Structural Dynamics
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Lecture - 4
SDOF Response to Harmonic Loads

Good afternoon, today we are going to be talking about the single degree of freedom response.

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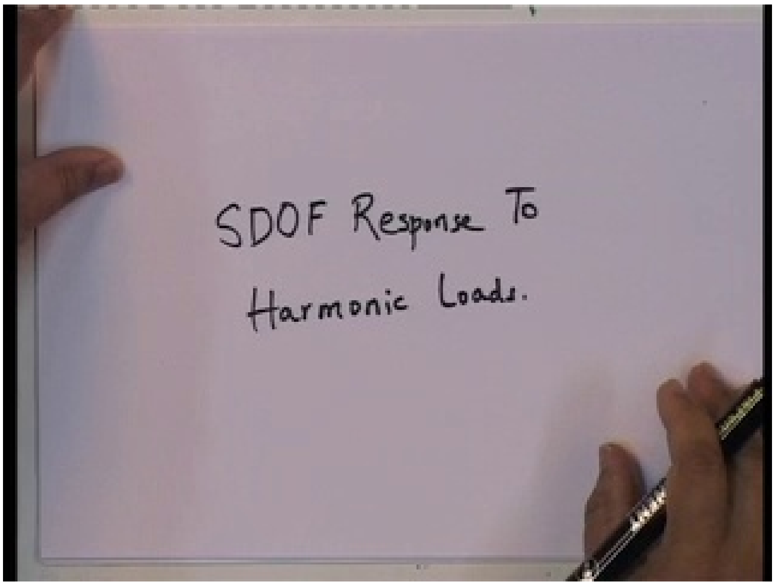

Single degree of freedom response to harmonic loads, if you remember we had discussed towards the end of last lecture what those different kinds of loads that structures are actually subjected to, and you know the simplest kind of load. And the kind of load that you get when you are rotating machinery with constant speed sitting on top of a structure is known as harmonic load. The harmonic load is of this of this form.

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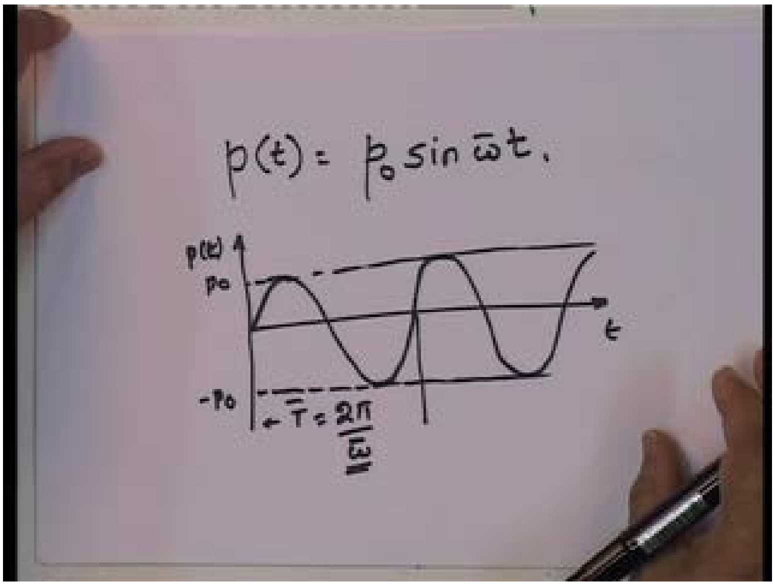

We will see that later on the harmonic need not necessarily be of this form, there are various harmonics that you can have, but this is the simplest kind of harmonic, and let us go with this. So, this kind of loading essentially is of this form. This is the periodicity of the load, which is of course given by this, where this is loading frequency of the harmonic load. The amplitude is p naught and minus p naught. So, this is the sinusoidal load that is given in this format. So, let us look at the solution of the response of a structure to this kind of loading. So, before we start looking at different kinds of structure. We will start of with the un-damped structure.

