**Structural Dynamics**

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**Lecture - 6**
**Response to Harmonic Loading**

Good morning in the last lecture, we were looking at the response of the single degree of freedom structure to harmonic loading. We will continue looking at the Response to Harmonic loading today, because there are several things that I wish to you know talk more about on this. In the last lecture, I stop with something called the argand diagram, which is essentially a representation of all the forces and the displacement velocity acceleration responses in a complex plane. Because, that gives you an idea of how equilibrium, at any instant of time t is maintained in the structure. So, but before I continue with argand diagram I would like to do a quick review of what we had looked at last time.

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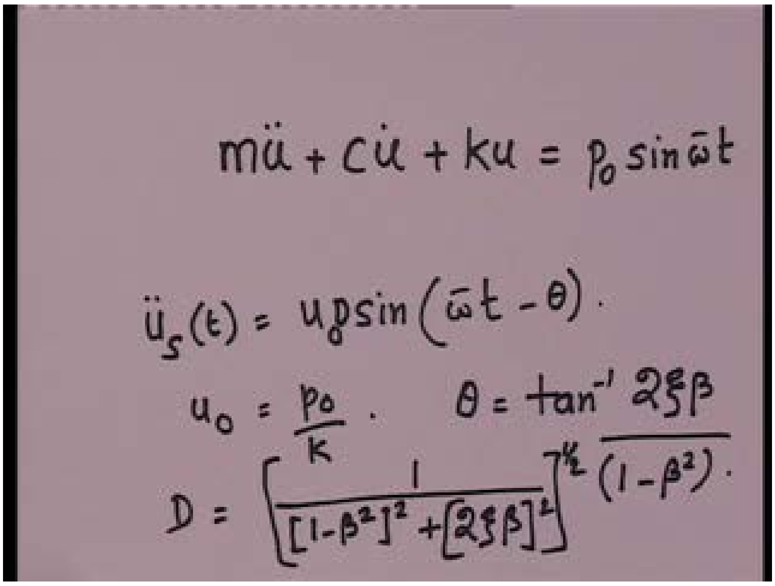


If you remember the equation that we solved, this was equation that we had solved. And what we said was well, this has two parts of homogeneous part and a particular solution and we saw that the homogeneous part was really a free vibration equation solve solution. And the particular part was what we called as the steady state solution and the steady state solution was what we were interest in, because we say that why and large, the transient dies out over a period of time as long as the structure has some amount of damping in it any amount of damping.

It just that the transient part takes longer to die out if the damping is small and it died on quicker, if you are damping is large, thus the only thing. So, therefore we saw that the solution came out to be of this form and there is a steady state solution, steady state solution came out to be, where u 0 is equal to p naught up on k and theta is equal to there is a D here. So, theta is tan inverse of 2 xi beta up on 1 minus beta square and D was equal to 1 up on 1 minus beta square the whole square plus 2 zeta beta the whole square and the entire thing under square root. And if we look at this D and theta, they took up the form as we look at in the next slide.

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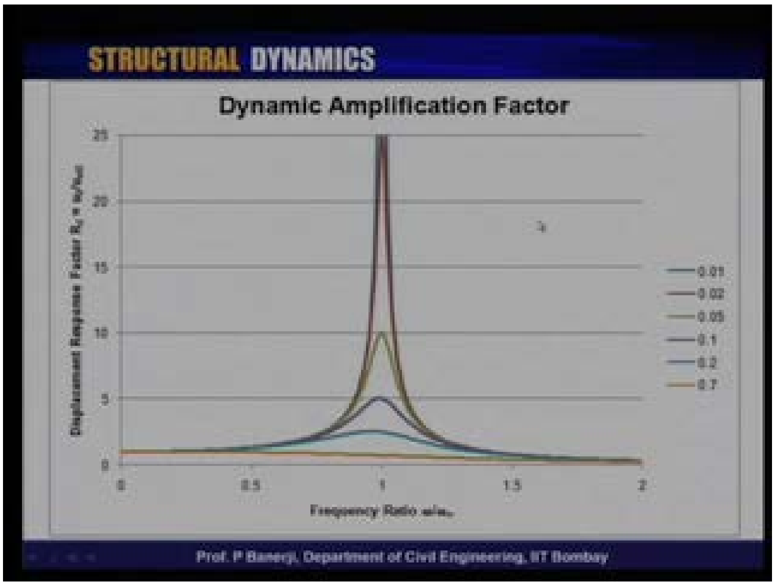


If you look at it, this is the dynamic amplification factor D. And if you look at it these are for different values of damping, this is for 1 percent damping 2, percent damping, 5 percent damping, 10 percent damping, 20 percent and 70 percent. If you look at it, if you look at 1 percent damping it is exceedingly high, for 2 percent damping the dynamic amplification factor quick dynamic amplification factor is 25. And if you look at 5 percent then you have 10 and this again is very obvious.

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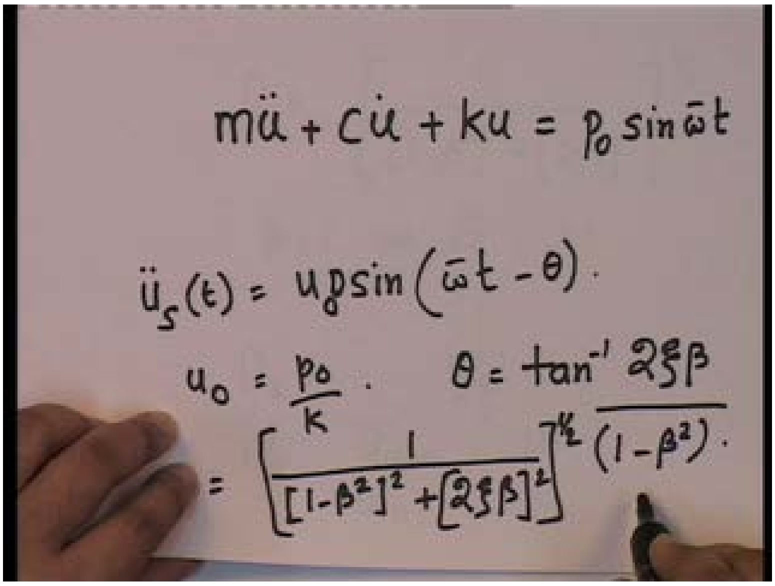


If we look at D is equal to 1 up on square root of 1 minus beta square plus 2 xi beta the whole square, square root. So, if I put beta equal to 1 which is, what we call as resonance where the excitation frequency is equal to the natural frequency of the structure, we substitute that D turns out to be equal to 1 up on 2 xi. And so if xi is equal to 1 percent D max by the way this is D resonance. So, it is not D max D resonance is equal to let us look at it 2 into 0.01. So, that it is 50 for xi equal to 5 percent D resonance is equal to 0.05, so 10.

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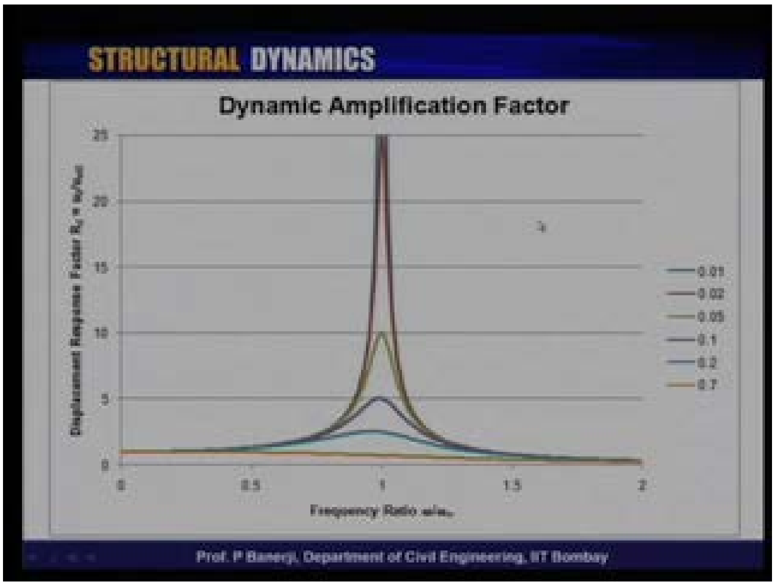

So, if you look back at this at the slide, you will see that at resonance this is equal to 10. Similarly, if you look at 20 it is equal to 10 percent it is 5 and 25, so this is how your dynamic amplification factor looks and if you look at the next slide this is nothing but this tan inverse 2 zeta beta up on minus this thing. And if you look at remember last time I talked about it and we had said that this theta.

