Structural Dynamics

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Lecture – 9
Dynamic Characteristics and Periodic Loading

Hello again, last time you know last few lectures, I have been looking at harmonic response of single degree of freedom systems and then we looked at you know how we can apply that to practical problems where we have force transmission, and you know vibration transmission and we saw that the transmissibility ratio could be used etcetera, etcetera, and I have solved a couple of problems on that. Now we are going to move on and what we are going to be talking about in this lecture are two things. One is about the concept of how to compute dynamic characteristics from the perspective of from the perspective of harmonic response. Let us not let me go back and look at how we can compute the dynamic characteristics. What are the dynamic characteristics? Omega and xi.

How can we estimate these parameters from measurements? That is what I am going to be looking at and essentially if you look at that, then what we are going to be looking at is, there we may going to look at response to periodic loading. What is periodic loading? Periodic loading look at harmonic loading. Is not it periodic? It is periodic. In other words, the harmonic load repeats itself after every instant of time which is given by t bar. t bar is what? excitation time period. So every t bar, it repeats itself. So in other words, if it be the sinusoidal wave, go like this and in t bar, it would go through one cycle and then the next one again is the same. So this is what a periodic loading is and this is what a harmonic load is and that is what a periodic loading is. A periodic loading is that every instant of time. Now the question is that look it is not harmonic.

In other words, it is not describing a sinusoidal or a cosine function or whatever any other trigonometric function it is not; however, what it is doing is, it is come here repeats itself that is called periodic loading. In other words, whatever is the form of the load in the during t bar is repeated after t bar. So in other words, from t bar to 2 t bar is the same load 2 t bar to three t bar is the same load and that is how that is what a periodic loading is. So now let us look at dynamic loads, dynamic characteristics. How to measure dynamic characteristics?

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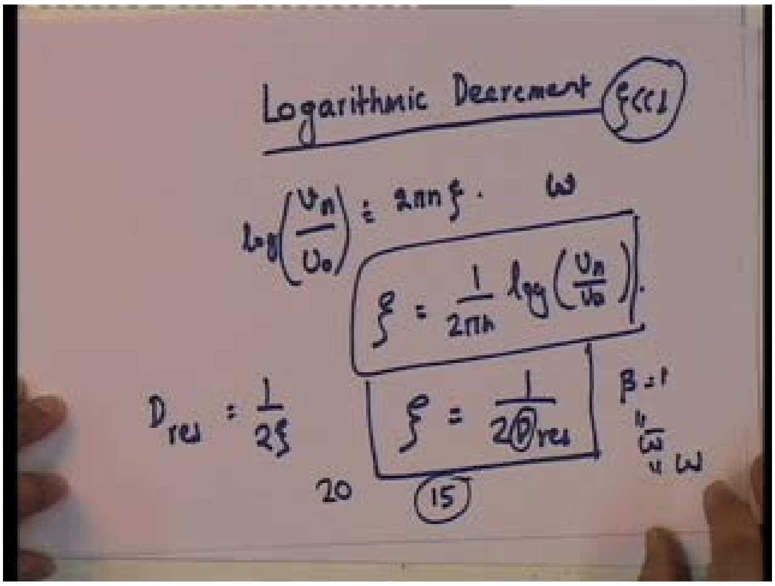

Think back, we saw that remember the logarithmic decrement. What was that? That was that v n upon v 0 is equal to 2 pi n xi. So therefore, xi can be obtained as 1 upon 2 pi lambda. It is approximately, because we are actually you know there is the omega upon omega d which we are neglecting.

It is going to be equal to log v n upon v 0 that was that we solved from free vibration. So you could estimate xi from that. We also saw another way, did not we? We saw that if you plotted the dynamic amplification, what was D at resonance? D at resonance was 1 upon 2 pi. So, xi could be obtained technically as 1 upon 2 D resonance. We could get omega here by looking at what is the frequency of the free vibration. That would give us omega because especially all of this is valid, when xi is very much less than 1 which is what it is in real structures. So that is why you would actually get omega d, but omega d is really the same as omega xi less than 1. So we can get omega from the free vibration and xi we can get from logarithmic decrement.

Then similarly, if we look at beta equal to 1 and we look at what is the value of omega bar. That omega bar value would give us omega and xi we get us 1 upon 2 d. So, if we get the d dynamic amplification factor, we could do it this way. The problems with both these methods are that the logarithmic decrement typically is for free vibration and in free vibration, it is very hard think of a building. How can you hit it hard enough for you to get a signal that where the signal to noise ratio is sufficient for you to be able to measure it. We will be going into by the way we will be going into the lab and I will show you with lab models how all of these kinds of things will be used some way along this lecture, may be after this lecture, we will go to that lab and we will have a look at what all of these things are. So we will actually solve real problems at that particular point of time. The harmonic you know the excitation when you actually exciting, excite a structure harmonically that is a load.

A load can be sufficiently large enough to provide significant signal. So therefore, this method which is you know xi upon 2 D res and omega bar is equal to omega seems to be a very good approach. The problem here is that in real life, when you have very small xi, the D becomes very very large and when you have very large dynamic amplification and you are exciting a system and it starts rotating. When you as you come close to this thing, this starts doing this and if it starts doing this, you get a feed back into the system and you essentially you know you start what is known as a feedback loop and you cannot really maintain resonance in real life. Because you start hearing, we will go back to we will see in the lab and you will hear a huge combustion and we have to really move fast through that.