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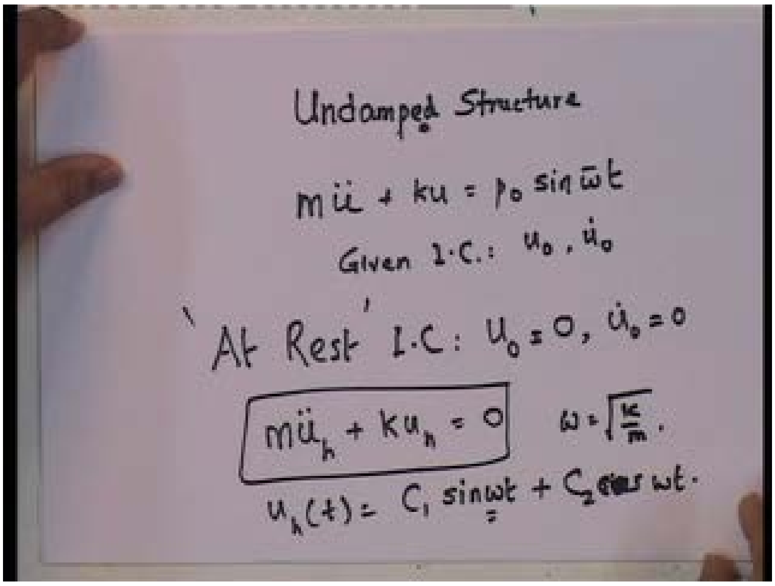

We will see later that the un-damped is a special case of damped, but we will start off with the un-damped structure. Then the equations of motion becomes m u double dot plus k u is equal to p naught sin omega bar t of course, given initial conditions which are u 0 and u dot 0. So, these are the given initial conditions. Now we do not want to be looking at free vibrations. If we did not give an initial condition you wouldn't have a vibration for example, if you did not give an initial displacement the structure would not vibrate in freely; however, for forced vibrations typically the assumption is that if you have the structures like this and then its hit with the load, and because the load it starts moving. So, by enlarge when we are looking at force loads, we are going to assume what are known as at rest initial conditions, which is u 0 is equal to 0 u dot 0 is equal to 0.

So, that is what we would assume, but of course, you know it is not necessary that you have to assume this if you do have a initial displacement and initial velocity. It will solve the problem you know that is not an issue, but let us just solve it for this particular problem. Now remember we talked about the fact that you had the homogeneous solution and the homogeneous solution was essentially, the solution to this equation we know what that solution to that equation is.

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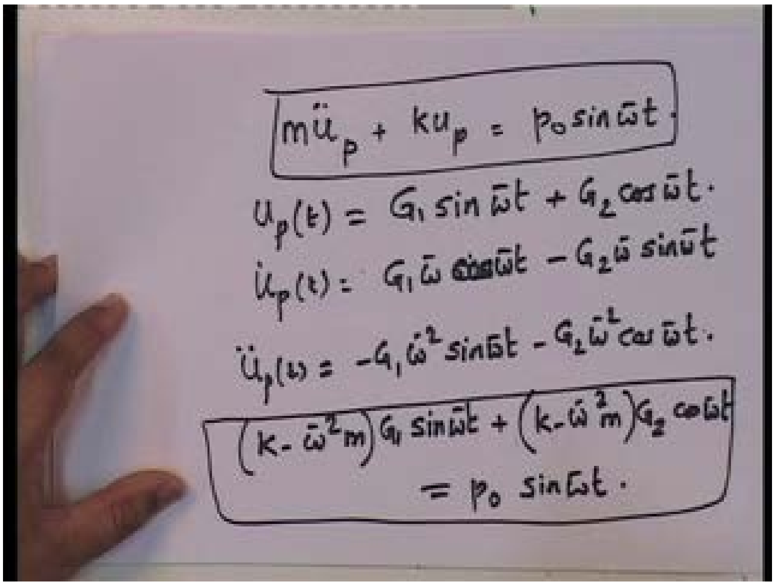

