What are those three degrees of freedom? Theta at b, theta at c (Refer Slide Time: 22:57) and delta (Refer Slide Time: 23:03); this delta is the same as this delta because bc cannot actually deform.

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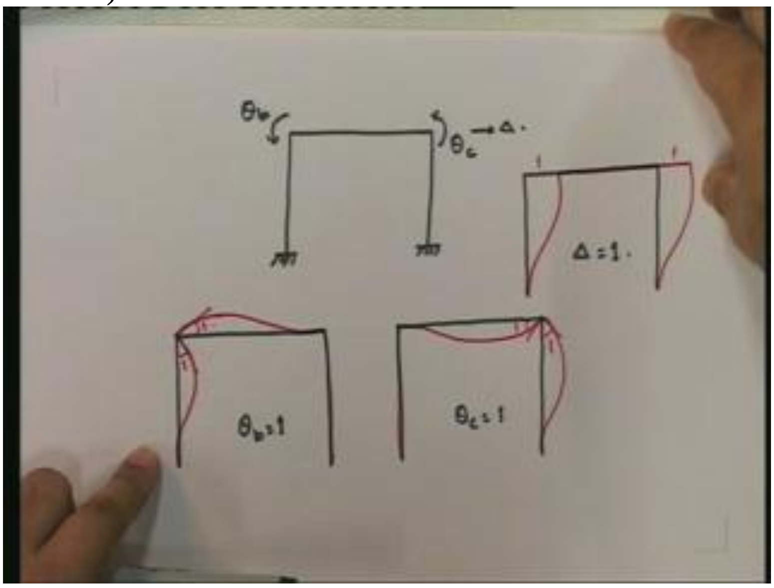

Let me do that again; I have theta$_b$, theta$_c$ and delta. Now, I am going to introduce some concepts and that is, what do those…? Before I write down the slope deflection equations, I have to be able to relate these (Refer Slide Time: 23:57) with respect to the member end deformations. For that, I am going to now start a new procedure by which I am going to find out, given each degree of freedom, how the displacement occurs, so that I can always write down in terms of their displacement what the member end displacements are; then, when I take all three of them together, that becomes a sum total.

Here, I am going to be drawing. What happens when you have…? theta$_b$ I am going to put…. Since I need to evaluate it together, I am going to draw it piece by piece; so, theta$_{ba}$ is equal to 1. How will the displacement look? theta$_{ba}$ is 1, so, it is going to go like this, this is going to go like this, this point cannot go because delta is equal to 0 and theta$_c$ is equal to 0, so this has to be fixed and then, this remains exactly this way; this is my 1, 1.

Let me write down all the member end displacements relative to that, but first I will draw all of them and then I will write them down. Let me put $theta_c$ is equal to 1. What is $theta_c$ is equal to 1? You will see that if $theta_c$ is equal to1, this will go like this, this will go like this (Refer Slide Time: 25:59), this will come here and finally, delta is equal to 1 and is going to be this way; note that when delta is equal to 1, $theta_b$ and $theta_c$ is equal to 0, so this is going to go this way (Refer Slide Time: 26:25). The reason why I am doing this is that… you know, if I show all of them together, it is going to look very very complicated; so, what I do is I take each one separately and then I know that all three act together. Once I know all the shapes for… this is equal to 1 and here, this is 1 and this is 1. Once I have this, I can write down each and every… for member ab, what are the deformations in terms of $theta_b$, in terms of $theta_c$, in terms of delta; and now, with this, look at how their deformations go.