So if you move fast through it, you know getting the value of d becomes a problem. For example, let us say it is 20 and you know we could not you know 20, it is impossible. So if we go so fast through it, the peak value we get is 15. Think of the error in xi value. If you use twenty, it becomes what 1 upon 40. So that is 2 point 5, 2.5 percent damping whereas, if it is fifteen, what is it becomes? It becomes 3.3 percent damping you have how much? you have almost a 33 percent error in the estimation of xi purely because of this problem that you could not exactly get the value at presence. So both these methods, actually although in principle, they are very good. In reality, they are not very good and that is the reason why we tend to use what is known as the half power bandwidth method.

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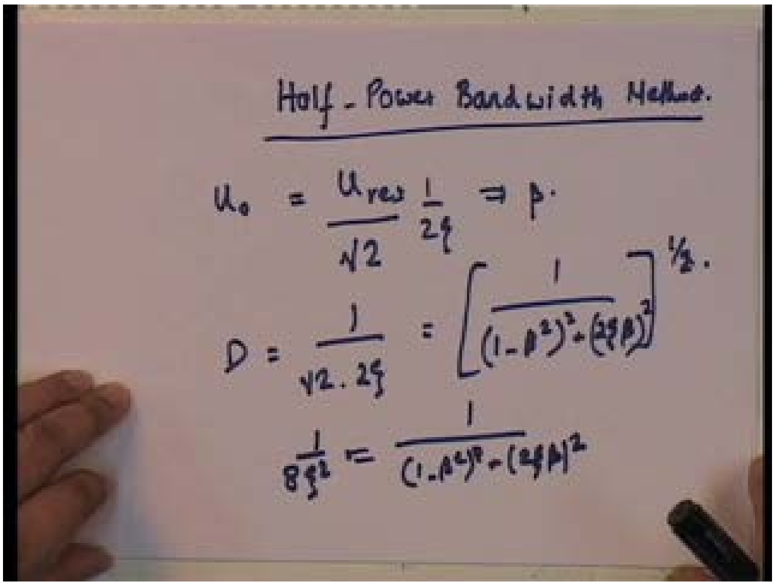

This is known as half power bandwidth method. What exactly is this method? If you look at the response, you know dynamic amplification of the structure.

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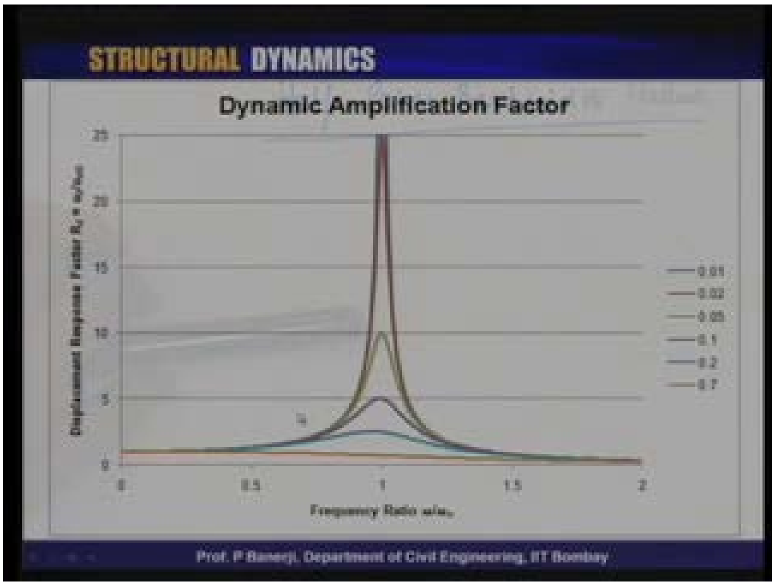

So look at it, I mean if you are 2 percent, you go so high that you really cannot get here. But let us look at a situation where we act we are at half power. Half power essentially means that u 0 which is you know, peak amplitude is actually equal to u at resonance up on root 2 because if displacement is root 2, then the power which is the square of the displacement is half power. That is the reason why we call it half power bandwidth. So essentially what I am looking at is, I want a situation and I want to find out the beta at this value. Note, if you have 25, 25 over root 2 would be approximately in this region.

So if you look, if you look at this region, you can actually you are able to generate this value and note that how do we get this value? Well very easy. Because what we are saying is that this is you know 1 upon root 2 u res is equal to 1 upon 2 xi. So this is equal to your dynamic amplification is one over root 2 into 2 xi which is equal to square root of 1 upon 1 minus beta square plus 2 xi beta the whole squared and square root. So if I do that, this becomes then and take square of both sides, what I get is, you know 1 upon 8 xi squared is equal to 1 upon 1 minus beta squared plus 2 zeta beta.

And now you know, I mean this basically if you look at it, it becomes a quadratic in beta squared and I am not going through the entire process here because it is a fairly trivial and you people can get it, if what I get is that this one can be solved for two values of beta and these can be approximately written as 1 minus zeta and approximately as 1 plus zeta. What I am neglecting in this entire scheme of things? Well, actually if you look at it this, this comes as this.

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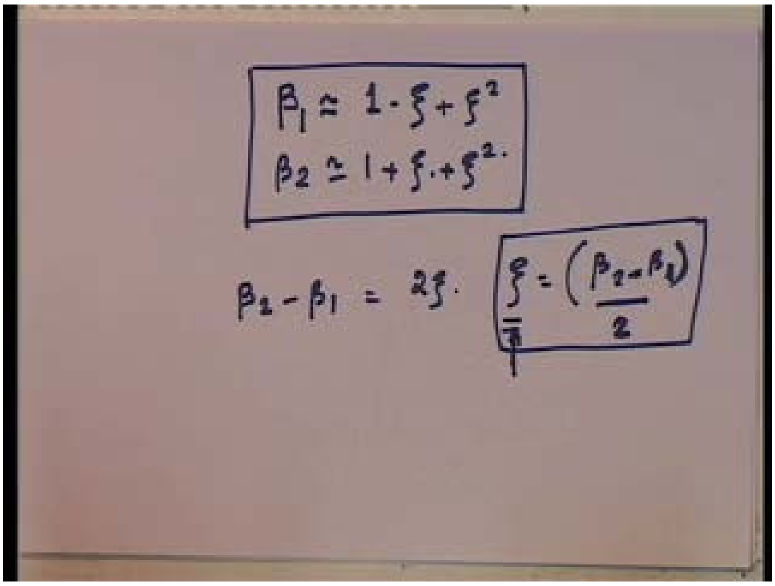

