Structural Dynamics

Prof. P. Banerji
Department of Civil Engineering
Indian Institute of Technology, Bombay

Lecture - 5
Response of SDOF Structures to Harmonic Loading

In the last lecture, we looked at the mathematical solution of the response of a single degree of freedom system to harmonic loading. And the specific harmonic loading that we looked at is p naught sin omega bar t.

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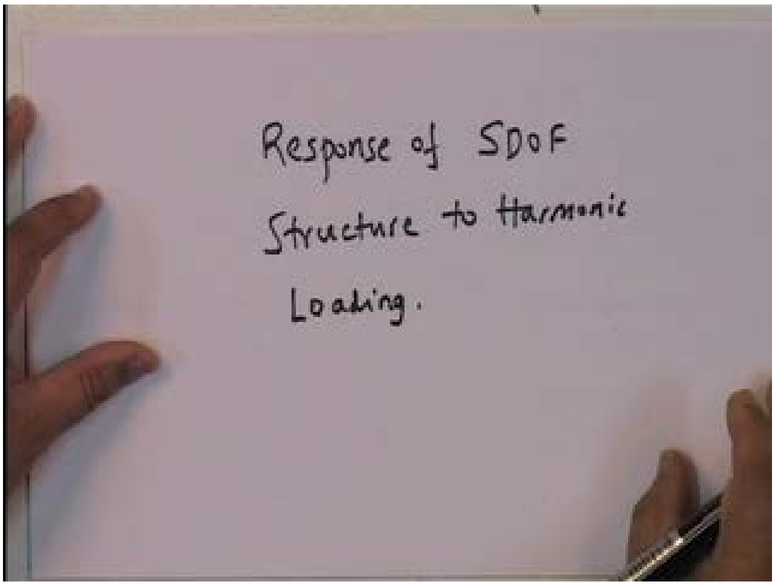

I would like to start of this lecture, and this is actually a continuation of, Response of SDOF Structure to Harmonic to harmonic Loading. So, what we did was in the end, we saw that for damped system the response u of t is equal to e to the power of minus zeta omega t C 1 sin omega D t plus C 2 cosine omega D t plus p naught upon k into 1 minus beta squared the whole squared plus 2 zeta beta a whole squared, inside 1 minus beta squared sin omega bar t minus 2 zeta beta cosine omega bar t, this was the solution that we obtained.

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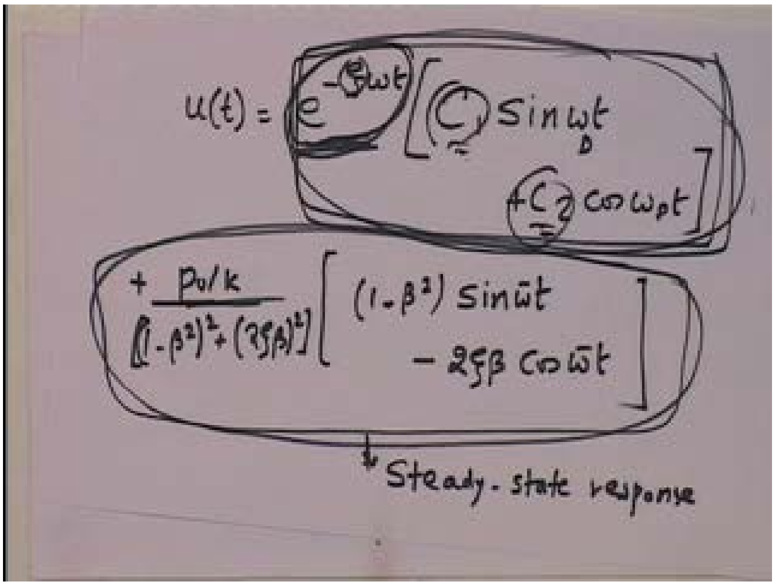

Now, what is the solution contain, I mean I said you know the C 1 and C 2 can be obtained from you know, the fact that initial conditions now but you know. So, therefore, the I am not still soft the entire thing, but the essence where you look at what this looks at, there are two parts to this problem, this is the homogenous part. In the homogenous part, if you look at it, what is a homogenous part depend on it depends on omega and zeta, those are the two quantities that determine this part.

And if you look at this part, what is this part depend on this part depends on p naught by k, which is the equivalent static peak displacement, p naught up on k remember p naught up on k is p naught is the peak you know amplitude. So, p naught up on k is the peak equivalent static, so that is gives you the static response, but the dynamic part depends on two terms. One term is xi which is the damping ratio and the other important parameter that it depends on, is beta, beta is the frequency ratio it is the ratio of the excitation frequency to the natural frequency of the structure.

So, in other words if you look at it, again it boils down that the response of a single degree of freedom system, depends on zeta and omega which are the dynamic characteristics of the structure, which we talked about in the two lectures ago. And the other part that depends on is the ratio of the excitation frequency to the natural frequency. So, in other words three parameters define, xi the damping ratio, omega the natural frequency and omega bar the excitation frequency, these three parameters define the entire response of the single degree of freedom to harmonic response.

The other interesting part if you look at this is the following. You see C 1 and C 2 depend on the these two conditions, but if you look at this part what is happening to this part, this part is a harmonic excitation sorry harmonic should I say part, which is modulated by e to the power of xi omega bar t. So, let me put it this way, that if we look at a sufficiently far, time far away time from the initial condition right. If we look far away, what is going to happen to this part, this part is going to diminish and it is going to go to 0 because of this term.

And, so this term is going to disappear after a sufficiently long window, how look that window depends on what this xi is, xi is small this window is large if xi is large this window is small it does not matter, the fact remains that this part is dissipated and what you are left with is this part. Now, what is this part, this mathematically is the particular solution, physically this part is called as the steady state response.

Why is this the steady state, that is that as I said this part comes from the initial conditions. Now, if you take a sufficient window of time, away from the initial conditions then this is what is remain. So, this is what is known as steady state, in the steady state once the transient is you know, damps out the steady state response is what remains and typically it is seemed that you know, by a large although we looked at it from this perspective, but look this part does look at the initial conditions. But, if you look at harmonic excitation, harmonic loading what harmonic what when do you have harmonic loading. Remember that when you start a rotating machinery, you start it from a particular RPM and then you build it up to a particular RPM and then if that RPM, you look at that response and so the actual if you really look at it.