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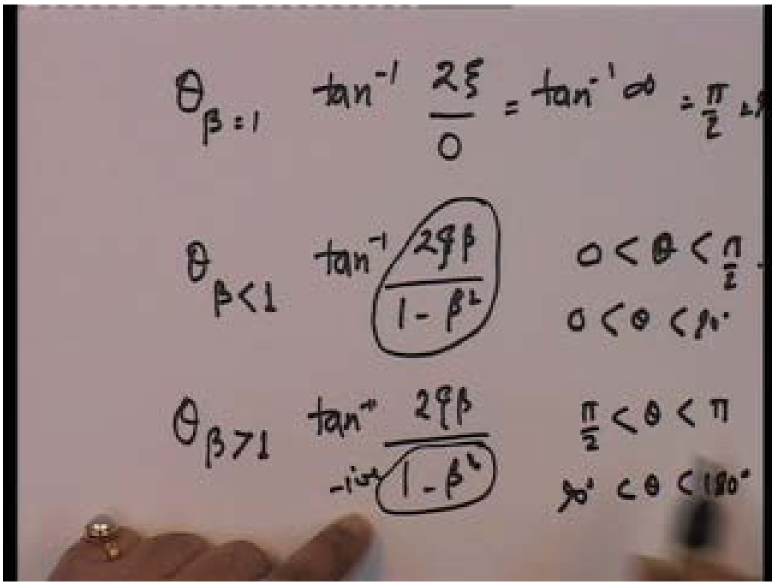

If you looked at theta beta equal to 1 this becomes tan inverse of 2 xi up on 0, so that tan inverse of infinity which is pi over 2 or 90 degrees.

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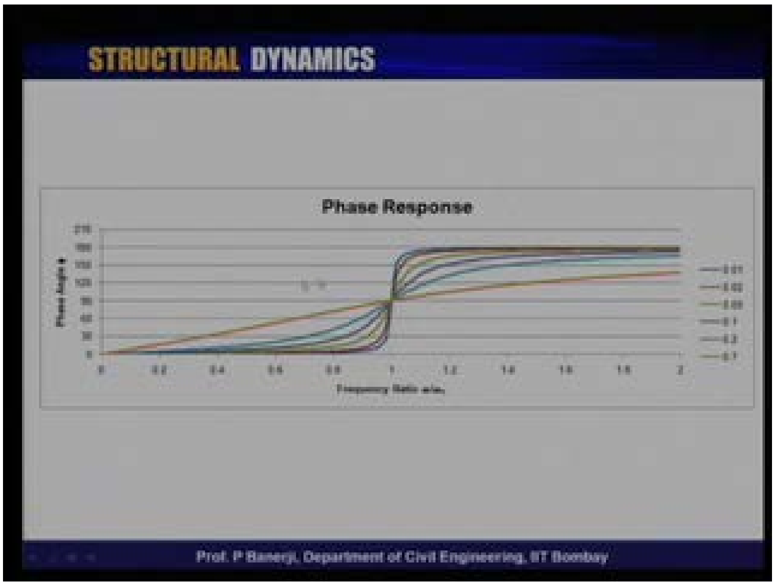

So, the thing that we saw, was that irrespective of what is the damping, at the resonance the phase angle is pi by 2 or 90 degrees ((Refer Time: 08:01)). And we also saw that if for theta beta less than 1, this is tan inverse of 2 zeta beta minus 1 minus beta square, this term is positive for beta less than 1. And therefore, if this is positive then theta is between 0 and pi by 2 or in degrees 0 and 90 degrees.

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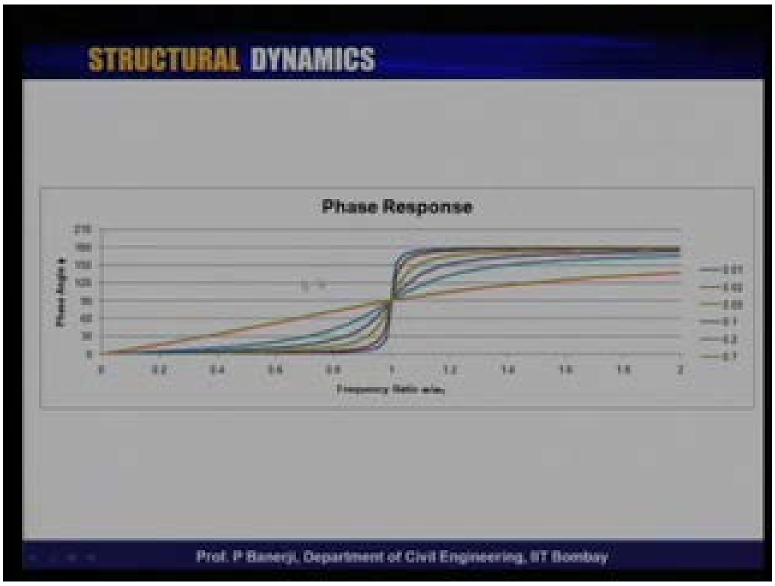


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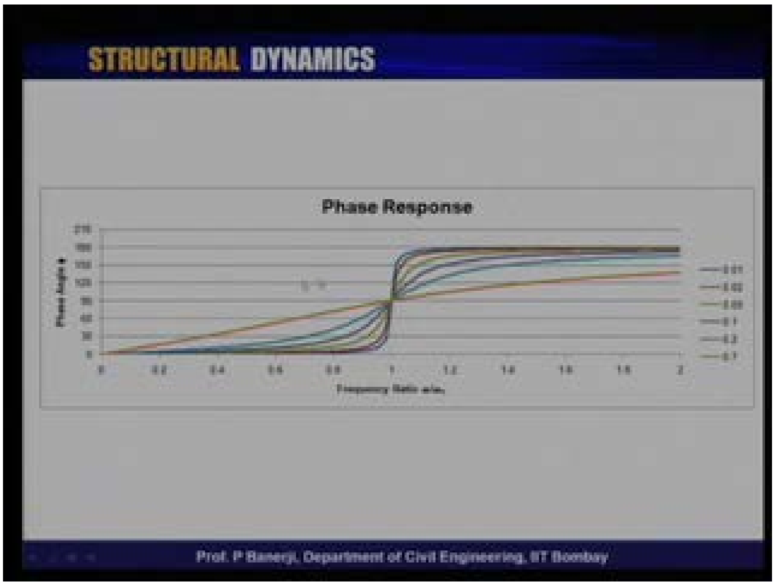


And this is what you see, in this that if beta is less than 1 the phase angle is less than 90 degrees between 0 and 90 degrees, at beta equal to 0 of course, it is 0 by 1, so theta is 0 irrespective of any by this thing. And please also note that seems this is tan inverse of 2 xi beta up on 1 minus beta square, if xi is small, irrespective of beta if xi is small.

This value will remain small for a larger value of beta, till it is only jumps up to 90 if the xi is larger, than it starts growing larger. So, you see for a particular value of beta if you look at it at 0.8 if xi is equal to 1 percent it is almost 0, xi equal to 2 percent almost 0, xi equal to 5 percent well it is about 10 degrees, xi equal to 10 percent it is about 20 degrees, 25 degrees and so on and so forth. So, the point here is that the phase angle when better less is between 0 and 90, when theta is greater than 1 then tan inverse 2 xi beta 1 minus beta square. Note that this term becomes negative, this term becomes negative then theta basically goes between pi by 2 into second quadrant pi by 2 to pi or between 90 degrees and 180 degrees.