Here, what I am assuming doing is that, if you have one upon you know 2 xi squared 1 square root of 1 plus 2 xi squared. I have neglected this xi square term because xi is very much less than 1 and so that becomes square root of 1 plus 2 xi square I take it as 1 and then assume the fact that xi's are small and so therefore, you know you know that 1 plus x square root where x is very small essentially is 1 plus x upon 2. The second order terms are all neglected. So doing all of those with those kind of approximations built in this is what we get as the 2 values.

What are these? beta 1 is the value on this side less than 1 and beta 2 you see, there are two values note that that is why you have two values. There are two values one which is on this side and one which is on this side of one. So 1 beta 1 is less than 1 and the other one, note that xi square is always smaller than xi. So this is actually less than 1 and this one is greater than 1. So if I do beta 2 minus beta 1 which is really the bandwidth. So in other words, you understand by the half power bandwidth come from. At half power, what is the frequency bandwidth? which is beta 2 minus beta 1. If you look at this, what do you get? You get it equal to 2 xi. In other words, xi is equal to beta 2 minus beta 1 upon 2.

This is the known as getting the damping from the half power bandwidth and let us look at it, if you look at this, there are how many values 1 2 and if you really look at it, you know bandwidth even at 1 and 2. If you look at these, the bandwidth is fairly stable. It does not depend on where it goes. If you look at this 1 percent and 2 percent, the bandwidth is almost the same. In other words, the point that I am trying to make is, that the estimation of xi from bandwidth becomes much more stable and the error in xi from the half power bandwidth method, half power bandwidth. This is, these are the frequencies at which the half power the power becomes half and so the bandwidth which is beta 2 minus beta 1 gives you the bandwidth.

So that is why the name half power bandwidth method comes from and this gives a very very stable xi and also because you are at half power, you can actually do not get sufficient feedback. So you can estimate that those values fairly accurately. So those values can be measured fairly accurately and getting the xi from there is a very very easy procedure. So, so much for estimation of dynamic characteristics, we are now going to move on to periodic loading. Let’s look at a periodic load. What is a periodic load look like and again over in periodic loading, we are going to assume, we are not going to take initial conditions. We are going to assume that the periodic loading has lasted for very long time and it will last for very long time.

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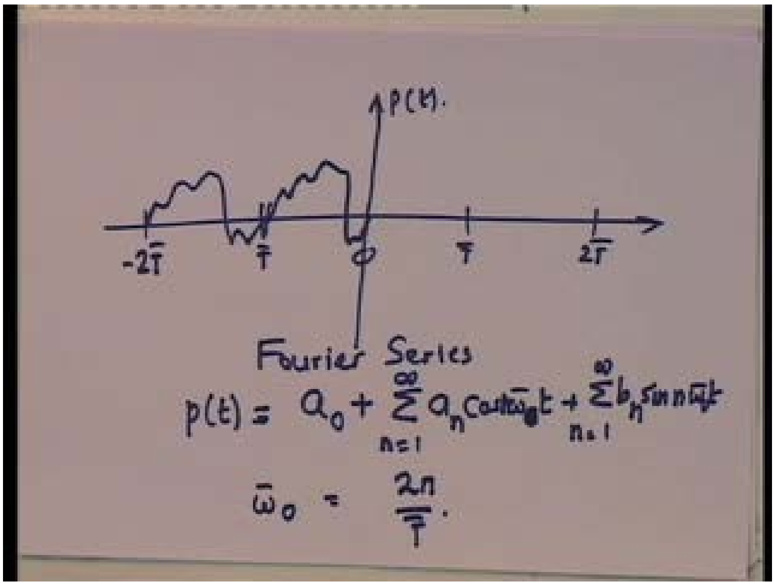

So in other words, my time t equal to 0 is really more a kind of a thing that I do rather than anything else and so I will say that well, I am going to kind of say that it repeats. So this is minus 2 t bar t bar 0 t bar 2 t bar. So and this is a load something some load. It repeats itself, it just keeps repeating itself. Now, when you have a periodic load, it is very interesting and that is that. So this is p of t. The load p of t actually can be written in a Fourier series and the Fourier series is of the following form a 0.

So this is known as a Fourier series. This loading can be written in terms of summation n going from 1 to 0 a n cosine omega n t plus sorry omega n omega t and b n sin n omega bar 0 t. This again going from 1 to infinity. This is a infinite Fourier series and any periodic loading can be written in this form. This comes from mathematics I am not able to deal with it. It can be written in this form where what is omega bar 0? Omega bar 0 is nothing but 2 pi upon t. So omega bar 0, the lowest frequency is really this is the time period. So this is the time period 2 pi upon time gives me the lowest frequency and so if you look at it, what is this become? A periodic loading can be written as a summation apart from a term, which is a 0. I will come to what a 0 is.

It can be written as a summation of infinite number of trigonometric functions where it the trigonometric, if you look at the frequency of the trigonometric functions, there are multiple frequencies. The lowest frequency is the 1 that is given by the time period t bar. 2 pi upon t bar is the omega bar 0 and every other harmonic is a multiple of that harmonic. So that is how the thing is and of course, we need to know what a 0 a n and b n are those I have to be defined and those are defined as a 0 is equal to 0.