About time that we start looking at, is really just a snapshot, actually the loading has lasted for ever and it continues and this is not drawn very well and it is actually a harmonic load. But, the point is that you know there is no initial condition, I mean you know the initial conditions, actually happens at let us say minus infinity. So, when you look at response, then all that you get is the steady state response that we are interested in so…

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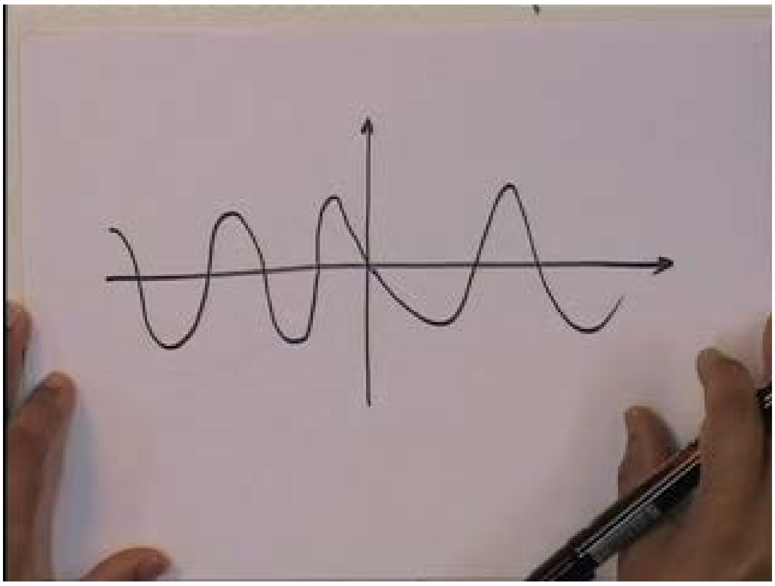

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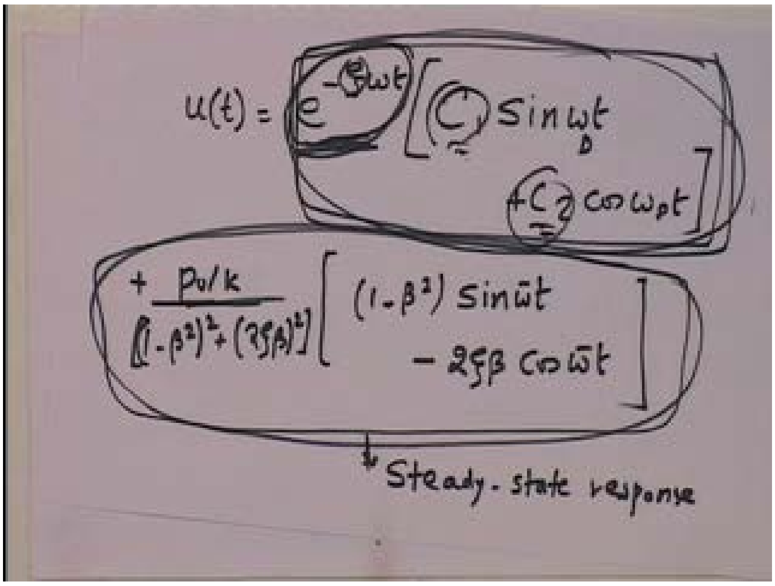

So, this is ultimately the response that how about note one thing, that if you had no damping this would never get damped out. And that in other words, you would never be able to damp out the transient part; however, even if you have very low damping at some point of time, the transient part will be damped out and you are left with steady state. So, although un damped system, does not have steady state, but you can take a situation where xi tends to infinity, sorry tends to 0 without it being un damped, do you understand my point.

In that situation you can define a steady state, a transient will go out and if this loading is being on for long enough. We can say that look we do not care about the transient and we are left with the steady state response. And which is why, in harmonic loads because the load is supposed to be there forever, the initial conditions are part of the transient, the transient is eliminated even if you have slight damping very small damping in the system it is eliminated. And what you are left with is the steady state response.

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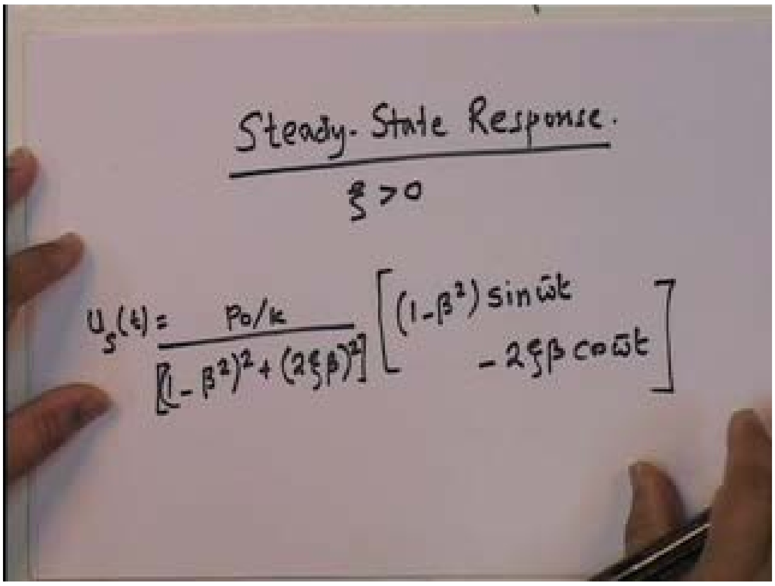

And therefore, the steady state response takes up a large role in determining harmonic response to single degree of freedom response to harmonic excitation, that is in other words, this is true as long as xi is greater than 0, it cannot be 0 steady state response make sense, and steady state response is given in this form. So, I will call this now steady state response 1 minus beta squared sin omega bar t minus 2 zeta cosine omega bar t. And if you look at this, see this is sin omega bar t cosine omega bar t. So, this system can be written as square root of 1 minus beta squared, square plus 2 zeta beta whole squared square root it can be written as sin omega bar t minus theta. So, if you look at this, in this system this equation can be written off in this fashion. And that is, that it can be written as a sin wave itself.