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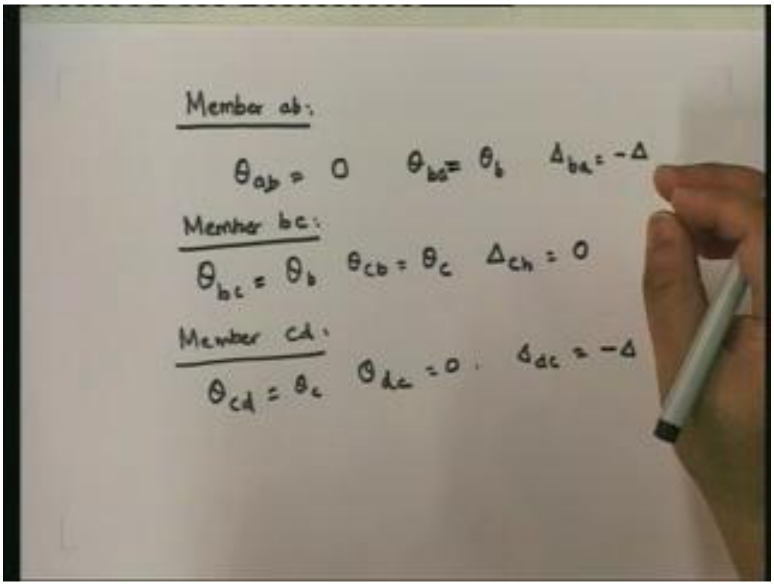

I am now going to write down the kinematic relationship and that is, for member ab, what is $theta_{ab}$ equal to? Look at this. $theta_{ab}$ is always going to be 0 because at that particular point, you have the theta will be 0. What is $theta_{ba}$ equal to? Let us see $theta_{ba}$. When $theta_b$ is equal to 1, $theta_{ba}$ is equal to 1 and so $theta_{ba}$ is equal to 1 into $theta_{b;}$ when $theta_c$ is equal to 1, $theta_b$ is equal to 0; when $theta_{delta}$ is equal to 1, $theta_b$ is equal to 0 and therefore, we are going to write down $theta_{ba}$ is equal to $theta_b$ only. Similarly, write down $delta_{ba}$. When $theta_b$ is equal to 1, what is $delta_{ba}$? 0. When $theta_c$ is equal to 1, what is $delta_{ba}$? 0. When delta is equal to 1, what is $theta_{ba}$? Let us look at this, $delta_{ba}$ (Refer Slide Time: 28:45), so ab has to go up this way (Refer Slide Time: 28:47) for it to be positive; it is going down, so it is actually minus 1, so it is minus delta.

Similarly, let us look at member bc. What about member bc? $theta_{bc}$ is equal to what? What is bc equal to? Let us look at it. bc is equal to 1 into $theta_b$ (Refer Slide Time: 29:24) plus 0 into $theta_c$ plus 0 into delta; that means $theta_{bc}$ is equal to $theta_b$. What about $theta_{cb}$? $theta_{cb}$ is equal to 0 into $theta_b$ plus 1 into $theta_c$ plus 0 into delta, so it is equal to $theta_c$. Similarly, $delta_{cb}$; c relative to b, 0 into $theta_b$ plus 0 into $theta_c$ plus 0 into delta; note that this has to move this way for it to be delta, so $delta_{cb}$ is equal to 0.

Now, I am looking at member cd, this is the way I am looking at it (Refer Slide Time: 30:28). I am looking here to here. What is member cd? delta$_b$, cd is equal to 0 into theta$_b$ plus 1 into theta$_c$ plus 0 into delta. So, theta$_{cd}$ is equal to theta$_c$, theta$_{dc}$ at fixed end is 0; what is delta$_{dc}$ equal to?

Let us look at it. delta$_{cd}$ is equal to 0 into theta$_b$ (Refer Slide Time: 31:05) plus 0 into theta$_c$ plus… now, let us look at this, what does this mean? cd, d has to move in this direction for it to be positive delta; however, since this has gone up (Refer Slide Time: 31:22), we can say that relative to c, this has actually gone this way (Refer Slide Time: 31:26), so it is negative delta, let us make it negative delta. We have now got all the relationships between the member displacements, the member end displacements and the degrees of freedom – theta$_b$ theta$_c$ and delta. Now, we substitute these into the equations, but before we substitute, we have to find out the fixed end moments; we have to find out the fixed end moments in ab, bc and cd due to the loads.