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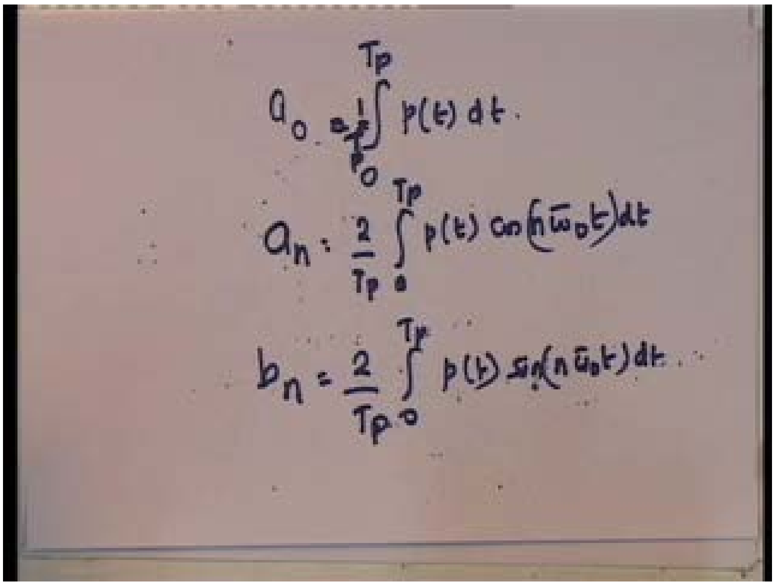

I am going to call this a little bit call it periodic loading. So instead of anything else, instead of a calling it t, I will call it t p or call this t p minus t p just to keep it. So this becomes t p. This is the notation. This becomes 0 to t p, p of t d t. What is this term? think of it. This is what? 1 upon t p there a 1 upon t p is outside. What is this term? Look at it. What is the integral the area under the curve, area under the load curve divided by the duration of the loading because it is periodic. So I take it from 0 to t p and I am dividing by that period. What is that?

This is the average value of the loading. The mean value of the loading is this. a 0 is directly the mean value. a n is equal to 2 upon t p 0 to t p p t cosine n omega 0 t d t, b n is equal to 2 upon t p 0 to t p p t sin n omega 0 t d t. So these are you're a 0. So these are in terms of p n and you know p t. So you can always evaluate a 0 a n and b n given any load. Now, what advantage do I get by taking p t to be a Fourier series? Well, let us see. Let me look at the response of a structure to a periodic load.

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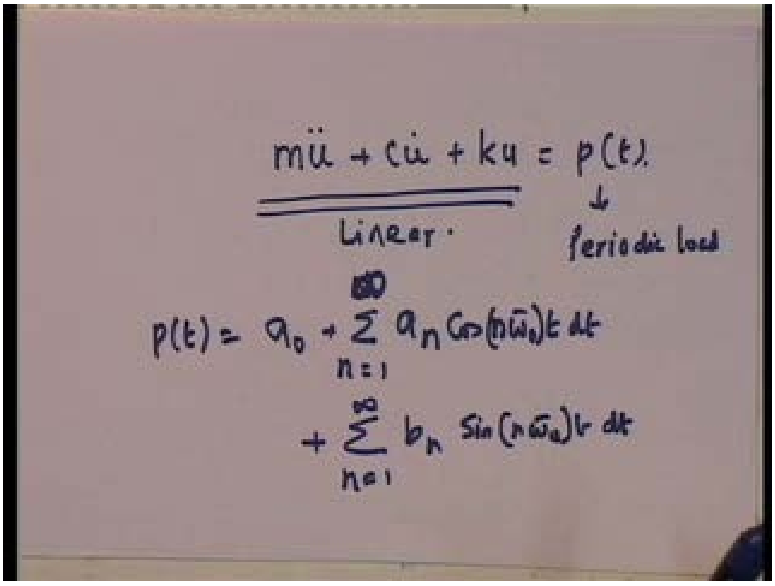

So in other words, what am I doing? I am going to say that well I have a periodic load. I know and I would not look at the response of a single degree of freedom system to this load now, where this is a periodic load. Now, if this is a periodic load and I want to find the response for it, note that I only need to find out the particular solution, a steady state solution because you know, this load lasted for ever. So you do not even know the initial boundary conditions were. Well, the initial boundary conditions they have been the transient is gone so. So therefore, we will look at the solution.

Now, p of t is given as a 0 summation n going from 1 to infinity a n cosine n omega 0 t d t plus this is p of t because it is periodic. We have already seen that we know how to calculate a 0, a n and b n. So now, look at this system. This represents a linear system. It is linear. If it is linear, remember something that principle of superposition is valid. If principle of superposition is valid, then I can say that look, the load is made up of various terms.

What I need to do is, I need to look at the response to a particular load, and then for I need to find out the response to each load and then if I add up the responses to each load, then I get the response to p of t. p of t consists of many loads, a 0 is 1 load. You know, a 1 cosine omega 0 t is another load. a 2 cosine 2 omega bar 0 t is another load. So these are all you know. So I am looking at a periodic load as a summation of a constant load plus many harmonics. So I need to only find out the solution, the response to each load and then I can just add them up. So let us look at the response to each load.