So, we know that u h of t is equal to C 1 sin omega t plus C 2 sorry C 2 cosine omega t, where omega as you know is the natural vibration frequency, which is given by this we already know this we have already evolved this I do not want to go into that now we go to how to get the particular solution how do we get the particular solution well. Let us look at what this is the particular solution has to satisfy. Now mathematics you know from mathematics we know that u of p t has to be of the form for the harmonic load. It has to be of the form G 1 sin omega bar t plus G 2 cosine omega bar t. This is the form that it has to take and G 1 and G 2. We can obtain by actually putting this into this equation that is the particular solution. So, if you look at this without going into the y of it this one turns out to be G 1 omega bar cosine omega bar t minus G 2 omega bar sin omega bar t and u double dot p of t is equal to minus G 1 omega bar sin omega bar t minus G 2 omega bar square cosine omega bar t. So, substituting these in here and you know putting them together becomes what putting the sin omega bar t together,

You get like u double dot minus G 1. So, it becomes k minus omega bar square m into g 1 sin omega bar t, and then what is the other one? It becomes the cosine if you take it becomes plus k minus omega G 2 cosine omega t is equal to p naught sin omega bar t. This is what we get by substituting these into this equation this equation, and substituting that into this and collecting terms, we get this equation now once you get that equation then automatically if you look at it what happens here, let us look at it sin omega bar t.

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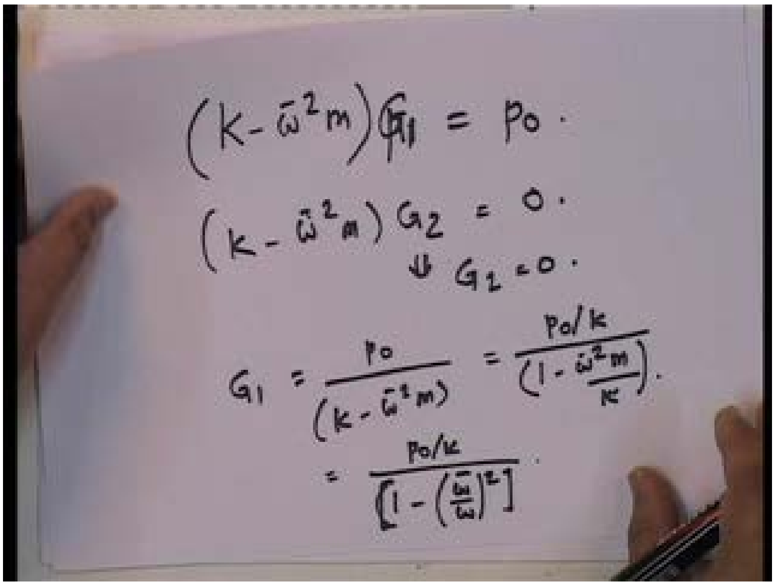

Sin omega bar t cosine omega bar t does not exist, which basically means this is equal to this and this is equal to 0; obviously, you know you have to the left-hand side and the right-hand side have to be the same for at any instant of time t. So, at any instant of time t since sin omega bar t and cosine omega bar t are different functions; that means, if this is sin omega bar t the coefficient of this has to be the same as this and the coefficient of this has to be always 0.

So, having given that what do we get substituting that we get K minus omega square m into sorry into g 1 is equal to p 0 that is one and the other one says k minus omega square m into G 2 is equal to 0 that is what we get. So, automatically this k minus omega bar square m need not be 0 at any time. So, therefore, obviously, this implies that G 2 is equal to 0, and from the first one we get G 1 is equal to p naught upon k minus omega bar square m, now what I am going to do is I am going to make this equal to I am going to take k outside.

So, this will become p naught by k. So, inside will be 1 minus omega bar square m upon K. Now note what is k upon m k upon m is omega square. So, what do we get? then this becomes G 1 becomes p naught by k into 1 minus omega bar this is what G 1 becomes. So, if you substitute that in then the fact that G 2 is equal to 0; that means, the particular solution turns out to be of the form.