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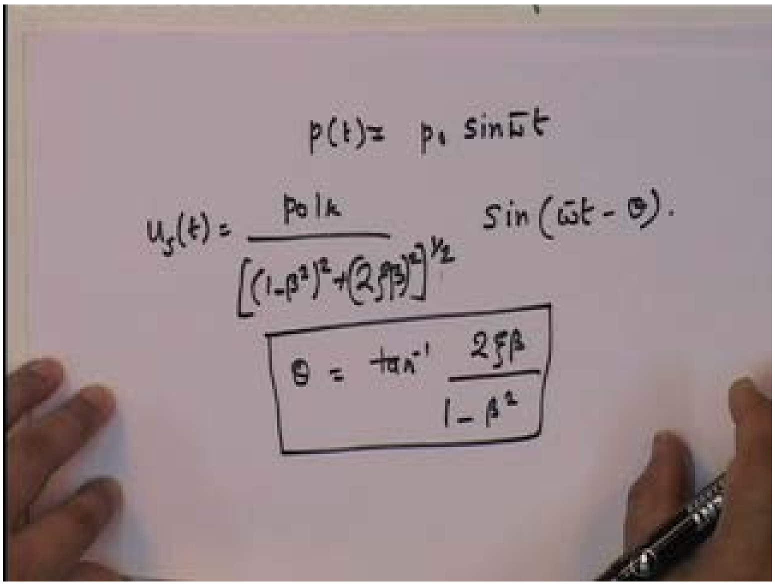

So, u s and note, that if you look at this, this squared plus this squared square root, now this one is already this squared plus this squared. So, if you put square root on the top all that happens is it becomes square root at the bottom. So, u s of t becomes p naught up on k the square root into sin omega bar t minus theta, where theta is given by tan inverse of 2 zeta beta up on 1 minus beta square. So, in other words the steady state response, of a structure to unloading given by p naught sin omega bar t, the steady state response is p naught up on k into this term, sin omega bar t minus t.

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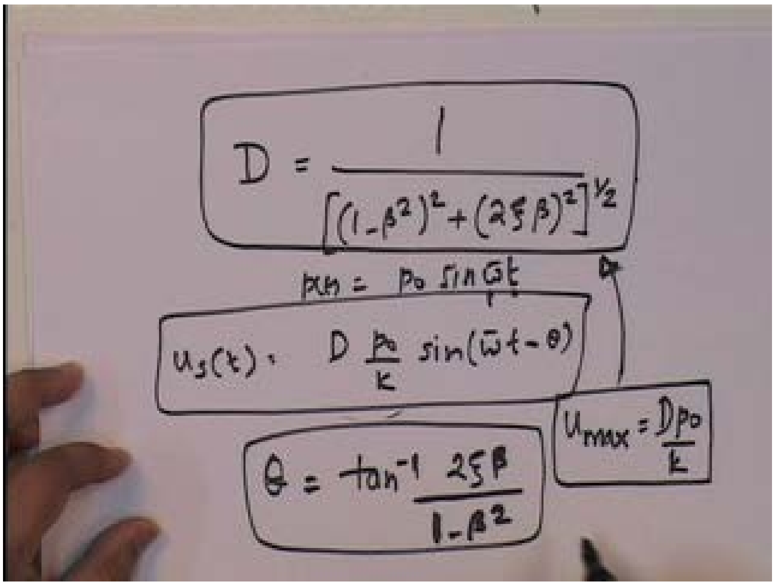

So, I can now say that look I will define, a term D which I will take as 1 up on 1 minus beta square the whole squared plus 2 zeta beta square root. If I define this, then I can define u s of t as D into p naught up on k sin omega bar t minus theta for a loading p of t p naught sin omega bar t where D is given by this term and theta is given as tan inverse 2 zeta beta 1 minus beta squared.

So, in other words if we go back what are we saying, we are saying that look at this, if you have a loading p naught sin omega bar t, then the steady state response is D p naught up on k. Note that, the amplitude of this is D into p naught up on k did you see this very interesting part, what is the peak steady state response become, D into p naught up on k, in other words where D is given by a particular parameter. So, note that if I substitute in this parameter and get D and I do a static analysis with D I can actually get, the peak response, you see the beauty of this.

So, therefore, although we have done a lot of mathematics behind it, now and this the steady state lags with minus theta, lags the load by a angle which is given by tan inverse 2 zeta beta up on 1 minus beta squared. Note that, this theta as zeta becomes very small also becomes extremely small, for almost all values of beta. Now, let us look at these two things, so this becomes a very trivial aspect now; that means, u max is equal to D p naught up on k where D is given by this term.

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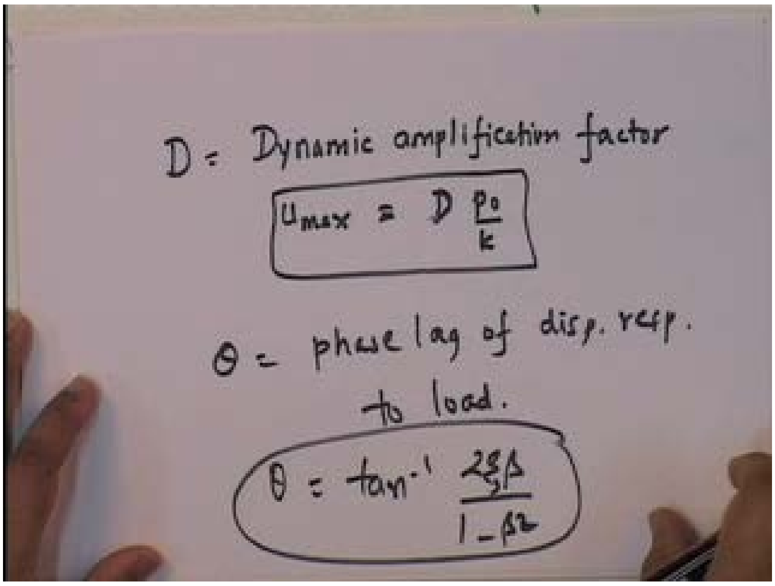

Since steady state response as I said, the transient response disappears, steady state response is what we are left with, let us look at the two parameters, the two parameters that define the steady state response is S and theta. So, let us look at D and theta very carefully.