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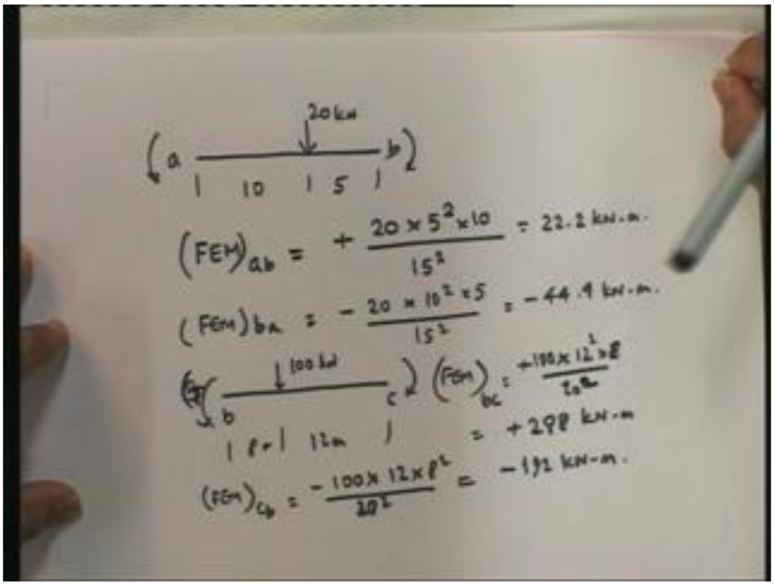

Member ab has a load. What is the load? This is a, b, 10 meters, 5 meters; at this point, we have a 20 Kilonewton force. What is this going to give me? The fixed end moment at ab is going to be equal to b and note that this is going to be this way (Refer Slide Time: 32:51) and this is going to be this way; this is positive, so this is going to be plus ((20 into 5 squared into 10) by 15 squared); this is going to be equal to 22.2 Kilonewton meter.

The (FEM)$_{ba}$ is going to be minus – it is negative – (20 into 10 squared into 5 by 15 squared) is equal to minus 44.4 Kilonewton meter. Let us look at what happens in bc. bc has 10 Kilonewton force acting; this is 8 meters, this is 12 meters (Refer Slide Time: 34:09), this is going to be this way, this way, so fixed end moment at bc is equal to plus ((100 into 12 squared into 8) upon 20 squared) – this is equal to plus 288 Kilonewton meter. Then, the fixed end moment at cb is equal to minus ((100 into 12 into 8 squared) upon 20 squared) – this is going to be equal to minus 192 Kilonewton meter. Now, look at cd; cd does not have any load on it, so the (FEM)$_{cd}$ and dc are going to be equal to 0.

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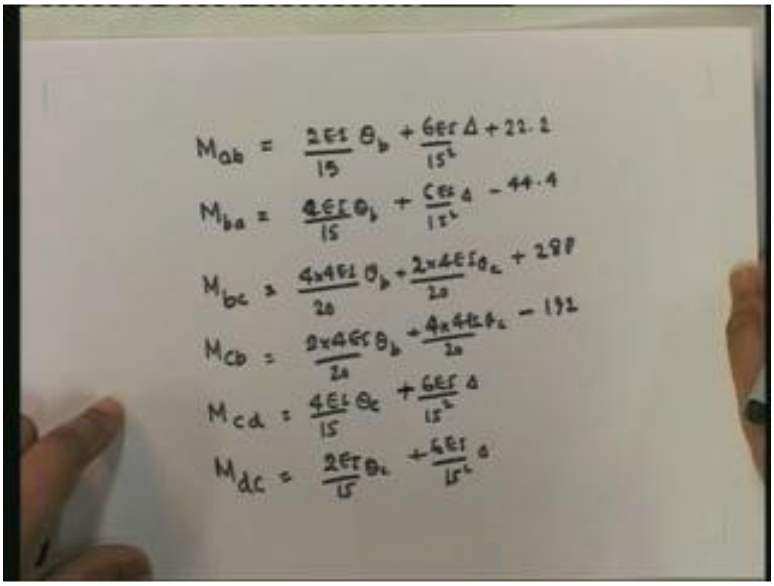