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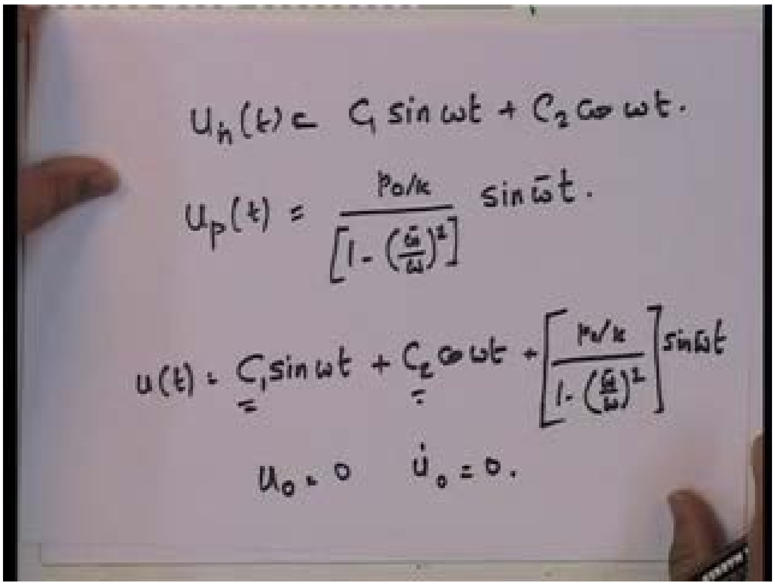

U p of t turns out to be p naught upon k into 1 minus omega bar on omega square this is G 1 sin omega bar t. So, you have we have seen that u h of t is equal to C 1 sin omega t plus C 2 cosine omega t and therefore, u of t therefore, becomes C 1 sin omega t plus C 2 cosine omega t plus p naught upon k 1 minus omega sin omega bar t. So, this becomes now the solution now how do we get C 1 and C 2 well, we substitute the fact that u 0 is equal to 0 at rest initial conditions remember that at rest initial conditions. So, we need to satisfy we need to plug-in these two terms into this and we can get the solution. So, let us look at what the solution looks like. So, if we substitute this in directly if we go back here, if you substitute t equal to 0 this one is 0. So, it becomes C 1 0 plus C 2 into 1 plus this term into 0. So, what do you get? You get C 2 into 1. So, C 2 is u 0 u 0 0. So, that implies automatically that C 2 is 0. How do we get C 1? Well.

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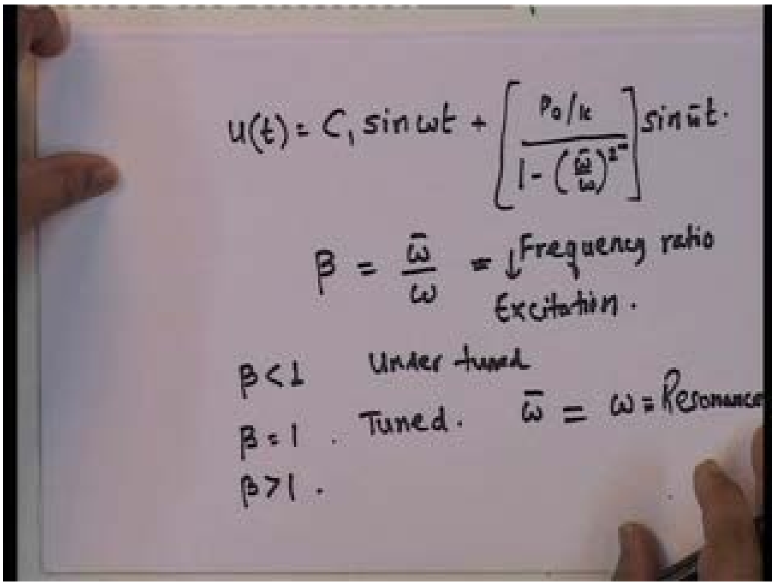

Now, since we know that C 2 is 0 this basically implies that u of t is equal to C 1 sin omega t plus p naught upon k 1 minus omega bar into sin omega bar t, now just for the sake of simplicity instead of keeping on writing this I am going to define a parameter beta, which is omega bar this is known as the frequency ratio. This is the excitation frequency to the natural frequency of the structure. So, this is the ratio of the excitation to the natural. So, this is actually more written as excitation frequency ratio note, that if you look at this if the excitation frequency, the loading frequency is lower than the natural frequency of the structure then beta is less than 1. If the loading frequency and the natural frequency are tuned to each other, what does that mean? Tuned this is known as under tuned, because excitation frequency is lower than the natural frequency. If beta is equal to 1 this is known as the tuned condition, where the excitation.