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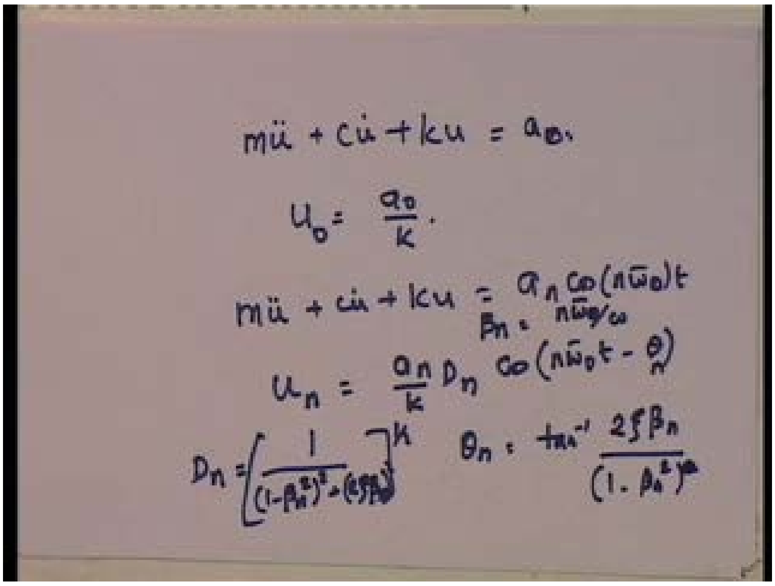

m u double dot plus c u dot plus k u is equal to a 0 and I want only the, remember steady state solution is a particular solution. What is the particular solution? U. Particular solution is a 0 upon k. Similarly, m u double plus c u dot is equal to a n cosine m omega 0 t .What is the solution to this? I know the solution to this. Note that, the solution to this is the following u. So this is u 0. u n is equal to a n upon k into d n. I will write down what d n is into cosine n omega 0 t minus theta I know that. What is d n? d n is equal to now here, I will define a term. What is beta n? beta n is equal to n omega 0 which is the excitation frequency divided by omega bar that is beta n. So d n is equal to 1 minus beta n squared plus 2 zeta beta and the whole squared square root.

Theta n is equal to tan inverse of 2 zeta beta n upon 1 minus beta n squared the whole I am sorry this is beta n square. So since I know omega 0 and I know omega. Omega is equal to square root of k upon n. I know everything. I can find it out. I can find out here. Similarly, the final one is the solution to the sin and the solution to the sin is what?

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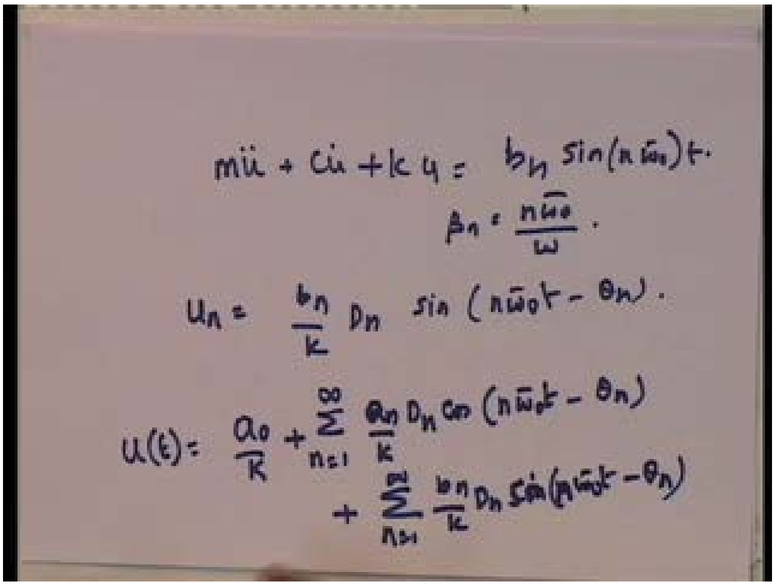

m u double dot plus c u dot plus k u is equal to b n into sin n omega 0 t. Again taking beta n to be the same value n omega 0 upon omega, my u n turns out to be equal to b n upon k into d n sin n omega 0 t minus theta n, but d n and theta n are exactly the same as given here. This is same we showed that. So now, what is my u of t equal to? It is basically a summation of these loads.

So u of t is equal to a 0 upon k plus summation n going from 1 to infinity a n upon k into d n cosine n omega 0 minus theta n plus summation n going from 1 to infinity b n upon k into d n sin n omega 0 t minus theta n. So that is what we get, that is the solution. So now, understand that if I am going to get p t, I can find out u of t. Now you know, these series functions are seemingly infinite series. So how does it help me to get the solution? Well, let us see what happens and let me take an example problem to show you that indeed in this particular case we do not strictly need to look at an infinite series, but we can just take the first few terms of the series because understand that if p of t, a particular load is given by a series solution, that series of course, has to be a convergent series and for a convergent series, what happens? The a n and b n actually reduce with increasing n. We will do it by looking at a particular problem. So, let us look at the particular problem.

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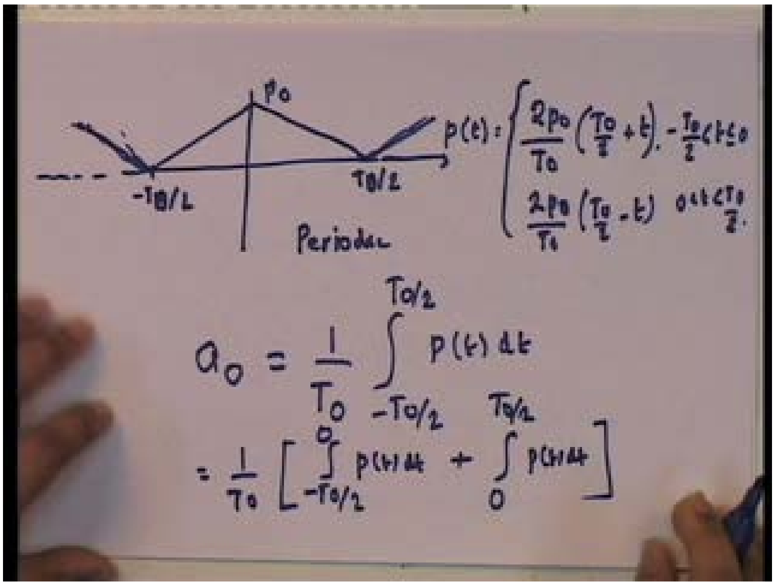

Let me take a specific problem and I am going to only define and note that when it is periodic, remember something, that if it is periodic, where I take my 0? whether I start this at 0 and go to t p by 2 and then t p. This is a triangular function which is being repeated, it is repeated. So whether I start at 0 or whether I start at you know minus t p by 2, it does not matter in this particular case. So let me just define it in this fashion and this is p 0. So, this is the periodic load. Mind you, this is you know it keeps going on and on. So in other words, p of t is defined in this form. It is defined by 2 P naught upon T naught into T naught by 2 plus t.