Let us look at the equilibrium equations. If you look at the equilibrium equations, what we get is that M_{ab} is going to be equal to 2EI upon 15 multiplied by theta$_b$ plus 6EI upon 15 squared into delta plus 22.2 Kilonewton meter. M_{ba} is equal to 4EI upon 15 theta$_b$ plus 6EI upon 15 squared delta minus 44.4. Then, we have M_{bc} is equal to 4 into 4EI (note that it is 4 into 4, the EI itself is 4EI; remember that) upon 20 into theta$_b$ plus 2 into 4EI upon 20 theta$_c$ plus 288 (note that delta is equal to 0, so it does not contribute). M_{cb} is equal to 2 into 4EI upon 20 theta$_b$ plus 4 into 4EI upon 20 theta$_c$ minus 192.

Finally, M_{cd} is equal to 4EI upon 15 theta$_c$ plus 6EI upon 15 squared into delta; note that it is minus again, so it is minus into minus, plus, and that is it, there is nothing else, there is no fixed end moment. M_{dc} is equal to 2EI by 15 plus 6EI upon 15 squared delta. These are all the moments that we have. We have been able to write down the member end moments in terms of the unknown displacements that we have. Once we have done that, then we have another aspect: equilibrium. Here, equilibrium is going to get really really complicated. Let us look at what equilibrium actually entails.

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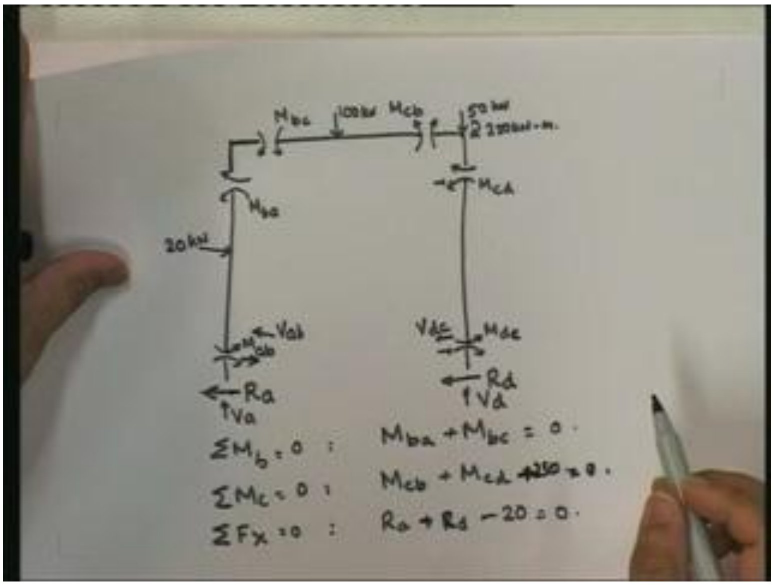

Again, I am going to separate it out so that you can understand what I am trying to say; I am separating out all the members from the joints and representing them separately. I am first putting down all the moments that we have computed and all the forces that we have. These are the moments that I have; I am putting them all in the positive; this is M_{ab}, this is M_{ba}, this is M_{bc}, M_{cb}, M_{cd}, M_{dc}; so, this happens to be M_{ab}, M_{ba}, M_{bc} (Refer Slide Time: 41:22), M_{cb}, M_{cd}, M_{dc} (Refer Slide Time: 41:32). These are the values that I have already put down just now; these are all in terms of theta$_b$, theta$_c$ and delta – the unknown displacements corresponding to the degrees of freedom; I have put them down.

Now, what I need to do is, I need to actually put down all the other forces that I have on the system. This one (Refer Slide Time: 42:02) has 20 Kilonewton here. Then, what else do I have? I have… let me just look back at the problem itself; I have 20, then I have 100; in addition to that, I have 50 (Refer Slide Time: 42:33) and 250 Kilonewton meter here. Anything else? No, that is it, these are all the loads.