So, therefore, D is known as the dynamic amplification factor; obviously, why because u max is equal to D into p naught up on k. So; obviously, this is a static and the u max, to the dynamic load is given by D into p naught up on k. So, this is the dynamic amplification factor that gives us the peak load, and theta is called the phase lag of displacement response to load. In other words, what we are saying is that if there is damping because note that theta is equal to tan inverse 2 zeta beta 1 minus beta squared.

In this damping there is a phase lag, there is a positive phase lag note this there is a positive phase lag of the displacement to the load. In other words, the load comes the structure thinks for a while and then response to it, although this is not strictly true because it is not really the loading is not starting, you know we are looking at a particular point of time, where the loading has been there for a while, the transient initial conditions the transient part, which was due to that initial conditions has been damped out.

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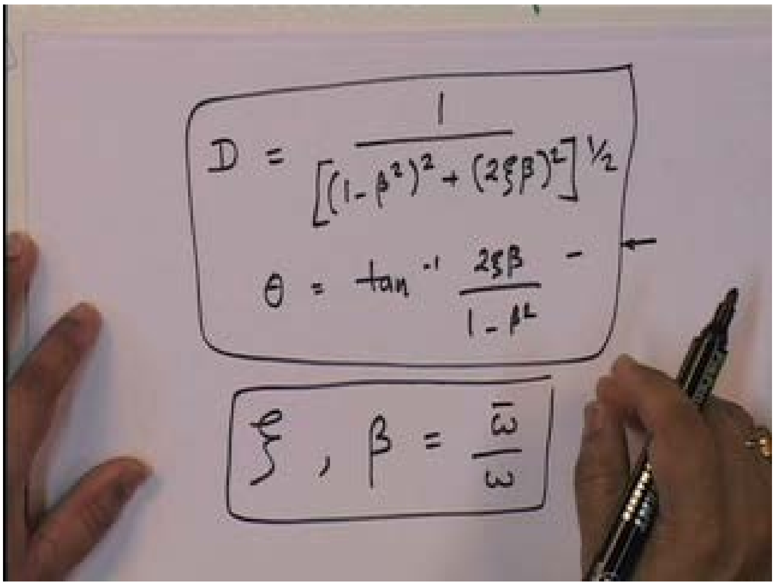

And all you are left with is the steady state part and what we are saying, is that the steady state part lags the load by a lag, phase lag given by this term. So, let us first look at delta, what is delta depend on, let us look at both the terms, in fact you know, dynamic amplification phase lag. So, I should now only talk about dynamic amplification phase lag from here on out, so if you do that.

So, dynamic amplification is given by, what is these terms depend on these terms depend on zeta and beta, beta is nothing but the excitation frequency to the natural excitation these are the only two terms, that this two things depend on. So, therefore, what we have is a situation, where if we look at this the D what does it look like, so this D depends on beta, so I will draw d with beta.

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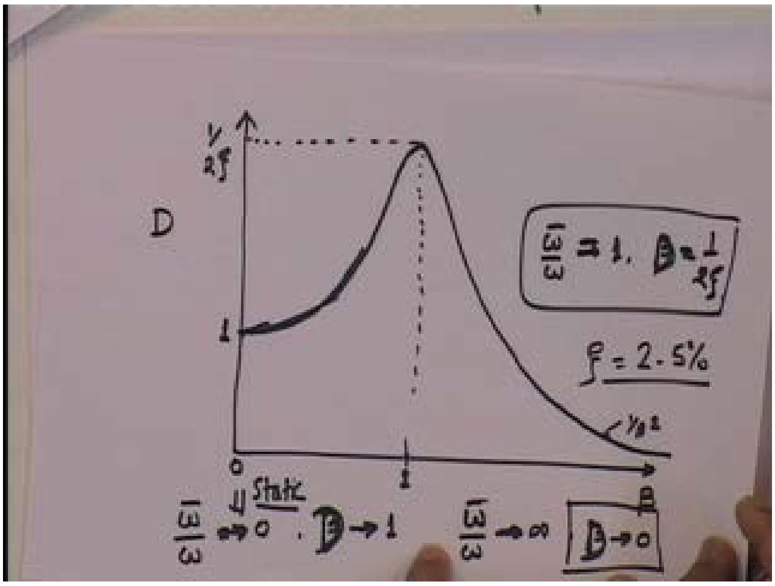

So, on this axis I have beta, on this axis I have D, let us see what D is let us start and this plot I will do for a particular value of zeta, I will do it for a particular value of zeta let us see let us start. So, I am going to plot this, but before I plot this let us see, put beta equal to 0 plug in beta equal to 0 what you get, you get this is 1 and this if you put it as 0 this becomes 0. So, D is 1, so d is 1 and note that D is 1 independent of xi, it is independent of xi.

Now, what happens let us look at it, at beta equal to 1 what happens, at beta equal to 1 this term becomes 0, this term becomes 2 xi squared square root. So, it becomes 1 up on 2 xi, so at 1 the peak value is 1 up on 2 xi (Refer Time: 21:40). Now, let us look at some large value, a large value of beta, what happens here when you start getting large values of beta, if you look at it, for large values of beta and small values of xi. Remember, what we are looking at structural damping right, structural damping remember that xi is typically less than 10 percent, which basically means xi is less than 0.1. Small values of xi right, for very large values of beta what happens is look at this term, let us take 10 beta equal to 100.

So, what happens here, this becomes 100 squared, so that is 10000 minus 1 who cares, you know if you look at this term, this term kind of completely you know goes over on this. And so at very large as beta tends to infinity D goes down, if you look at this is very small, so I can ignore this, so this whole thing becomes 1 up on beta squared. So, in other words, here it goes down in a hyperbolic part.

So, if you look at this, this is what it looks like. And here, it goes down as almost 1 up on beta squared, so in a sense if we look at it what are the key aspects, the key aspects is dynamic amplification at beta equal to infinity is 0, that is one thing that you see from here. So, that is very interesting, what does it say that if you have if omega bar up on omega is tends to 0 then beta tends to 1, if omega bar up on omega tends to infinity beta tends to 0. And omega bar up on omega as it tends to 1 beta tends to 1 up on 2 xi.