I am calling this as T naught when I am going from minus T 0 to 0 and it goes from 2 P naught upon T naught T naught by 2 minus t when it goes from 0 to T naught by 2 and these are actually I can look at it as n T naught you know minus n T naught to n minus 1 T naught. Understand this periodic. Please understand that this is not a load. In other words, if I continue with my time, this will be going like this. So it is a periodic load which is continuing and I am looking at only the period and I am defining it for that particular period.

So let us see what do I get? Well let us look at a 0. a 0 is equal to 1 upon T naught since I am going from this to this p t d t. I can take it as 1 upon T naught outside. This integral going from minus T naught to 0 p t d t plus 0 to T naught because they are two different functions p t d t and if I look at this particular thing that you know just do the integrals for those functions.

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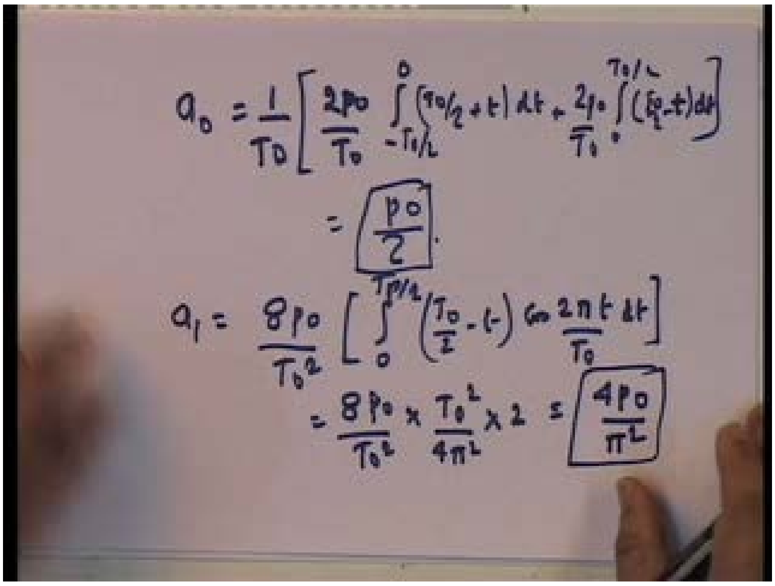

I mean, I can I am going to put it down and this is going to be equal to a 0 is equal to 1 upon T naught into 2 P naught upon T naught integral minus T naught by 2 to 0 of T naught upon 2 plus t d t plus 2 P naught T naught 0 to T naught by 0 T naught upon minus t d t and after doing all of those, this works out to be P naught by 2. So a 0 is P naught by 2 and this indeed, if you look at this, it represents the average value. So that is a naught.

Similarly, I can go through the process for a 1 and I am not going to write down everything. I will just write it down since it is note something that since it is actually symmetrical, I can actually do go from here to here for cosine and then I will go ahead and do it and so this is going to be equal to 8 p naught upon t 0 squared 0 to t p by 2 because note that this is the even function, this is the even function, cosine is the even function. So, when you have an even function and even function, what happens? You can just do this into this and this becomes just twice of that. So that is what I have done. So I have just taken one of them and just done twice the other.

So I get T naught upon 2 minus t into cosine 2 pi t upon T naught because that is omega naught 2 pi upon T naught is omega naught into d t and if I go through this process and go through it by paths, you know I am sure you people know how to integrate. It is going to be 8 P naught T naught squared. If you do not know how to integrate, just go through the process T naught squared upon 4 pi squared into 2.So this becomes essentially 4 P naught upon pi squared that is a 1.

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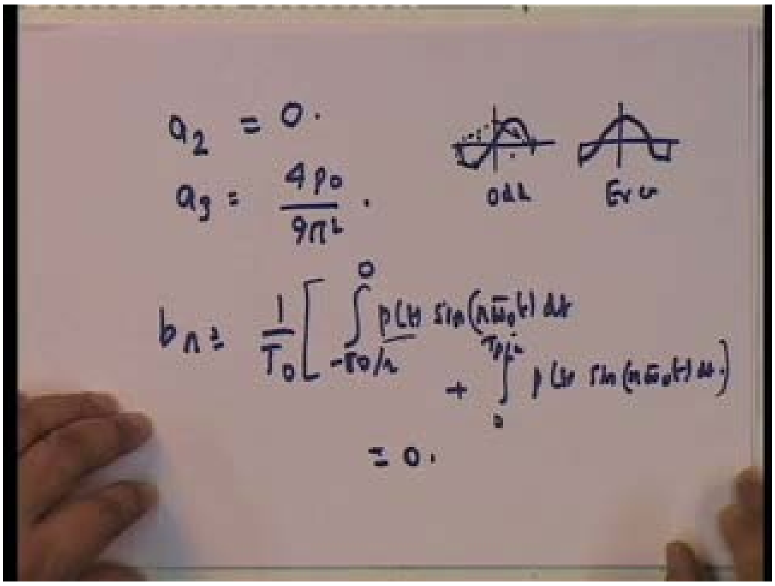

In exactly the same way, we can show that a 2 is going to be equal to 0, a 3 is going to be 4 P 0 upon 9 P naught. So if I extrapolate from a 1 a 2, what I get is that and note something very interesting and that is that how do I get b n? b n is equal to 1 upon T 0 minus T 0 by 2 p of t sin n omega 0 t d t plus 0 to T p by 2 p t into sin n omega d t. Now note, that p t we saw was an even function and sin is an odd function. It is an odd function why is it an odd function, because sin is of this form. This is how sin goes cosine goes how cosine goes. So cosine is even, this is an odd function. If you multiply by odd function with an even function, note look at this, this one is this way, this one is this way.