Frequency is equal to the natural frequency of the structure this is also known as resonance. The reason why we call it as resonance is the fact that they tune to each other, and when you are tuning you will see that there is a beating phenomenon you know about that, but anyway beta greater than one implies that the excitation frequency is higher than the natural frequency of the structure. So, these are the possibilities. So, now, let us look at what happens let us see how to calculate C 1. So, let us now see the situation, where I am going to substitute now I am going to actually differentiate that term. So, let me differentiate that term. What do I get?

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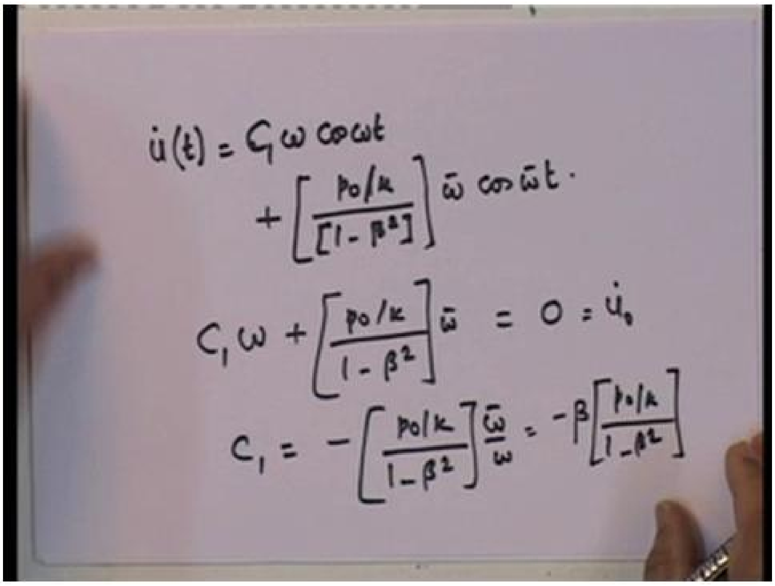

I get u dot of t is equal to minus sorry C 1 omega cosine omega t plus, now we have p naught by k upon 1 minus beta square. The beta is the excitation, the excitation ratio of the excitation to the natural frequency into omega bar cosine omega bar t now I am going to substitute what am I going to substitute? I am going to substitute the fact that u of t at t equal to 0 cosine omega bar t is 1 cosine omega bar t sorry cosine omega t for t equal to 0 is 1, and cosine omega bar t for t equal to 0 is also 1. So, what we get is that c 1 omega plus p naught upon k into 1 minus beta square omega bar is equal to 0 why , because the initial velocity is 0 at rest initial condition remember that. So, from this we can get c 1 and c 1 turns out to be equal to minus p naught by k 1 minus beta square omega bar upon omega, which is also beta. So, its equal to minus beta p naught by k into 1 minus beta square, and then having got that now we can write down the final form of u of t, the response the displacement response of a single degree of freedom.

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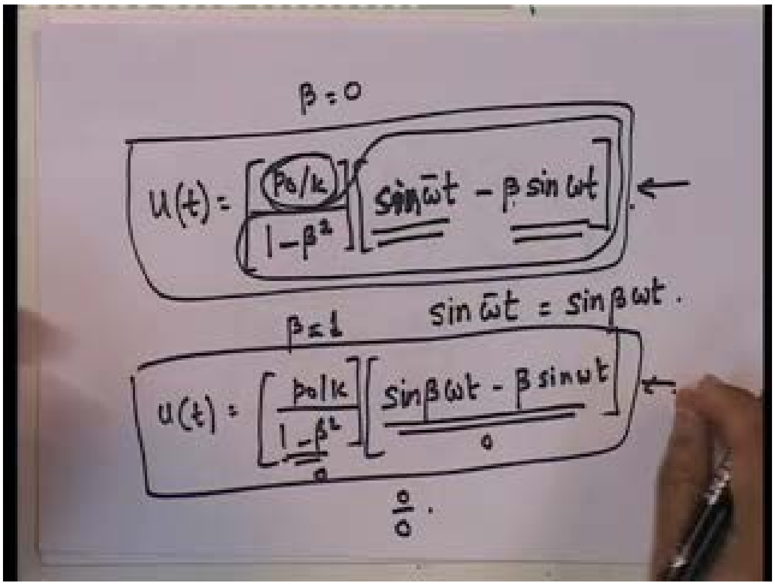