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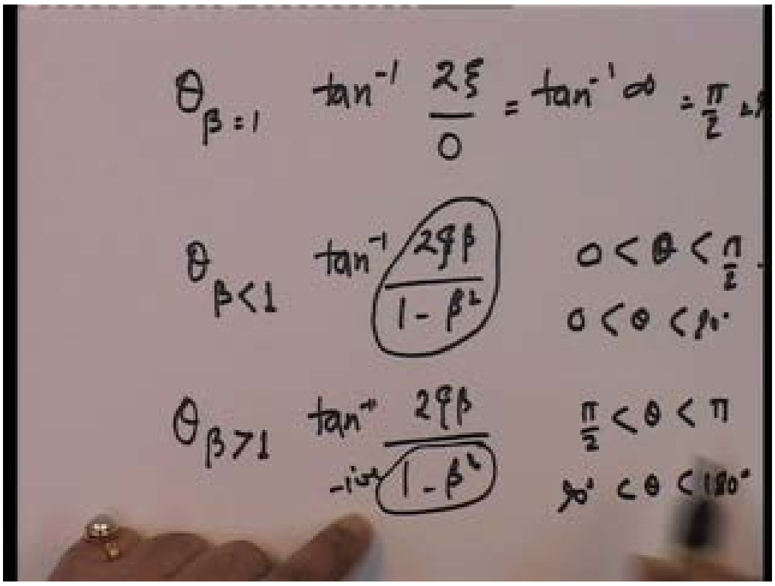


So, that what happens that if beta become greater than 1 then this aspect essentially becomes that, it goes beyond 90 degrees and if you take 1.2 for instance you will see that, all of them are between 90 and 180 degrees or between pi by 2 and pi. And the point again to note is that, if you have a load if you have a very low damping note, what happens it is 0 till it comes close to 1 it at 1, it is 90 degrees a pi b y 2 and write after it goes beyond, it becomes practically again 180 degrees.

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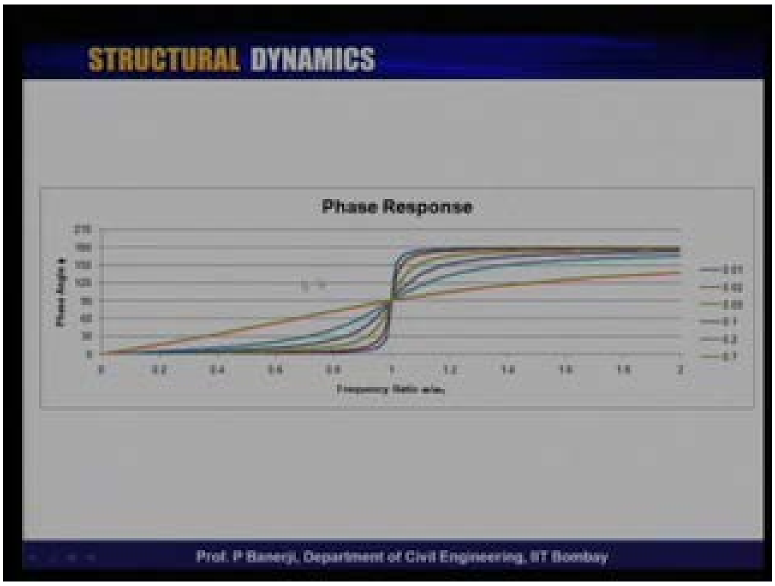


So, it flips and that is understandable because when it flips this is going to almost 0 and if it goes to 0 from the negative side limit of a tan inverse 0, going from the negative side is pi by definition, so that is what you have. So, this phase response is a very important aspect because irrespective damping this phase between ((Refer Time: 12:10)) the steady state and the loading, the phase between this and this becomes 90.

So, if we use these two channel, two jacks one input with the loading and one input with the response it is a steady state response, you will see that essentially these being 90 degrees out of phase, implies that they become circle. So, if you look at it is very easy to see, when resonance occurs and in fact you can find out the natural frequency in that manner, for low damping values is by looking at this x y plot. And the x y plot remains like this, when it is they are in phase.

And so assume a very low damping 2 percent damping structure, it remains like this till its resonance then it becomes like this. And as soon as it crossed resonance it goes back to something like this, so you see the resonance can be picked up very, very easily and we saw that resonance is where the excitation frequency is equal to the natural frequency of the structure. So, this is actually a beautiful way of finding of the natural frequency of a structure.

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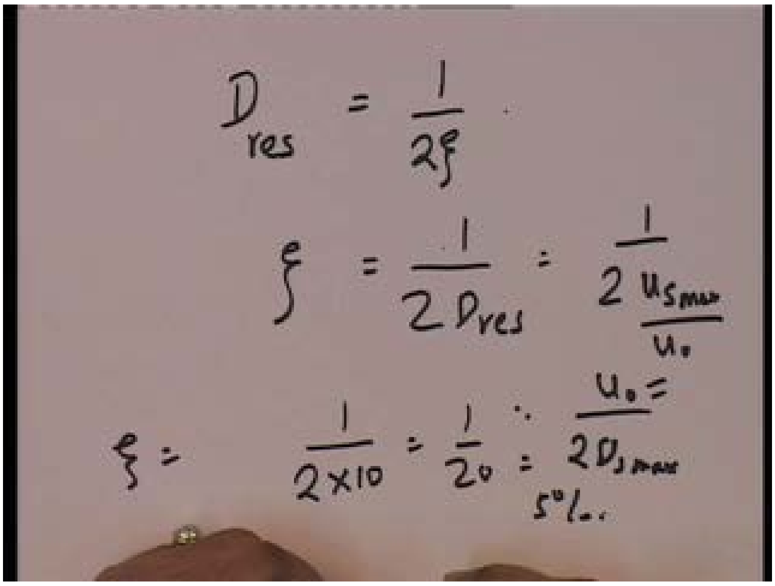

So, now, the other question that becomes is since we saw that at D at resonance is equal to 1 up 2 xi actually you can do xi is equal to 1 up on 2 D at resonance. And what is D, D if you look at, it is nothing but 1 up on 2 the u s max up on u 0, so this basically becomes u 0 up on 2 u s max.

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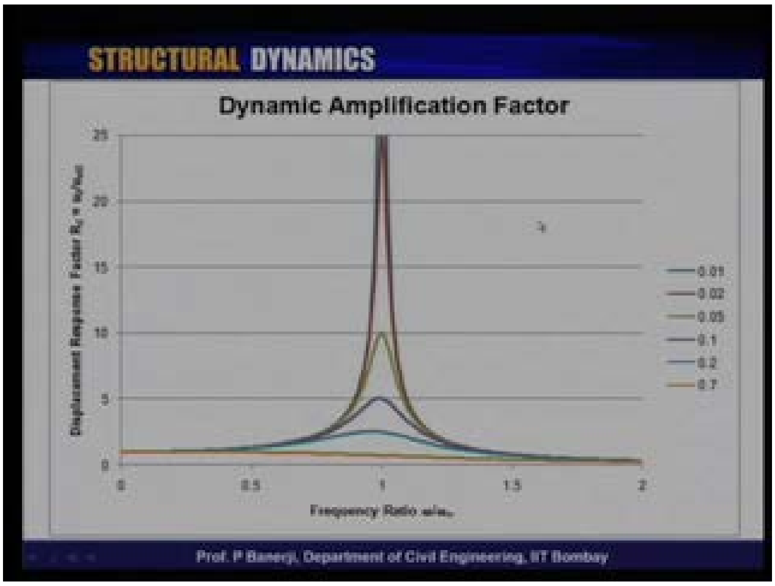


So, in other words going back to the previous one, if I find out if I look at the displacement response and I find out this value, this is nothing but this value here. And I keep looking at the dynamic amplification factor and I go like this and I see what the maximum value is read of the maximum value. So, here for example, in this structure I read of the maximum value I see the maximum value is 10 times u x ((Refer Time: 15:12)).