Now, I need to be able to do is write down the equilibrium equations. What are the equilibrium equations? One equilibrium equation obviously is moment equilibrium, so summation of all moments at b is equal to 0 – this is going to give you that M_{ba} plus M_{bc} is equal to 0. Similarly, one is moment. What is this moment (Refer Slide Time: 43:27)? Corresponding to the rotation; this is corresponding to this rotation. Sigma M_c is equal to 0 and that is going to give you that M_{cb} plus M_{cd} is equal to 0. Is it? No, it is not. M_{cb} plus M_{cd} plus 250 is equal to 0. Note that when we are taking equilibrium, the external loads also have to come in, so it is plus 250 is equal to 0. These are the two equations corresponding to the two rotations (Refer Slide Time: 44:09).

Now, I have a displacement; corresponding to that, what do I have? I have to write down an equation which says reactions have to be equal to 0 and what reactions have to be equal to 0? Let us see. If I take the reaction that I get here, let us see what I get. This is going to be like this, so the loading will be like this (Refer Slide Time: 44:37); let me just put it; so, this is going to be this way, this is going to be this way.

Similarly, you will get over here, this goes this way, so this has to go this way, this will go this way, this has to go this way and this has to go this way (Refer Slide Time: 45:06). The other equation that we get is going to be that this force – the reaction at a plus this force – the reaction at d…. The other equation is corresponding to this delta, so it is going to be sigma fx is equal to 0 of the entire thing and that is going to be give me R_a plus R_d minus 20 (this is the force (Refer Slide Time: 45:54)) is equal to 0; this is my third equation, which essentially is this.

Look at the other reaction. This is V_a (Refer Slide Time: 46:04), V_d. The other equation V_a plus V_d is equal to 100 plus 150 is not going to give me an additional equation; you will see why. You will always see that since you have a rotational degree of freedom, you have a moment equilibrium at b; since you have a rotational degree at c, you will have a moment equilibrium at c; rotations go with moments and displacements go with forces. So, since the displacement is in this direction, I need a sigma f_x is equal to 0 to be able to generate that and that is the reason why I have that. If I had any degrees of freedom in this direction (Refer Slide Time: 46:58), then I would have to take V_a plus V_d. That is the overall scope of things. Once you have these, I know I can substitute them in; this one, I have to derive, I can find out? How do I find this out? I can find out that this is going to be equal to V_{ab}; R_a is equal to V_{ab} and R_d is going to be equal to V_{dc}. I can find out these from this equation (Refer Slide Time: 47:32).

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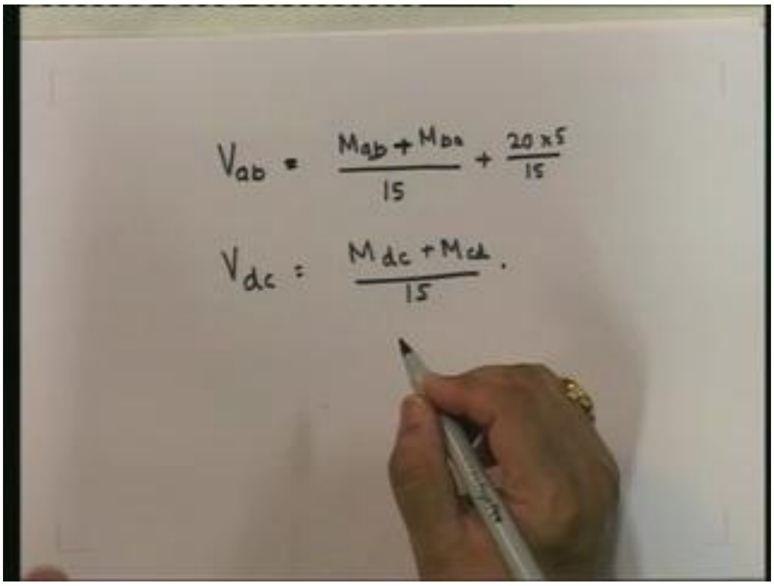

What is V_a equal to? Let me evaluate that. V_a is going to be equal to…. V_{ab} is going to be equal to – I can easily evaluate it – (M_{ab} plus M_{ba}) divided by 15 (these are the moments, reactions due to this) plus 20 into 5 upon 15; that is going to be V_{ab}; V_{dc} is just going to be equal to (M_{dc} plus M_{cd}) upon 15; and you know M_{ab}, M_{bc}. If you put all of these together, what you get is M_{ba} plus M_{bc} gives you … let me write it down.