Un-damped structure to harmonic excitation becomes this p naught by k into 1 minus beta square, and then inside we have cosine omega bar t minus beta sorry this is sin u t is sin omega bar t into beta sin omega t. So, this is the response , you see there are there are two path to it there is one path that is the excitation frequency path, and there is one natural frequency path so; obviously, you know these are two different harmonic functions an u of t is a sum of two different harmonics. So, there is a beating phenomenon and a constructive, and destructive beating.

Now let us look at the situation let us look at the particular situation, where beta is equal to 1. if beta is equal to 1 what do you get? Let us look at it if beta is equal to 1 substitute it by the way remember that sin omega bar t is nothing, but sin beta omega t right. So, if you look at it u of t equals to p naught by k 1 minus beta square sin beta omega t minus beta sin omega t right. So, this is the expression that is there for any time. Now substitute beta equal to 1 in here what happens if you substitute beta equal to 1 this goes to 0 and for note that this also goes to 0. So, you have the form 0 upon 0. So; obviously, you cannot there does not exist a solution directly. It is of the in determinant form 0 upon 0. So, what do you have to do? Well we know that when you have 0 upon 0 as a solution you have to use l’hospitals rule.

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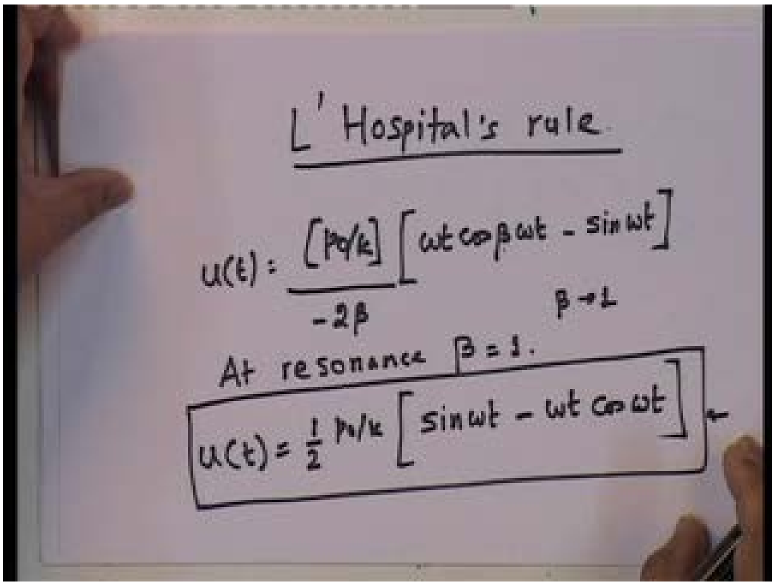

What is l'hospitals rule is that well, you know when you have a 0 by 0 in determinant form you differentiate both. So, see what is happening is this is this here the parameter beta, which is in the limit going to 1. So, essentially the parameter beta, so you have to differentiate both the numerator and the denominator with respect to beta and keep doing this till you go out of the 0 by 0 form. So, that is the important aspect of this situation. So, let us apply l'hospitals rule here. So, what do we get note that the p naught upon k is a constant. This does not change if you look at this the denominator 1 minus beta square when differentiate with respect to beta becomes 2 beta and the top.