So, this becomes, so this is 1 up on 2 into 10, so this is 1 up on 20, so xi is equal to 5 percent, I know that the xi is 5 percent. So, in a way this is another way of finding out the omega because this when you get resonance, you get this to come like this. So, you know that the phase angle is 90 degrees that tells you, what value of omega at which this is, so this one gives you the omega the natural frequency of the structure and this gives you the damping. So, in other words this is another way of finding out the dynamic characteristics of a structure.

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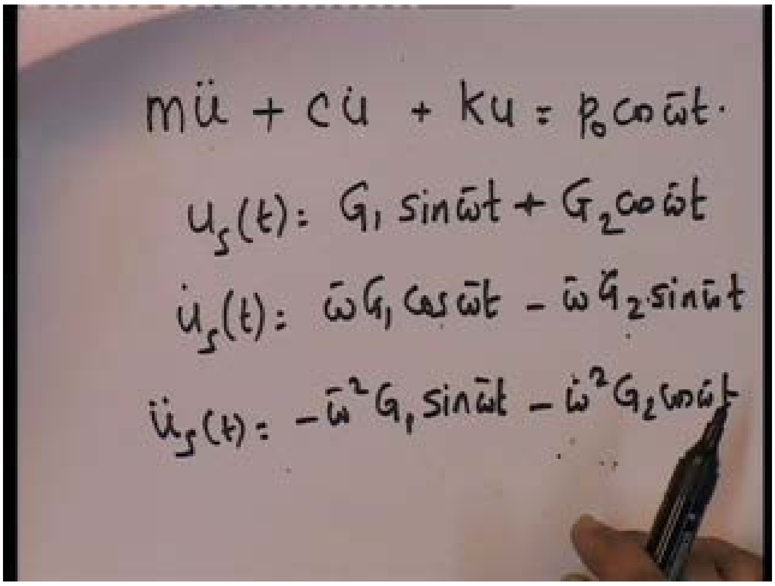

Now, what I will do before I continue, I will remember last time I said that, what if you have structure which has the following is subjected to this kind of loading, the loading it is a harmonic loading, but it is subjected to p naught cosine omega bar t. Remember, I said and I said that you should do it, through this to yourself now, what I will do is let me just solve this problem for you, you must have already tried solving it. But, I will go through this steps very, very easy steps, I am only going to do this steady state solution.

So, steady state solution what formed as a take up, remember that it always takes up the form G 1 plus G 2 cosine omega bar t. So, this is the form that it takes up and if I you know the calculate u s of the is equal to omega bar G 1 cosine omega bar t minus omega bar G 2 sin omega bar t and u double dot s of t is equal to omega square G 1 sin omega bar t minus omega bar square G 2 cosine omega bar t. So, if I substitute all of these into this above equation.

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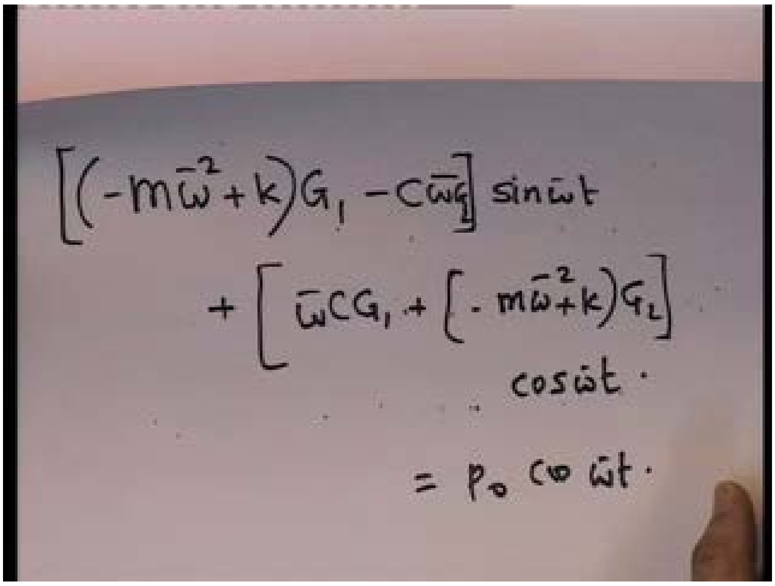


What I get is the following and then I just call the sin's and cosine. And then I get minus m omega bar square plus k G 1 minus C omega bar 2 G 2 sin omega bar t plus omega bar C G 1 plus m omega square plus k G 2 cosine omega t. So, this is the left hand side the equation, and this is equal to p naught cosine omega bar t.

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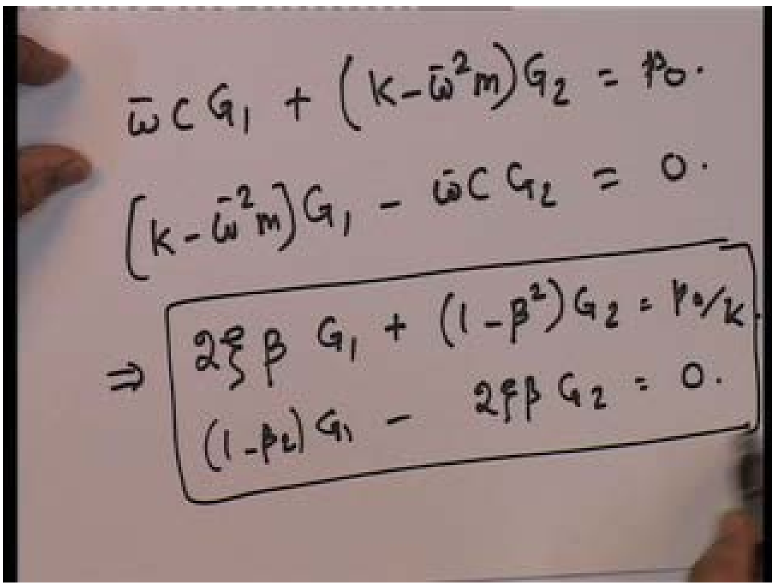

So, that is the solution then what happens we get the following, we get that omega bar C G 1 plus k minus omega square m G 2 is equal to p naught, and we get k minus omega square m G 1 minus omega bar C G 2 is equal to 0. Now, remember I had said last time that divide both of the equations by k and what did we get, I am not going to go in to the details we have already done it last time please review it.

This will imply that, I get that 2 xi beta G 1 plus 1 minus beta square G 2 is equal to p naught and 1 minus beta G 1 minus 2 xi beta G 2 is equal to 0. So, if these are the two equations then I solve them simultaneously, by the way this is p naught up on k because I have divided both by k 0 by k 0. So, if this is the equation these are the two equations that I solve for.

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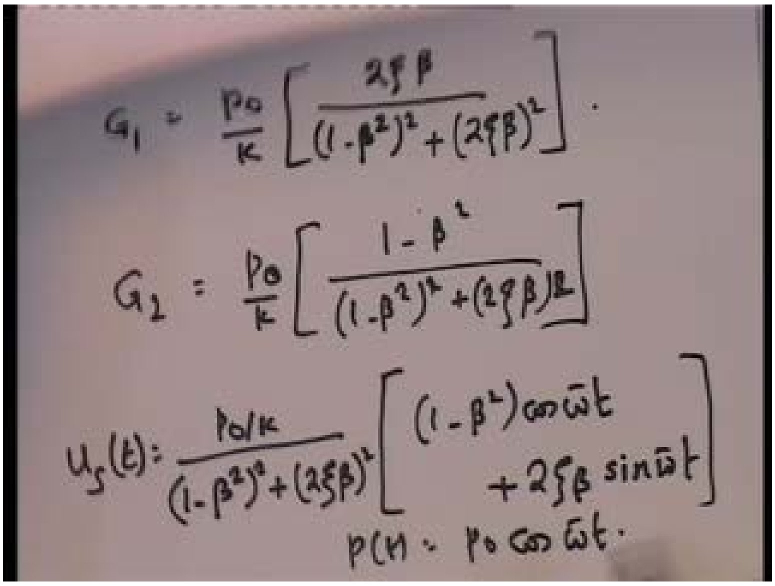


And if I solve for them, I get that G 1 is equal to p naught up on k 2 xi beta up on 1 minus beta square plus 2 xi beta square that is G 1 and G 2 is equal to p naught up on k into 1 minus beta squared up on 1 minus beta square plus 2 xi beta the whole squared. So, that G 1 and G 2 and therefore, the solution then becomes the following, solution becomes u s of t is equal to I am going to put p naught by k and 1 minus beta squared plus 2 xi beta the whole squared outside.