So, this 1 from 0 to t p, if it is being multiplied, see this is nothing but the product and its area under the curve. So, if you take this product and take the area under the curve and you take this product and take the area under the curve, note that this one turns out to be negative of this one; obviously, so when you do this, this is negative of this. So the total one turns out to be 0. So all b n are 0 and similarly, one can show that all even terms are going to be equal to 0 for cosine.

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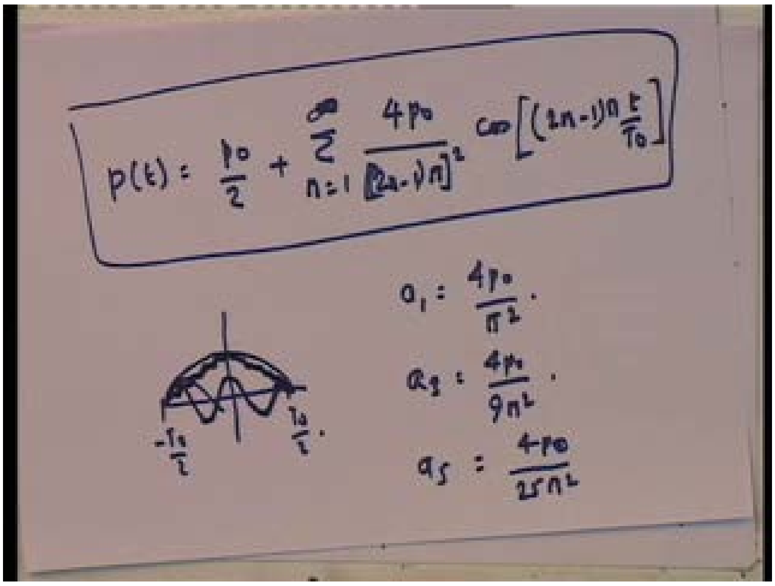

So now, ultimately what we are left with is the following and that is that p of t is equal to p naught by 2 plus n going from one to infinity 4 p 0 2 n minus 1. Why am doing two n minus 1 to get the odd values only. Remember that I am doing all throughout. So this way I will only get the odd values because even values are 0 and so this one also becomes 2 n minus 1 pi t upon T naught and of course, the other one sin term does not come in. So this is the representation of the saw tooth load and note something very interesting and that is, that a 1 was 4 p 0 upon pi, a 3 is equal to 4 p 0 upon 9, a 5, if you do it, you will see that this is equal to 25. Note the values. If this is one, this is 1 upon 9, 1 upon 25. What is it going down as? 1 upon n squared and that is what it is.

It is going on as 1 upon n squared; obviously, that is a convergent series and; obviously, the point is that we do not need to consider all the terms to bring into ensure that we get some amount the first one if I take only the first one, I get something like this. If I take the third one, I will get. So this superposes on this and as you keep superposing, you will get something. So in other words, if I just took five terms, it would like this rather than. So it approximates it close enough. Now let us look at the response of this to I mean response to this p t, we will look at the response to this p t. Then what do we get?

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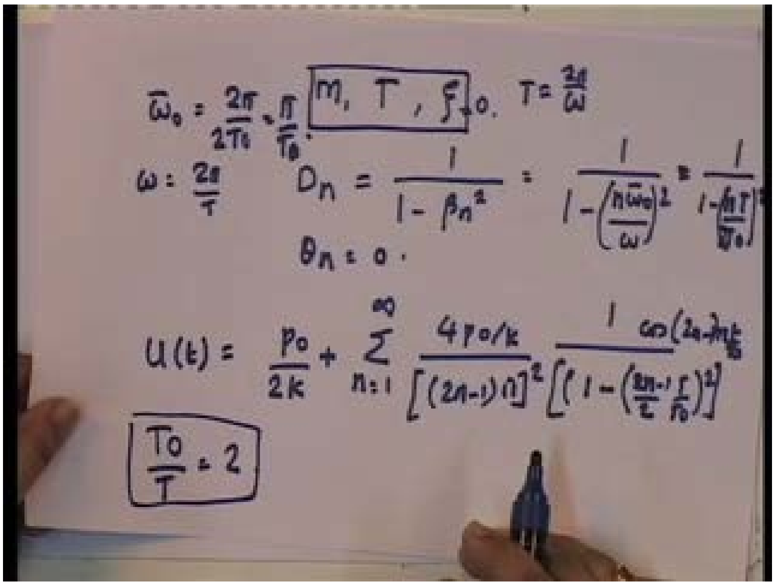

Well, let us look at it, I am going to take l k m k etcetera and with t as the time period. So I am defining the structure by its M T and xi. These three define everything. They define M, K, C everything. So this is what I am defining. But T is nothing but time period. So I am giving time period. So this represents the single degree of freedom structure. Well let me find out the response to this. Well, the loading is given in this fashion. So all I need to do is, take out how it is going to be and what I am going to do is, I am going to take xi equal to zero for now undamped system and show some numbers. So if I look at the solution, then what does D n becomes? D n becomes 1 minus beta n square which is nothing but 1 upon 1 minus. Now this is n omega bar 0 upon omega. Now what is omega bar 0 in terms of T 0? It is equal to 2 pi upon T 0. What is omega in terms of T? It is 2 pi upon T.