In fact, let us not say tends to it is equal to 1, beta is equal to 1 up on 2 xi, these are interesting points a very interesting points here. And that is that, what is omega bar up on this implies static loading because the dynamic the omega bar is 0, in other words the load, the excitation is in cyclic at all it is in a kind of a constant p naught and therefore, beta this is not beta this is D tends to 1 I am, so sorry I have made a mistake here, this D equals D tends to 1.

Well if you have a static load the dynamic amplification tends to 1, what is that mean well all it means is when if you have a static load, there is no dynamic amplification well it is a tautology right. The more important thing is, when the excitation frequency is exceedingly high, relative to the natural frequency remember that, you see this is all relative omega bar up on omega. So, if omega let us say, is 0.1 hertz then omega bar which is 5 hertz or even 1 hertz.

Now, you would say that omega bar 1 hertz is not a very high frequency, but relative to the flexible structure because 0.1 hertz is a 10 second time period it is a very, very flexible structure. So, if you have a very flexible structure a 1 hertz loading, so if you have a 10 second structure and if you have a 1 hertz loading, what do you have D tends to 0. Whereas, let us say if you have a 1 hertz structure, a 1 hertz structure omega bar 1 hertz is residence.

So; however, it is always relative. So, when the excitation frequency is very high relative to the natural frequency, then the dynamic amplification factor is 0, what is that mean there is no response, that is really interesting is not that, there is loading, but there is no response, look at to this I will explain the physical aspect to this, shortly. And that is that and let us look at the other situation when you have resonance, what is D 1 up on 2 xi.

Think of xi typically in a structure is in the 2 to 5 percent; that means; take 2 percent here, what you get 1 up on 0.4 25. So, for 2 percent structure, dynamic amplification at resonance is 25, for 5 percent it is 10 still very large, so therefore at resonance you have large amplification of response for harmonic loads, for any kind of normal structure there is a normal structure, the damping is always less than 10 percent and so you have that. Now, let us try to look at these two aspects and let us see what this mean because this has a physical relevance to what I have talked about earlier and that is let us see.

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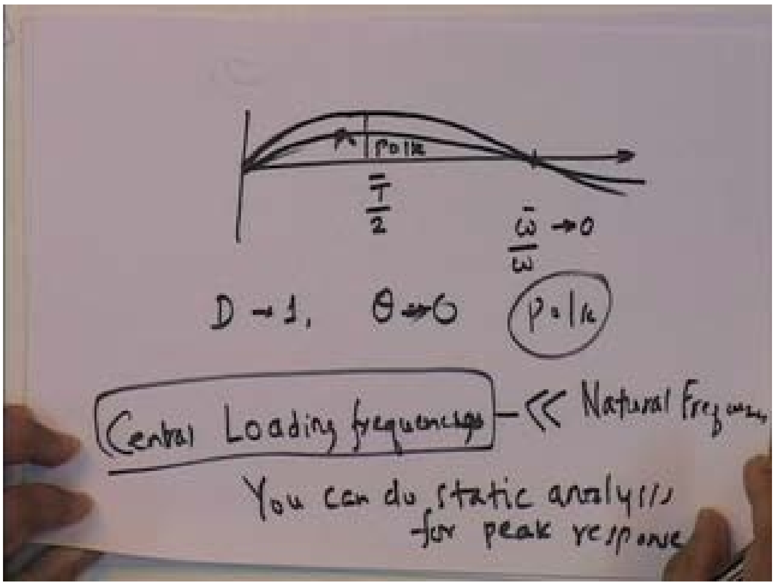

Relative to the structure, this is actually T bar by 2 this is half cycle. So, it is T bar by 2, so if you have omega bar up on omega tending to 0, it basically means that as far as the structure is concern this load is a very, very slowly varying load, a very slowly varying load. If it is a very slowly varying load what happens, for practical purposes, if we look at it let us substitute this in. So, let us see what dynamic amplification, is tends to 1 and let us look at what happens to theta it will be instructive to look at theta, we are not going to look at theta right.

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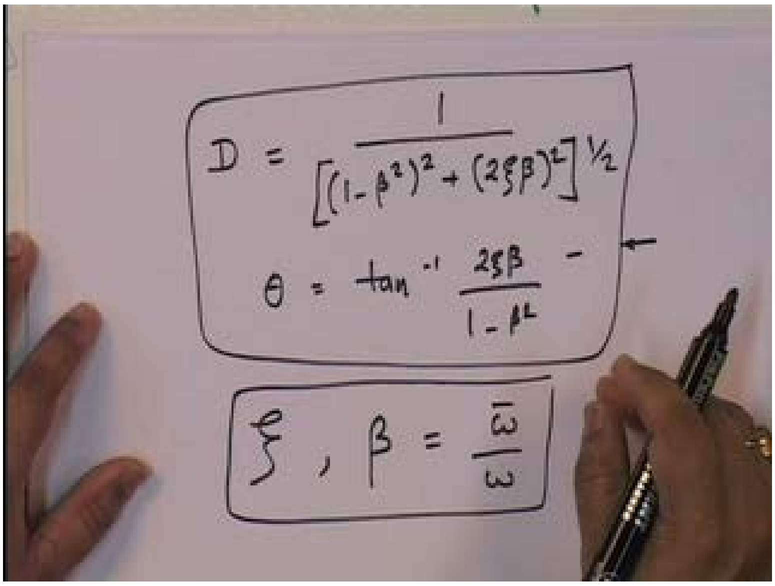

Now, but let us instructive, theta is tan inverse of 2 zeta beta 2 xi beta 1 minus beta squared. Now, beta is equal to 0, so this becomes 1 and this becomes 0 tan inverse of 0 is 0, so theta is 0; that means, what there is no phase lag. So, when you have a very, very slowly varying load how does the response go the response follows.