And what I get inside is 1 minus beta squared cosine of omega t plus 2 zeta beta sin omega t, this is the solution this is steady state solution of when you subject it to p of t equal to p naught cosine omega bar t. Now, note something very interesting and that is that once we do that, what happens if you look at this particular solution, remember what is cosine x minus y, cosine x minus y is cosine x cosine y plus sin x sin y that remember that. Once you remember that, then you will see that this solution becomes obvious.

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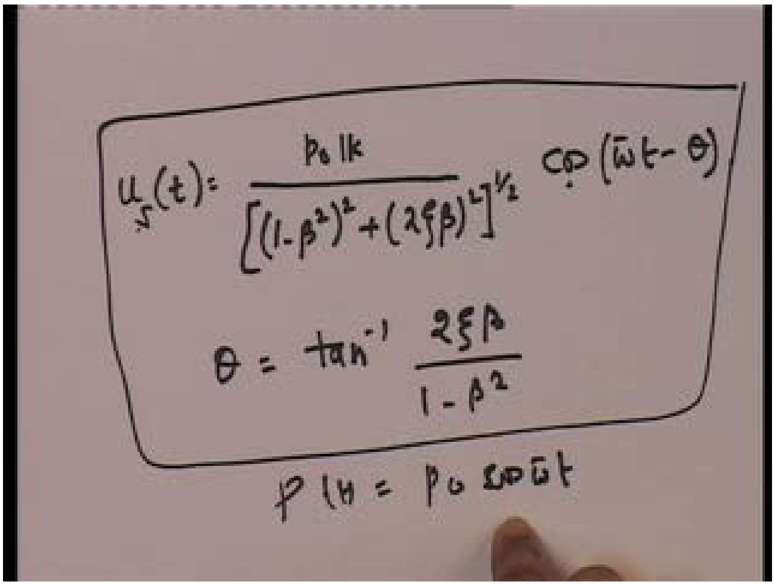

This solution becomes the following u s of t is equal to p naught by k up on 1 minus beta squared plus 2 zeta beta the whole squared ((Refer Time: 24:28)). And let us look at, look let us look back at this right this is cosine x cosine y plus sin x sin y, so this is sin x this is cosine x and if I look at the two of them together, it becomes what it becomes 1 plus beta square plus 2 zeta beta the whole square half, so this 1 gets cancel, so this becomes again half.

And what we get is cosine omega bar t minus theta, where by definition theta is equal to sin x up on cosine x. So, it is that tan x and so that is tan inverse of 2 xi beta up on 1 minus beta squared, look at this solution excepting for the fact that, this is p of t p naught cosine omega bar t and therefore, you have cosine here. But, if this was sin you would have sin here, with exactly the same theta and exactly the same, dynamic amplification factor because u s up on u s max up on p naught up on k is equal to 1 up on this, which is again D the same as, what we have shown here.

So, note that this dynamic amplification factor and this phase response are both equally true, irrespective of whether you are you know it is sin omega bar t or a cosine omega bar t or a linear combination of the two. So, therefore, for any harmonic response this is the dynamic amplification factor and this is the phase response. So, all though I had originally developed it last week for sin omega bar t, this week I have shown it to you that this is equally valid for cosine omega bar t. So, once we have this now let me go back to my the argand diagram that I was looking at. So, I would like to again go back to the argand diagram, and in the argand diagram let me again restate the solution process.

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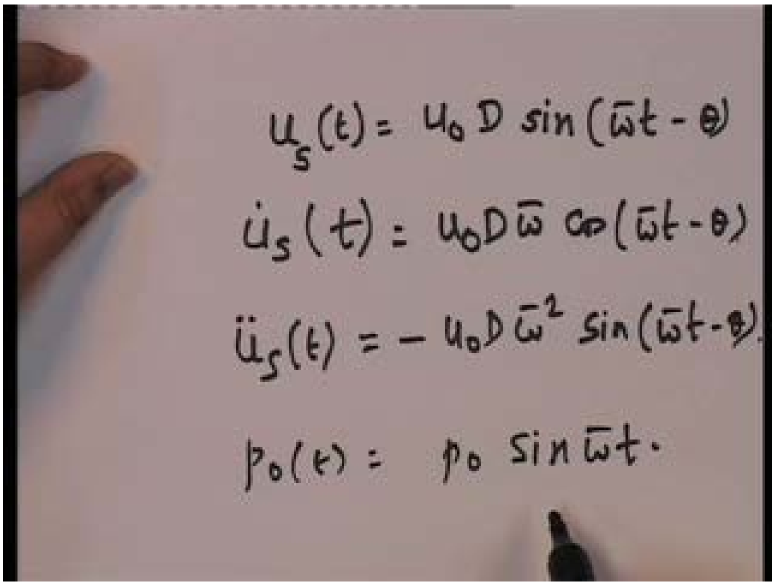


If you look at it the I will just do this for now sin omega bar t or you know, I mean does not matter I will do it for sin omega bar t. So, u s of t is equal to I am going to say u 0, where u 0 is into D into sin omega bar t minus theta, so note that now this solutions that I have although I have sin here. This is equally valid for any arbitrary only thing that is going to happen is that the response, displacement response is going to lag the forcing response, which is the load response by theta that is a given on that theta, we know what that theta is that is this phase response that I have shown over here.

So, now, now the question then becomes this theta is a known value. So, let me just go back that you have this becomes what, let us plot it this becomes u naught D omega bar cosine omega bar t minus theta and u double dot s of t becomes minus u naught D omega bar square sin omega t minus theta. Now, these are the displacement, velocity and acceleration and let me look at p naught of t, p naught of t is sin omega bar t right. So, let us be very, very clear about this, so if I would plotted on the argand diagram, how will it look. Now, I am coming back to the argand diagram.

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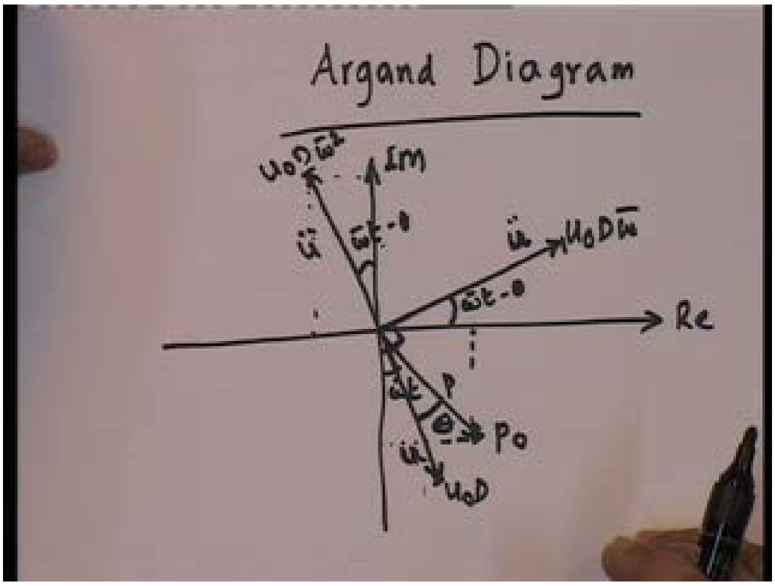


How do is it look well x plotted, this is the real axis this is the imaginary axis and what did we say was the argand diagram well, the argand diagram is such that this value which is a real value, is really the if I plot a vector. So, if I plot a p naught sin omega bar t, so I am going to plot, so vector of magnitude again what is the magnitude p naught. So, I am going to plot a vector p naught and what we say is that look, the projection on the this axis. So, in other words when the real axis, so projection on the real axis of this vector should be the this.