So if I substitute this in, what do I get? I am sorry this is squared. What do I get? This I am going to get equal to 1 upon 1 minus n T upon T 0 the whole squared. So now, given that fact I am sorry omega 0 is not 2 pi. What did I get? I just want to go back to that particular if you look at this. So this is going to be equal to 2 n. The omega 0, the first frequency which basically is this one actually is 2 T 0. So it is 2 T 0. So it is actually n by 2 T 0 omega 0 is this is 2 T. It is actually pi upon T 0 because T 0 only represents half. It only represents half of that. So therefore, if you look at it, we get it equal to the following.

So D n is this so and then you know theta n is 0, if xi equal to 0.So therefore, the u of t turns out to be equal to P naught upon 2 k that is P naught upon 2 k plus summation going from 1 to infinity 4 P naught upon k upon 2 n minus 1 pi squared into 1 upon 1 minus 2 n minus 1 upon 2 T upon T naught and over here, cosine 2 n minus 1 pi over t upon T 0. So this is the solution. Now let us look at some issues. We know that, you know, I am going to take the situation where T naught upon T is equal to 2. Let us take the specific situation of this and try to see what kind of a solution we get for this particular equation. So if we plug in, let us plug in t upon t naught 2, what we get that?

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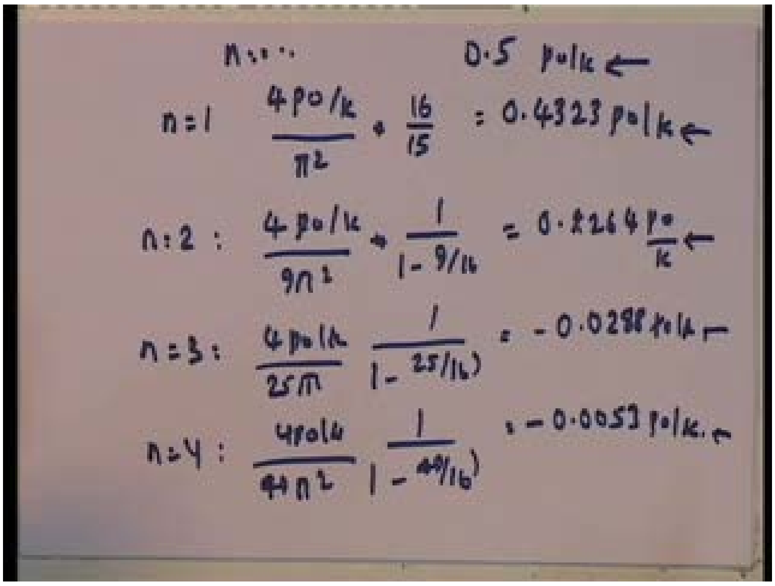

We get that the first term n equal to 1 is going to be equal to 4 P naught upon k into pi square upon into well now you need to do this that this is 2 n minus 1. So this becomes pi and then you need to do this one also. So that becomes 16 upon 15. So if I put that in, I get it equal to 0.4323 P naught upon k. n equal to 2, 4 P naught upon k upon 9 pi square into 1 upon 1 minus 9 by 16. So this becomes 0.9264 P naught upon k. n equal to 3, 4 P naught upon k upon 25 pi square 1 minus 25 upon 16. So this becomes minus 0.0228 P naught k. n equal to 4, by the time you get into 4, you get 4 P naught upon k into 49 pi square 1 upon 49 by 16. So this becomes – 0.0053 P naught upon k and n equal to 0 is what? It is equal to 0.5 P naught by k.

So you see, what we are saying is the following and that is that if you look at this solution, this is a 0. This is the u one, the response that we are looking at only the amplitude. So this is the second one, this is the third one, this is the fourth one. See where are we? By the time we reach the forth one, we are so low amplitude that it does not matter; whether it adds on or not. Even for practical purposes, if you look at these three, this one is almost negligible.

So in other words, you see, this is a peculiar situation in which just the first two terms, first two cosine terms and the average term alone represent the response very very well and note, that in this particular case, beta is actually you know is 2. So therefore, you know the third one, n equal to two terms actually gives a larger value of the amplitude than the first term. But typically, the frequencies are of the order of the same as that the periodic loading frequencies of the same order as the loading and therefore, pretty much what happens is, you know a 1 is the largest amplitude a 2, a 3, a 4 and by the time you go to a 4, you do not really need to worry about it.

So all you need to worry about are two things. You need to find out what is the fundamental frequency of the periodic load. That is omega 0 and you need to find out the fundamental period of the structure. If whatever n omega 0 n omega 0 bar upon omega, you remain the significant value. We have to consider that many terms for us to get an accurate estimation of the response to a periodic load. So therefore, you see although you know, you might require to represent the loading, you might require more than 5 or 6 terms in the periodic in the Fourier series to represent it accurately, but you see the response is always you know much fewer terms than you would require in a traditional to represent the loading itself.

So the response actually requires lesser number of terms than the loading itself requires. Loading require to represent the loading properly, you require many terms. But to represent the response to that periodic loading, you do not require that many terms. So that is typically how it works and therefore, the response to periodic load actually although you know it is an infinite Fourier. Fourier series is an infinite series. But in reality, we do not need to consider very many terms. Sometimes the first two or three terms are good enough to represent the response very very accurately. So so much for looking at periodic loading. Next time, we start looking at some very very more arbitrary kinds of loads.

Thank you very much. Bye.