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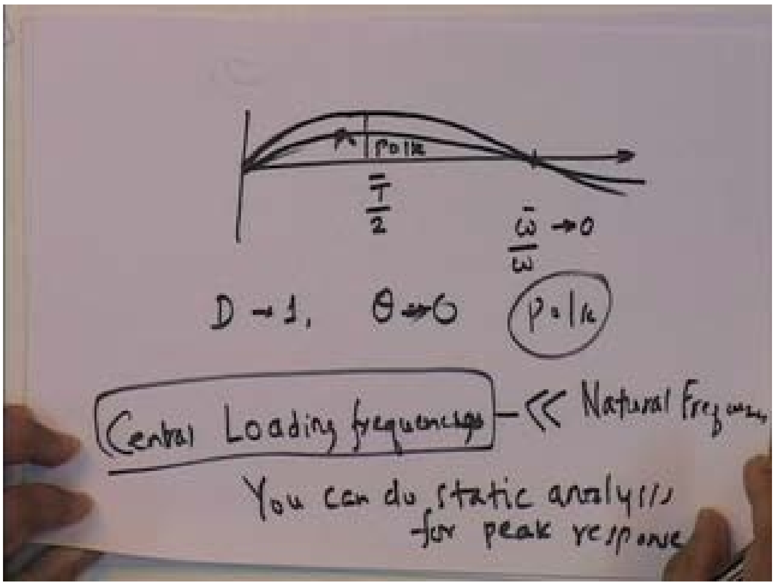

So, when it is a very slowly varying load, the response there is no phase lag it follows exactly what it is and of course, this is p naught up on k, where this is this is p naught. So, we will look actually plotting it in two different you know scales, but the point that I am trying to make is that for a slowly varying load, as far as the structure is concerned it is being subjected to a static load.

So, if the excitation frequency is very slow, compared to the natural frequency of the structure, you can for all practical purposes not have to do any dynamic analysis. Because, what is the peak response p naught up on k, you might have then do a static analysis for the load amplitude p naught, very important now this can actually be extended to all kinds of loads, all kinds of loads is later on I will discuss it further. If you look at loading frequencies, we will see later on that loading frequencies and we have a central loading frequency, these are terms. If you look at the central loading frequency, you can obtain this by doing some performing some mathematical operations of load, if this is very small compared to the natural frequency of the structure. You can do static analysis for peak response very, very important.

If the loading frequency and you know I mean in this particular case it is a harmonic load. So, we know what the loading frequency is, but for any kind of loading by performing some mathematical operations, you can get the central loading frequency you know. The loading frequencies in which the maximum, you know energy of the load is once you do that and if you look at those and those are very less, compared to the natural frequency of the structure. So, you have very rigid structure, if you have very rigid structure any loading, is going to be a static load for the rigid structure.

So, in other words if you have a very rigid structure, what can you say, you do not need to do dynamic analysis at all you can just do static analysis and get away with it, is that clear. So, that is the beauty of this approach now, the question becomes what happens as omega bar up on omega goes to infinity, we saw the dynamic amplification goes to 0. What is that, well let us look at it we just saw that for slowly for a as omega bar up on omega tends to 0, it becomes slowly varying load and therefore, you know it is a static load as far as you be concerned.

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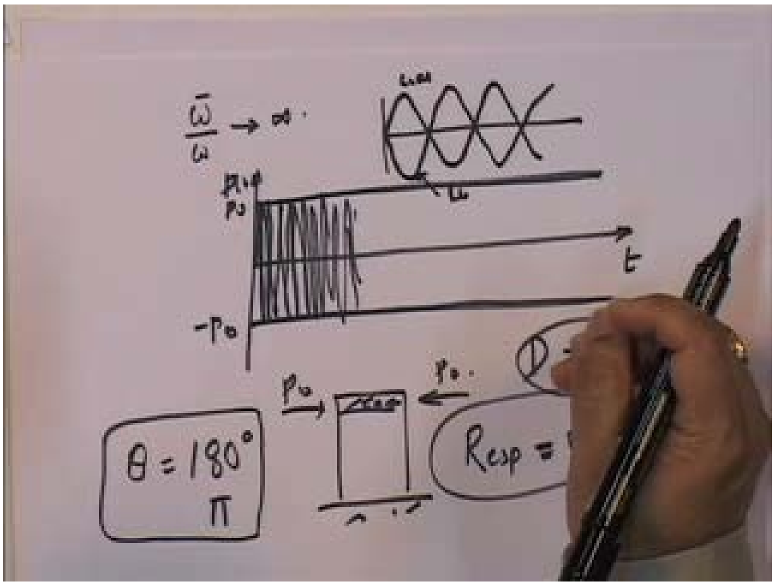

Now, what happens when you have omega bar up on omega tending to infinity, what happens when it is tending to infinity, when it tends to infinity let us look at what happens. This is t, this is p of t p naught minus p naught p naught minus p naught, this is what is happening I mean I am not going to draw it, but what we are saying is that omega bar is excitation frequency is very high, excitation is very high.

What is the structure see you know that the variation is, so fast all the structures actually seen, is a load p naught and another load minus p naught acting on it because the time variation is, so quick that as far as the structure is concerned all it can see is the p naught and minus p naught. So, if you subject a structure to p naught and minus p naught, what is the response. As far as the structure is concerned right, it is not seeing any load at all, see this because it is being subject to two equal and opposite loads. So, if you subject to two equal response the displacement response of the structure is 0 and that is why D is equal to 0 not very obvious. In many instructive to look at what happens to theta as beta tends to infinity.

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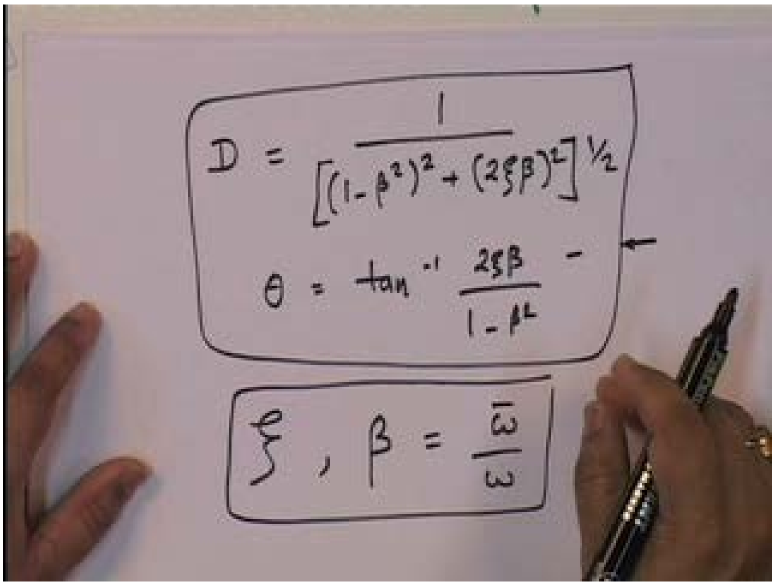

Theta is given by this term, as beta tends to infinity note that, this term disappears right. So, what do you have, you have beta up on, so basically it becomes tan inverse of 2 xi up on minus beta right. So, it becomes 2 xi up on minus beta, so it becomes minus 2 xi up on beta right, so it is tan inverse of minus 0 think about it, as beta tends to infinity this tends to minus 0 this term.