So, how was that then they was just look at it, if I call this as omega bar t, look at it this represents the real plane and that is p naught omega sin omega bar t understand because this is 90 degrees this is a vector. So, in other words remember again that the argand diagram is nothing but our representation of the you know, each response quantity in a complex plane such that, the real part on the real axis, the real projection of the vector always represents the vector the real vector that you have.

So, similarly let me put it this way, so if this is p naught p naught sin omega bar t, this is a vector that lags that by theta. So, what is that mean that is a vector that lags and what is that vector u 0 D, so this vector is u 0 D and it lags by theta, note that if I plot this becomes omega bar t minus theta. So, then if I plot, if I look at the real projection, you get u 0 D sin omega bar t minus theta, which is what this is, so this vector represents the displacement vector.

Now, let us look at what this is, look at this vector it is cosine omega bar t minus theta. So, if I would to look at this and look at, remember cosine always leads cosine leads the sin by 90 degrees, so you forward to look at cosine plot, it this would be in this fashion it would be 90 degrees to this plane, where what is this one, this one is omega bar t minus theta and this value is u 0 D omega bar.

So, what is this, so this vector represents the load, this represents the displacement, this now represents the and look at, it if you look at this the projection on the real axis of this is what, u naught xi D omega bar in to cosine of omega bar t minus theta, which is what we have here. So, this represents this vector represents u dot, this represents u, this represents p and finally, if you look at this one, all this is the acceleration is minus of sin omega bar t.

So, minus means till the opposite direction and what is the value the value is equal to u 0 D omega bar squared. So, this represents u double dot and if I look at it, if I take the plot where what is this one, this one is omega bar t minus theta and if you look at this, if I look at the projection of this it becomes what, this one becomes u naught D omega bar square into sin of omega bar t minus theta.

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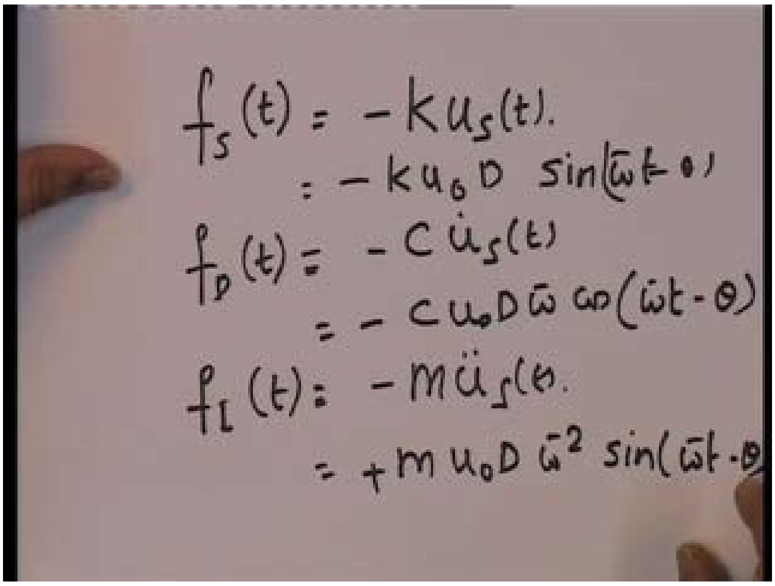


But, since is on the negative real axis it is negative and it represents this. So, the first argand diagram that I have drawn represents what, represents the displacement, the velocity and the acceleration. Now, next argand diagram I am going to draw, is going to be purely on the forces force equilibrium. So, if I look at it, what is it become then let us look at this particular aspect.

If you look at this u s is equal to this, what is the elastic force I will call it f s of t, f s of t if you look at it the if displacement is in this direction, the force opposes it, so it will always be minus k in to u s of t. Let us look at the damping force, damping force if you remember I talked about it, it damping force is damper a viscous damper, where if a velocity is in one direction, the damper opposes that velocity.

So, that is c in to u s and finally, the fictitious inertia force that we talk about, which is if anybody has mass, it resists, it you know if a body is accelerating in a particular direction. What is Newton second law says well there is a equal and opposite you know kind of an inertia force, that is proportional to the to the acceleration, but oppose is it and that I call as inertia force and that is equal to m u double of t.

So, we put this and then I plug in the values and the directions, what do I get this is equal to minus k u naught D sin omega bar t minus theta this is equal to minus c u 0 D omega cosine omega bar t minus theta and this is equal to minus m u naught D, note it is not minus because this is this itself is minus. So, minus minus becomes plus m u dot D in to omega bar squared sin omega bar t minus theta. So, these are the forces note that, this plus this plus this is equal to p. So, now, let me just go through a few steps here, I am going to plug in the value of u naught n here, u naught is equal to p naught up on k.

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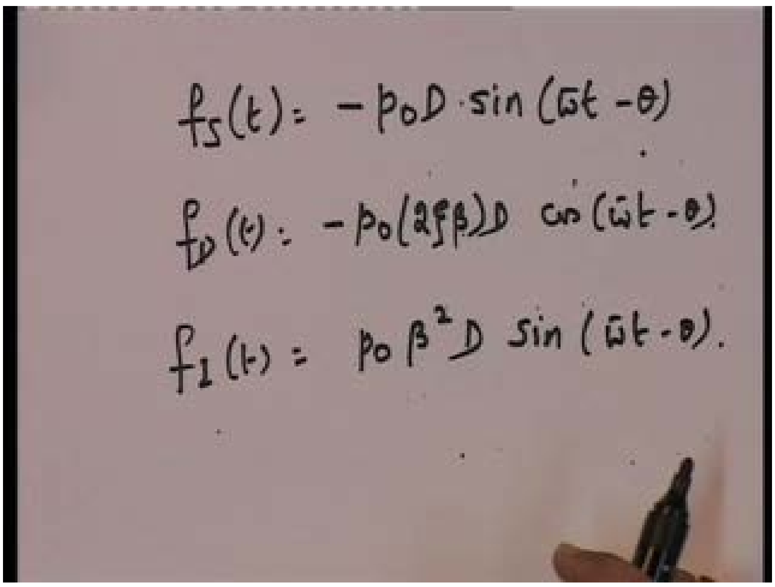

So, this becomes then our f s of t becomes equal to minus p naught D into sin omega bar t minus theta. Now, again I am going to plug in p naught up on k here and remember c omega bar up on k is equal to what, 2 xi beta we already derived this derive it again. So, basically f D becomes equal to minus p naught into 2 xi beta D cosine of omega bar t minus theta and f I of t well let us look at this. This is p naught up on k, so this becomes m omega bar squared up on k which is beta squared, we already seen that, so this is equal to p naught in to beta squared d sin omega bar t minus theta.

So, having return these let us look back at this, you note that if you look at u s of t and it f s opposes that. So, if u of t is in this direction your f s is in the opposite directions, similarly if you look at u s u dot s, if u dot s is in the particular direction cosine this is minus of course, sin. So, if u s is in this direction f D is in this direction. And finally, if you look at it, I am going to show this if you look at let me just show this, if you look at f I it is sin omega bar t it is in this particular direction, where this is in the opposite direction. So, if you look at this, this is in the direction of u s and f s is along the direction of u s. So, if I plot the argand diagram now, what do this becomes?