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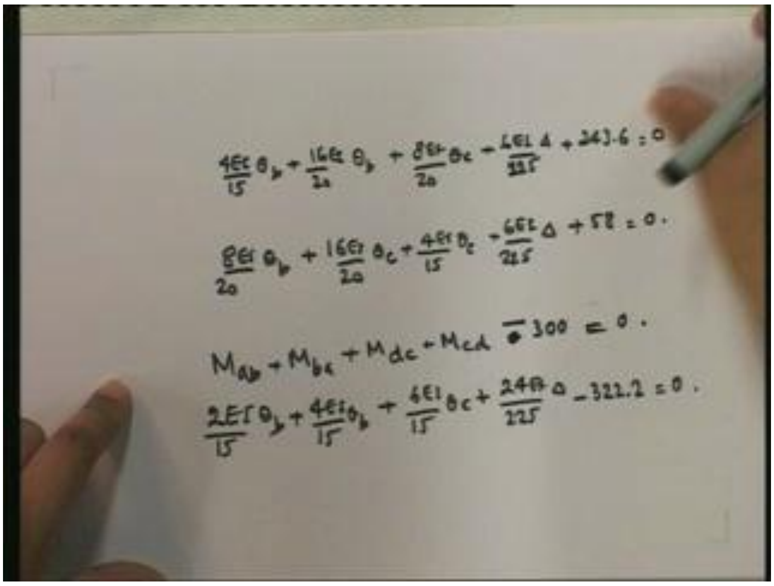

M_{ba} plus M_{bc} gives me 4EI upon 15 theta$_b$ plus 16 upon 20 EI theta$_b$ plus 8 EI over 20 theta$_c$ plus 6EI upon 225 into delta plus 243.6 is equal to 0 – that is my M_{ba} plus M_{bc}. If I look at M_{cb} plus M_{cd} plus 250 is equal to 0, M_{cb} is going to be equal to 8 EI upon 20 theta$_b$ plus 16EI upon 20 theta$_c$ plus 4EI by 15 theta$_c$ plus 6EI upon 225 delta plus 250 minus 192, 58 is equal to 0 – that is my second equation. My third equation is going to give me that this plus this (Refer Slide Time: 51:24) minus 20 is equal to 0.

Now, you can evaluate that, I am not going to go into that in detail, I will leave it up to you. In fact, what I can do is, if you look at this (Refer Slide Time: 51:28), since these are all divided by 15, I can just multiply by 15 over here and I will get M_{ab}. If you look at this, from this equation, I get M_{ab} plus M_{ba} plus M_{dc} plus M_{cd} minus 300 is equal to 0 and all I need to do is substitute all these terms in, let me do that. I am going to put that equal to 2EI plus 15 theta$_b$ plus 4EI upon 15 theta$_b$, so this is 6EI upon 15 theta$_b$ plus M_{dc} and $M_{cd,}$ I am going to just put it in, 6EI upon 15 theta$_c$ plus M; let me just go back.

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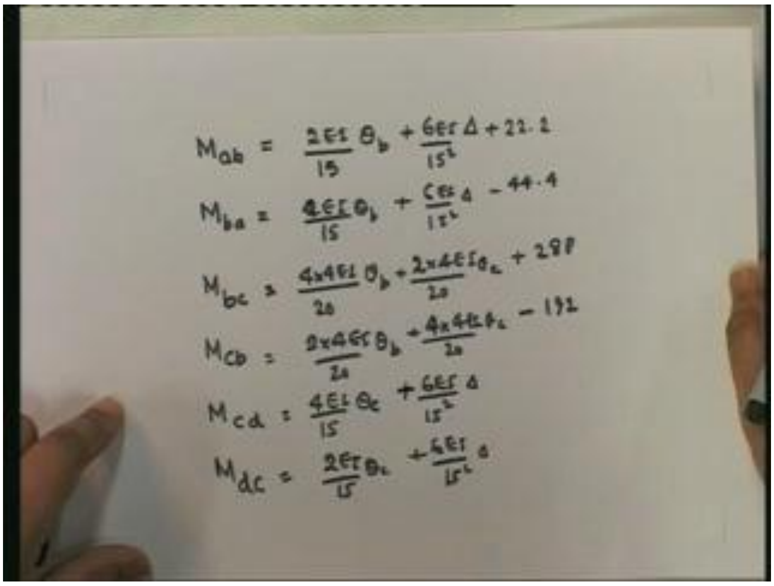