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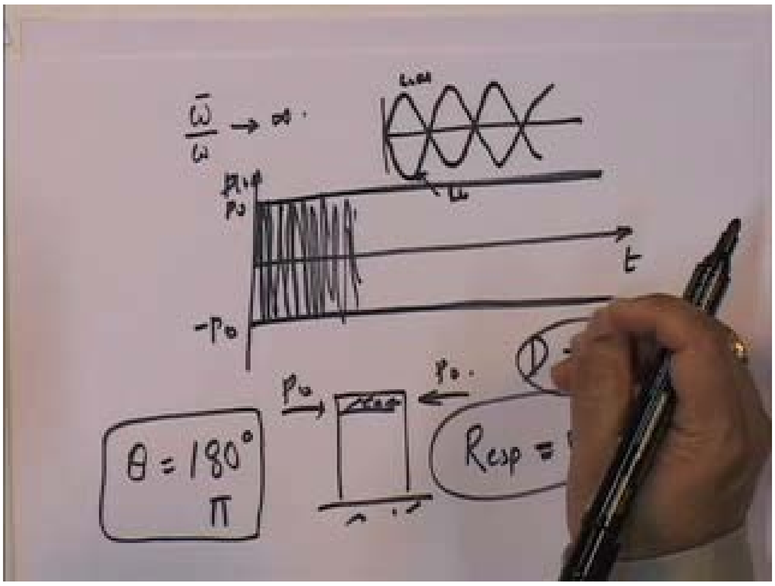

So, if it tends to minus 0 what is tan inverse of minus 0 tan inverse of minus 0 is pi or 180 degrees interesting is not it. So, if I look at the load you know I mean the load is very high dynamic amplification goes to 0, but you know let us assume that it is not, it is not tending to infinity it is slightly less than infinity. So, the lode goes like this and how do the response go, the response goes like this interesting is not it, response load. The load completely is out of phase, the response is completely out of phase with the load.

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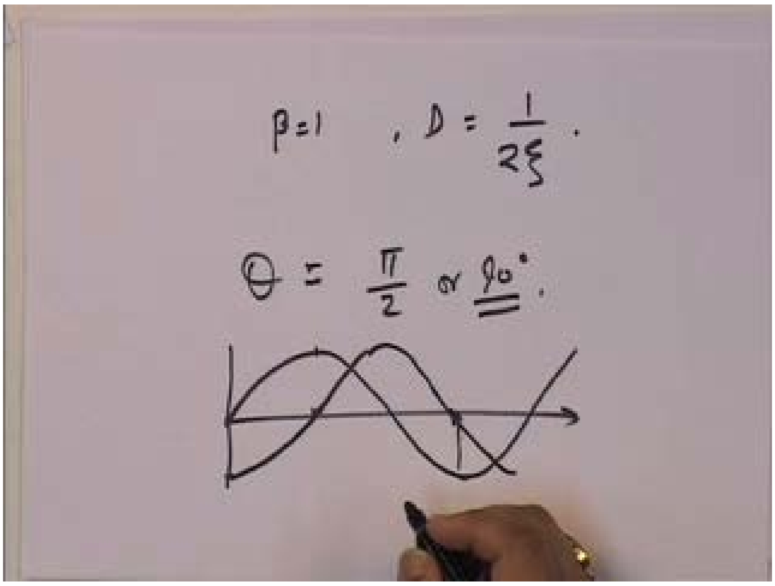

What is that, what is happening you know what is happening. Now, let us also see what happens we have beta equal to 1, we saw that D was equal to 1 up on 2 xi, what is theta equal to let us see let us plug in to beta is equal to 1. So, this is 1 minus 1 is 0 and this is 2 zeta up on 0, what is that tan inverse of infinity, what is tan inverse of infinity, if you look at tan inverse of infinity it is pi over 2 or 90 degrees, what it means is that if this is, the load then the response is of the following type, where this is maximum this is 0.

So, response is of where this is maximum this is 0 again. So, that is the response if is 90 degrees of out of phase with the loading, so when it is extremely high, when it is extremely low the response is in phase with the load because it is static. When beta is one which is resonance, dynamic amplification is 1 up on 2 xi, in other words it is extremely the response is extremely large and it is 90 degrees out of phase, it is exactly out of phase.

So, in other words it is almost like being sin omega bar t is the loading, then minus cosine omega bar t is my response that is what it says because it is you know that sin and cosine is 90 degrees out of phase with sin. So, if it is lag of sin omega bar t is minus cosine omega bar t, if it is extremely high frequency, dynamic amplification tends to 0 and response is completely out of phase, with the loading. Now, you know this is what we get, can we see is there a physical the mathematics tells us this, is there a physical aspect to this particular thing. And for this we need to go back to the fundamental aspect of what is u s.