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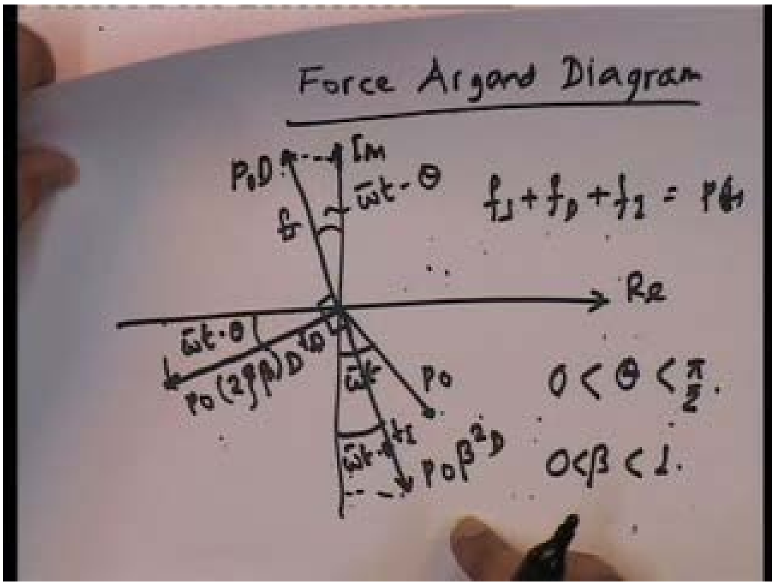


Now, I am going to draw the force argand diagram, first I draw the when you draw the force argand diagram. And let us plotted again, real, imaginary and remember p naught is in this direction with omega bar t, then this was the direction of u. So, therefore, just the opposite direction would be f s and what are the units, what is the dimension magnitude,

magnitude to this p naught D. So, it is going to be this is going to be omega bar t minus theta and this the magnitude is going to be p naught D.

Now, remember this was a direction of the velocity. So, this will opposite, your damping force is going to be equal to p naught in to 2 xi beta t, where this is omega bar t minus theta. And finally, remember f I is sin omega bar t p naught b naught, so therefore, this one will be in this direction and it will be p naught beta square D, where this is omega t minus theta. Now, note I mean an assumption here, what is the assumption that theta is between 0 and pi by 2.

In other words a priory I made thus assumption in this argand diagram, that beta is less than I am sorry theta is less than pi by 2 with basically means that beta, is between the 0 and 1. Now, let us look at this particular one, how does it look, now if you look at this what we are saying is that, this try where this is 90 degrees, this is 90 degrees, these of forces which this is f s this is f D and this is f I, f s plus f D plus f I is equal to p of t.

So, this is f s of t, so in other words if I take a snap shot at a particular instant of time t, this is snap short these three together equilibrate this. So, this plus this, so vector addition of this plus this plus this is going to be equal to the it is going to give me a vector which is along this line. Now, then the question becomes, how do we tackle this problem, how do these three and note that this as t varies, this t would imply that this will go like this, but these three are connected two p and will also move with it.

So, in other words this interplay that, this is going to be phase lag now, the inertia force phase lags, the applied force by theta. The elastic force opposes the inertia force and this opposes is at 90 degrees because it is proportional to the velocity and opposes this velocity. And remember, that velocity is 90 degrees out of phase, with acceleration and displacement we seen that. This is the very interesting thing now, I will show its very hard for me to show that this three meet together, but if I would look at it, in the real plane let us write to see what this represents in the real plane, this representation on the real plane is going to be equal to let look at this, this is the representation of the real plane.

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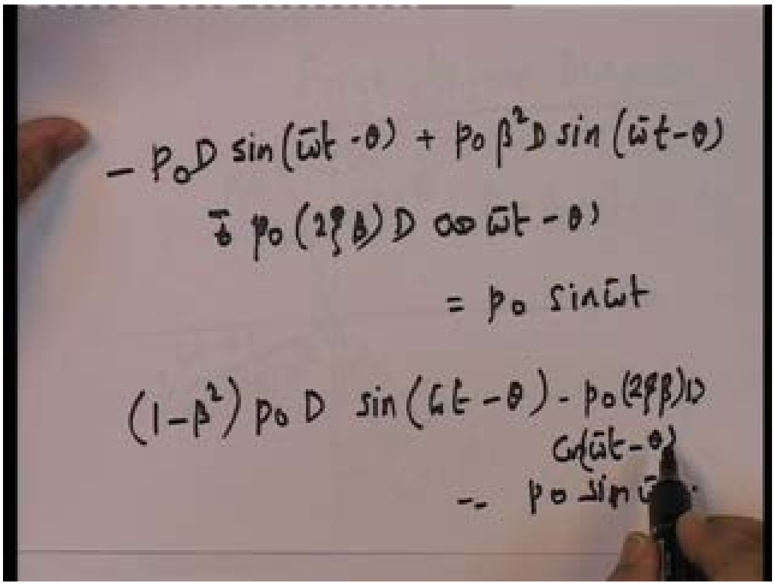

So, representation on the real plane if I plot, the f I representation in the real plane, what’s that going to be that is going to be equal to p naught D sin omega bar t minus theta, this is minus because it is along the negative. So, this is the this represents the real part of the spring force negative right, now let us look at this one, this is it and so that is going to be equal to p naught beta square D into sin, note that this is plus.

So, I put this plus this plus this. So, this one is going to be equal to plus p naught it is going to be, p naught into 2 xi beta into D cosine omega bar t minus theta and note that this is on the negative plane. So, this will actually be this should be negative, at this is equal to p naught right the because you know we say that all of these are together, now let us see it make sense, does it make sense sin omega bar t sin omega bar t goes together.

So, then this becomes then what does this become, this will become 1 minus beta square p naught D sin omega bar t minus theta minus p naught 2 xi beta cosine omega bar t is equal to p naught sin omega bar t is equal to p naught sin omega bar t. Now, let us look at this, how do we this is cosine omega bar t minus theta, now how do we show that this is equal to this, well let us go one step further. So, this argand diagram I am proving to you, in around about manner and that is let see how do I get these two together. If I look at this two to together I am going to plug in the value of theta in here.

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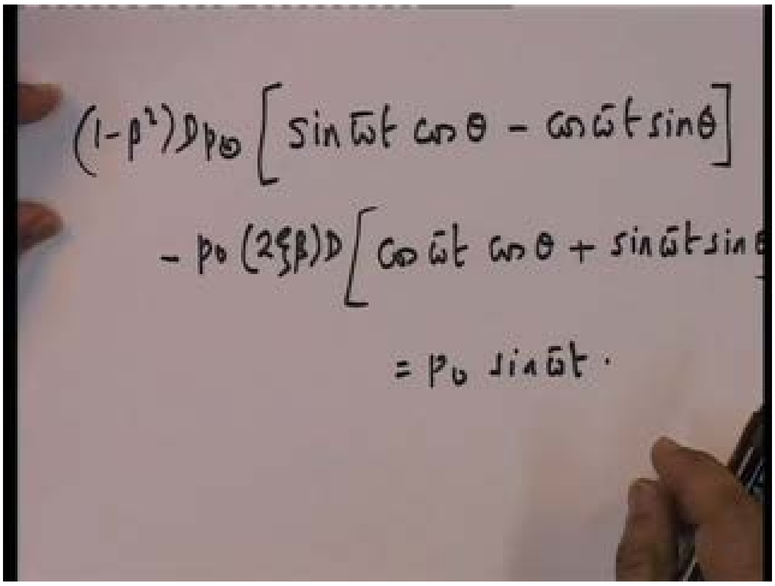

So, if you look at this I am going to expand these expand these, what do I get, I get 1 minus beta square in to Din to p naught is equal to a well on this side I get, sin omega bar t cosine theta minus cosine omega bar t sin theta. And on this side I will get p naught 2 xi d in to cosine omega t cosine theta plus sin omega bar t sin theta and this is equal to p naught sin omega bar t.