M_{ab} as 6 plus 6, then I have M_{cd} has 6 plus 6, so this is going to become equal to 24EI by 225 delta and then, I have 22 minus 44, that is minus 22, minus 22 minus 300, so this is going to be equal to minus 322.2 is equal to 0. Now, I have three equations. I will just put it down finally in an equation format.

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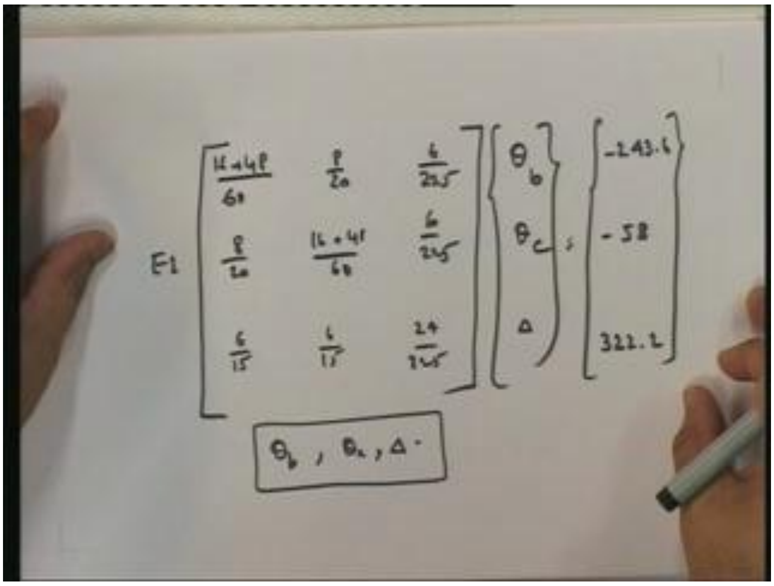

We look at it in an equation format. This is going to be $theta_b$, $theta_c$, delta and on the other side, I am going to have the values. It is going to be minus 243.6, minus 58, plus 322.2 and in here, you will have all the expressions; for example, in this plus this, (Refer Slide Time: 54:53) I can take EI outside and so, it will be 4 by 15, plus (16 by 20), 8 by 20, 6 by 225. So, 0.8 plus this thing gives me 380 and this is going to be 16 by 15. So, this is by 60, this is 16 plus 48 (Refer Slide Time: 55:25), so 64 by 60; this is going to be 8 by 20, this is going to be 6 by 225, this is going to be 8 by 20, this is going to be again the same 16 plus 48 upon 60, this is going to be 6 by 225, this is going to be equal to 6 by 15, this is going to be 6 by 15 and this is going to be 24 upon 225.

If you solve for these equations, you can get $theta_b$, $theta_c$ and delta. I am not going to go through these details, I will leave it up to you to solve it. Once you get $theta_b$, $theta_c$ and delta, you can find out these values (Refer Slide Time: 56:41) and once you find out these values, you can always draw the bending moment diagram for the given frame. The only difference that we had in a frame was that in a beam, you only have rotations as degrees of freedom but in a frame, in addition to rotational degrees of freedom, you also have displacement degrees of freedom and therefore, corresponding to displacement, when you take equilibrium corresponding to rotations, you take a moment equilibrium equation and corresponding to a displacement, you take a force equilibrium equation – that is how you go about solving it. I am going to spend some more time over the next couple of lectures looking at a few more problems and seeing how we can solve those.

I hope I have been able to explain a little bit over the last two lectures on how to use the displacement method to solve beam and frame problems. Thank you.