If you look at it we you have sin you know look this is what you are differentiating? right you differentiate this with respect to beta and you differentiate this with respect the denominator 1 minus beta square differentiate with respect to beta you get minus 2 beta. So, that is minus 2 beta and this term sin beta omega bar t what is it become? it becomes you are differentiating with respect to beta. So, what you get is omega t cosine beta omega t, and the other one is note that if you differentiate this term with respect to beta then what you get is minus sin omega t. Now let us substitute the beta equal to 1 in this particular case, because even now if you get 0 upon 0 you have to differentiate it further, but let us see what plug beta? Let beta tend to 1 right. So, plug in beta equal to 1 this becomes minus 2 and this becomes 1. So, it becomes cos omega. So, it becomes omega t cosine omega t minus sin omega t this is not 0 either. So, we have reached a stable solution, so therefore at resonance, which is beta equal to 1 u of t is equal to p naught by k I will call this as half, because 1 of 1 to be minus I will take inside. So, this becomes sin omega t minus omega t cosine omega t. this is defined the response of an un-damped system at resonance. If you look at this form what is this form look like? This form actually looks like this.

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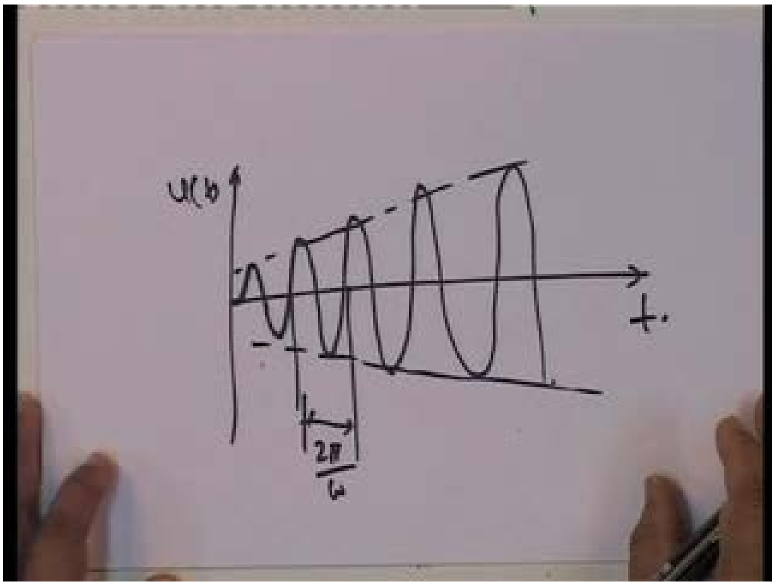

Note here that there are two harmonics sin omega t cosine omega t; however, there is an omega t associated with the cosine omega t. So, this explodes with time and. So, what you have is at this point since you have at rest conditions u 0 equal to 0 u zero u dot 0 equal to 0. So, it goes like this, what happens? Is it slowly explodes, and what happens is that this function is increasing with t. So, at infinite time you are going to have infinite displacement an un-damped structure is subjected to resonance.

Excitation is called resonance excitation why what is resonance excitation? Where the excitation frequency tunes to the natural frequency of the structure. So, if you have an un-damped system, which is subjected to resonant excitation harmonic excitation ultimately the displacement will become infinite and that is the major issue that you have of course, here this is 2 pi upon omega there is you know there is there is what is known as constructive beating. You know that the two frequencies are the same. So, the excitation frequency and the natural frequency and what happens is that the response keeps increasing with time this is u of t and this is t. So, that is what happens at resonance lets go back this is the displacement response at resonance, and this is the of peak or off resonance response it is interesting to know something and that is that you know you have a loading, let us just go back what kind of a loading did you have? You had a loading that had the form.

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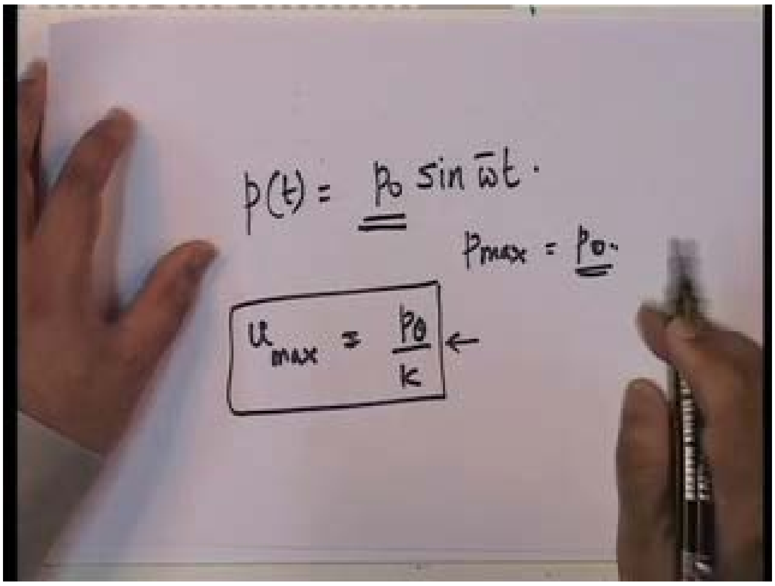