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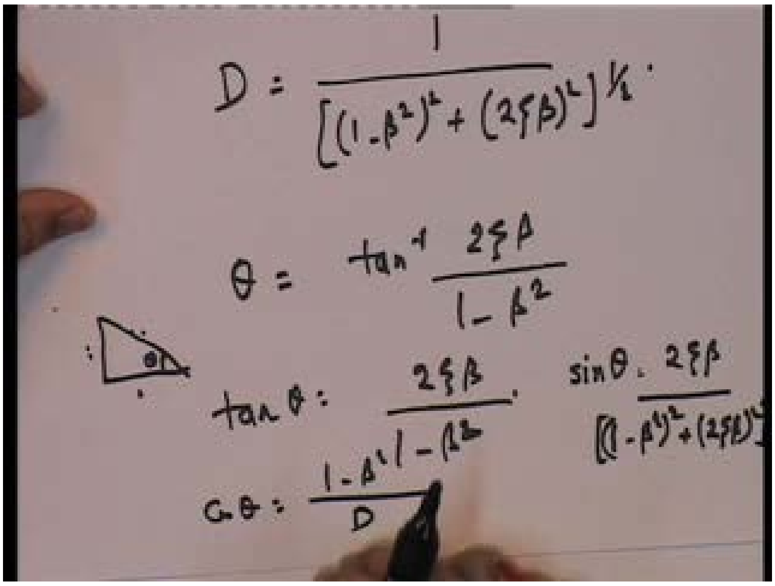

Now, let us just go through with this, if you look at it what we get here, what are D, D if you look at it is equal to 1 up on 1 minus beta square plus 2 xi beta half and theta is equal to tan inverse 2 xi beta up on 1 minus beta square. So, look at it tan theta is equal to 2 xi theta up on 1 minus beta square that is tan theta. So; that means, if we look at this, then what becomes sin theta.

If tan theta is this let us look at the thing, this is theta tan theta is what this up on this. So; that means, we are saying that this 2 xi beta this is 1 minus beta square and so what is sin theta, sin theta is this up on this. So, sin theta is 2 xi beta up on square root of 1 minus beta square plus 2 xi beta square root, Pythagoras theorem this square plus this squared square root of that is this.

And, so what is cosine of theta, cosine of theta if you look at this is this up on this. So, that is 1 minus beta squared upon, if you look at this, what is this, this is nothing but D right. So, I am going to put this as D, so having put these in let me plug these in to these equations that I have here, so this is nothing but let us look at this, so this becomes what cosine is 1 minus beta square up on D and sin is 2 xi beta up on D. So, D D cancels out and so what am I left with I am left with the followings.

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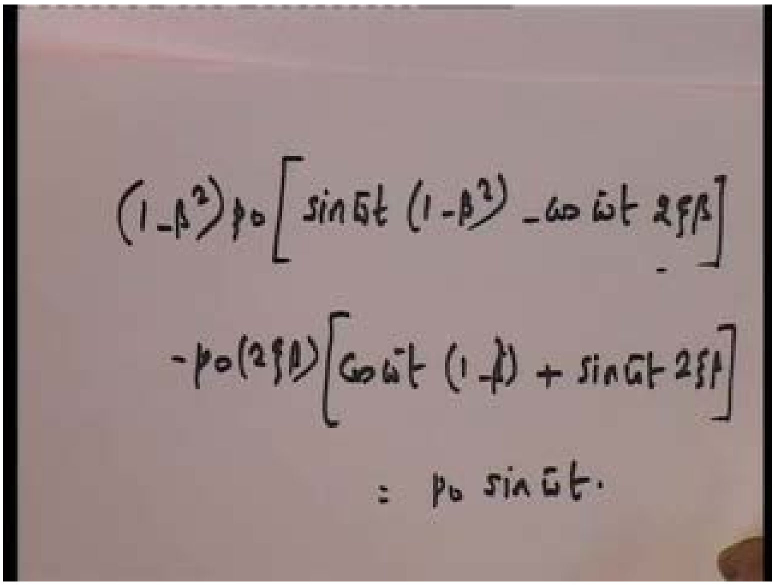

I am left with this 1 minus beta squared the D goes out because D is come out. So, sin omega bar t into what is this one, cosine, cosine was 1 minus beta squared up on this D is gone out and other one is, cosine omega bar square in to 2 xi beta up on again that is gone. So, this is what it is minus p naught in to 2 xi beta the D goes inside, so the D goes inside this becomes cosine omega bar t and what is cosine theta 1 minus beta square right the D cancels out and the other one is sin omega t into sin. So, that is 2 xi beta the d cancels out, so this is what I get and is equal to p naught sin omega bar t. Now let us look at this, let us put the sin's in cosines together what do I get, I get the following, I get I am just going to rewrite this I am going to put it all together and so the this is going to give me p naught 1 up on beta the whole squared.

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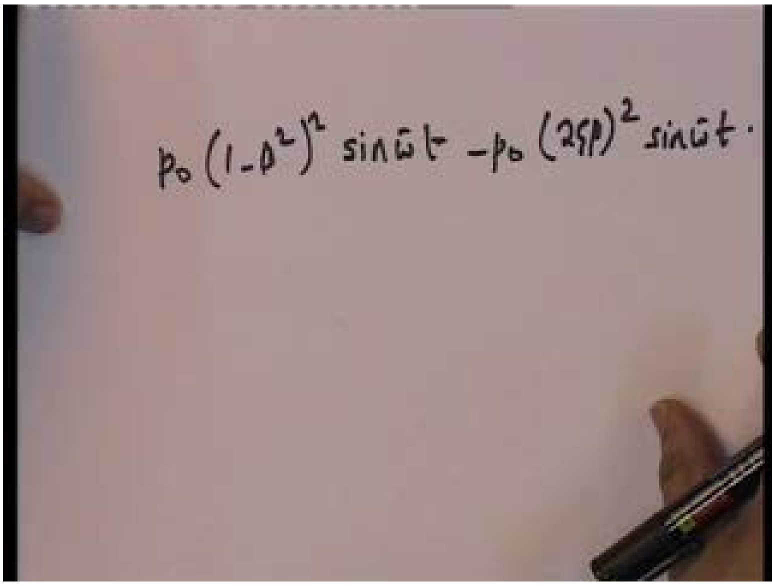

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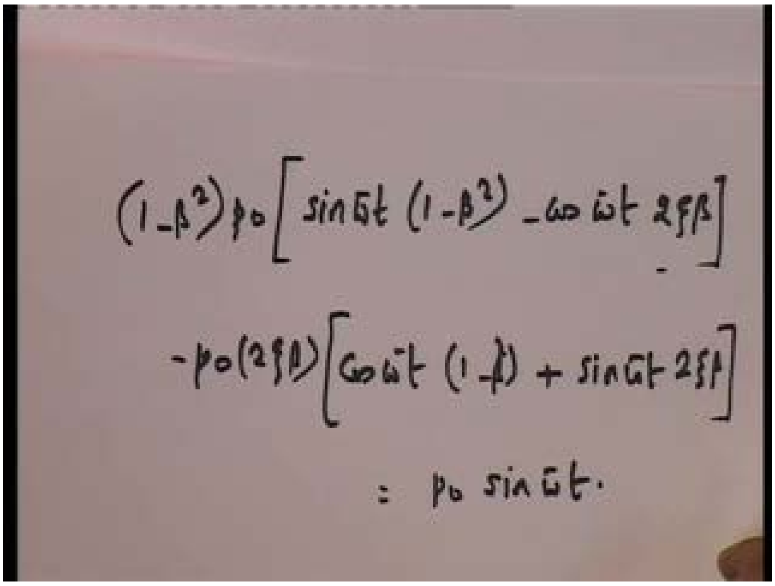

So, I am going to put that In, I am going to get p naught 1 minus beta square the whole squared sin omega bar t then I am going to take this one sin omega bar t, which is nothing but minus p naught right it is in minus p naught. So, this one becomes 2 zeta beta square, so this becomes then minus p naught into 2 zeta beta squared sin omega t.

And then I will put together the other terms from here, I am going to incorporate these terms from here and so what I get is cosine becomes then minus 2 xi beta in to 1 minus beta squared and minus p naught into 1 minus beta, you see the cosine term disappears. So, cosine term disappears and what I am left with is that 1 minus sin squared that is what I am left with. So, if you look at this ultimately this term, then becomes equal at all times to p naught. So, I have shown in a broader sense by taking the real one, I am able to show it you that p naught is equal to p naught that is what we get. So, they are in need equilibrium now, I am going to stop here today and I am next time I am going to take up and show to you, that for specific situations of when beta takes some specific values what happens to the argand diagram.

Thank you, bye.