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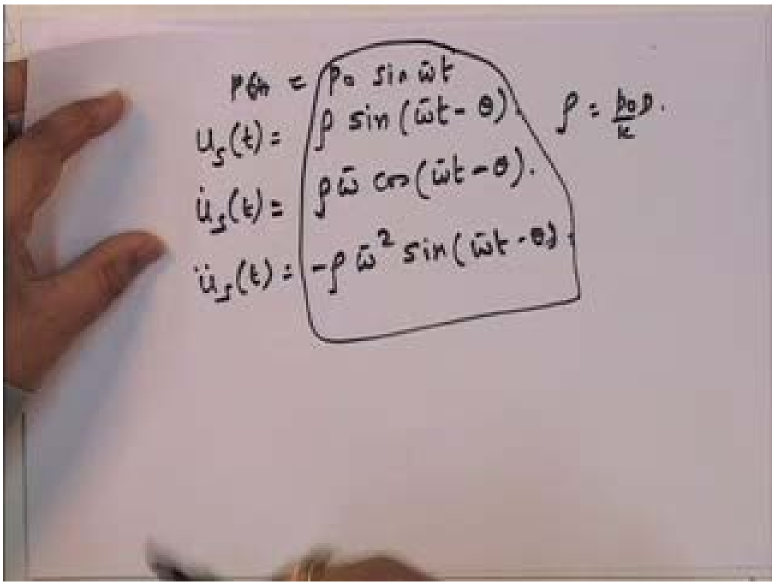

U s of t is rho sin omega bar t minus theta right, that is this term of course, p naught is sorry p of t is p naught sin omega bar t. Now, note that rho is equal to p naught up on k D, so if I differentiate this u dot s of t becomes rho omega bar cosine omega bar t minus theta. And if we look at it u double dot is equal to minus rho omega squared sin omega bar t minus theta, this is what we get, you know I mean you can see it very clearly.

So, now, let we on a complex plane plot these terms, so let us look at this, now note that you know if I am going to plot it on a complex these are real quantities. So, what I am going to be plotting is I am going to plot a complex vector, whose projection on the real axis represents this. So, that is what I am going to do, please note that let the vector by itself means nothing, the vector by itself does not represent anything. These are represented that the component on the real axis is what represents all these terms. So, let me now draw this thing and what I am drawing now is known as the argand diagram.

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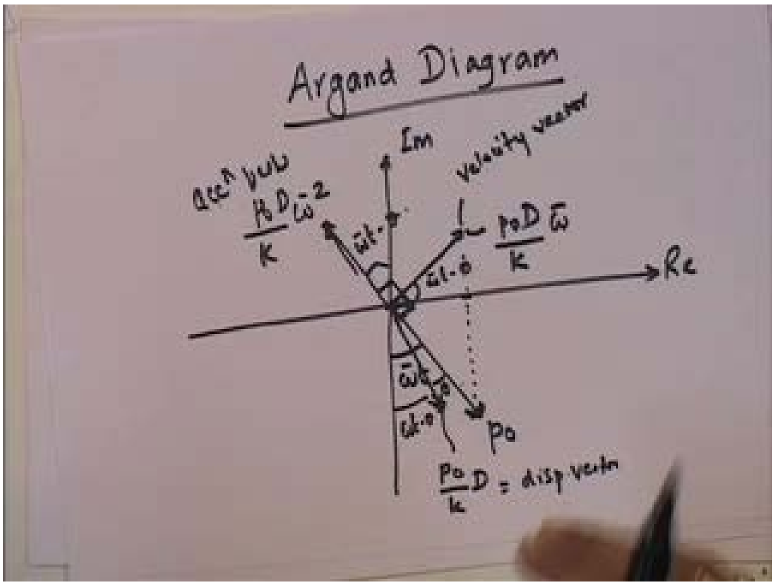

We going to draw argand diagram and how is the argand diagram look let us just plot it, So, this is the real axis, this is the imaginary axis, now, I have a p naught sin omega bar t how will that look, that looks like this, it looks like p naught this is omega bar t let us look at this, let us look at the real projection. Remember I said that, it always a real projection, so let us take the real projection, what is the real projection of this, if this is omega bar t; that means, this is omega bar t right.

So, what is this term, this term is if this is p naught, this is p naught, this from here to here is p naught if this is p naught then; obviously, this part is equal to p naught sin omega bar t. Remember, that is what it is the load is given on the real axis right, so this is, so in other words this vector in the complex plane, represents p naught sin omega bar t because it is real projection is p naught omega bar t.

In the same way, I am going to plot now the u s of t, let us see what this u s of t is. It is given as p naught d up on k sin omega bar t minus theta. So, to represent this I can actually plot it here, these are wait I mean this is not the you know I am just plotting p naught the p naught should be plot plotted on a force complex plane, and these should be the displacement complex plane. I am just drawing them on the same, you know axis it does not really matter, where this represents theta and the amplitude is given by p naught up on k into D.

So, this is your displacement, so displacement is this direction, I mean you know if you look at this is omega t minus theta and you know if you take the same thing, you will see that this is indeed p naught d up on k. So, this is the displacement, this is you know p naught D omega bar up on k cosine now, how does cosine look, cosine if you look at it. It actually leads it always leads, so it is leading see this is lagging, this is leading let us plot this and this one I am going to plot it in this direction.

So, this is the direction of the displacement vector, this is the direction of the velocity vector and what is the value of this, the value of this is p naught d up on k into omega bar, that is the velocity and let us see whether this make sense. Now, if this is omega bar and this is 90 degrees which is my omega bar, now if you look at it omega bar t because this is omega bar t minus theta and this is 90 degrees to it.

So, which is my omega bar t minus theta, let us see this angle would be omega bar t minus theta is not it. And let us look at it, is real projection it is real projection is equal to this parameter into cosine omega bar t minus theta, see that is it that is how it is and if you look at this is just negative of sign, if you look at this the acceleration is in this direction. The acceleration is in this direction and what is the acceleration equal to p naught d up on k omega bar square.

So, this is my omega bar t minus theta and you will again see that this is nothing, but sin omega minus sin. So, this represents these three represent the displacement vector, the velocity vector and the acceleration vector. And these the or this is you know, in this direction the velocity vector, leads the displacement vector and the acceleration vector is completely out of phase, with the displacement vector. And these vectors, with t keep rotating and they are all 90 degrees to each other.

So, this is the very important aspect, in the next lecture I am going to carry on this to show you how the entire load, is carried by the various forces you are going to do force equilibrium Eigen diagram and we going to very interesting because if you look at it this is the displacement. So, elastic force opposes the displacement, the inertial force will oppose the acceleration and the damping force will oppose the velocity. So, these are interesting points and we will look at this in the next lecture.

Thank you very much.