Right, you had this form the amplitude is p naught, now let us assume for a moment that we ignore this time variation we say that oh p of t is p naught then what would be the response? If you look at it the response would simply be nothing. So, you know p max is equal to p naught and if we subjected this static load to the structure u max would be p naught upon k; obviously, this is trivial. So, in other words if the p naught was static load, then the u max would be p naught upon k right, but look at this what happens here? If you look at in terms of u max look at this p naught upon k exists here.

So, this is like the static response. So, we can look at this as the dynamic parameter right, and look at this the generic parameter is 1 minus beta as the term 1 minus beta square this explodes think of think of the situation, where beta is equal to 0 what is beta equal to 0 means? Beta equal to 0 means that the load frequency is 0 loading excitation frequency is 0 excitation frequency is 0. What does that mean? That means, that if you look at it that this is nothing, but like a p naught load acting on the structure. If the loading was acting p naught loading was acting on the structure then what would be the response p naught upon k lets substitute in here what do we get sin omega bar t. So, this is 0 this is small this becomes omega bar t this beta is 0.

So, this disappears. So, essentially if you look at it this is 1 minus beta square beta square is 0. So, u of t becomes p naught upon k omega bar t that is the response and again omega bar is 0. So, for all practical purpose this tends to 0 what is the response? The response is p naught upon k well, you see beta equal to 0 is a static loading and the response is p naught upon k you get that automatically. So, the point I am trying to make is that you know this later on, we will see is a very important aspect of we will see how we can get a situation, where we can get what is known as the dynamic amplification factor that gives u max as p naught upon k times dynamic amplification factor.

Today you see I am although I am giving you some overview of the physics of the problem what we are going to be doing is essentially grinding through the mathematics for harmonic loading, because once you grind through the mathematics for the harmonic loading , we can then start looking at the physical aspects that are important for harmonic loading.

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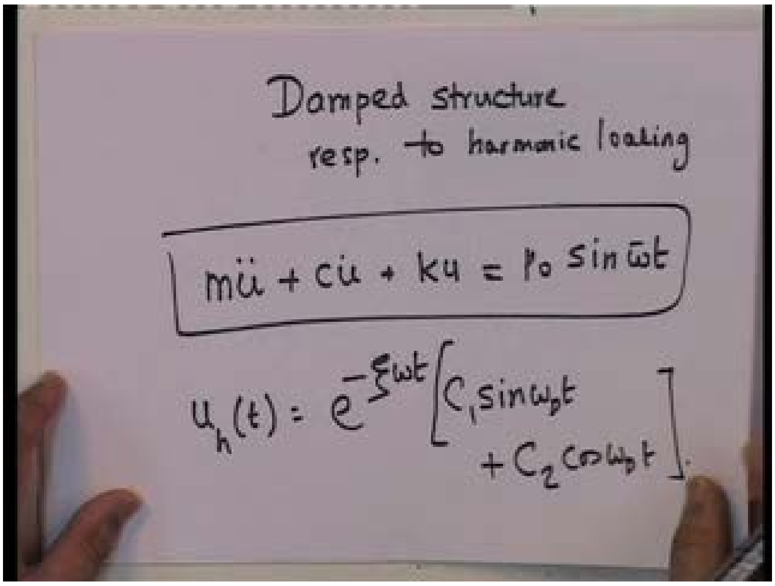

So, what I am going to do now today right now look at we have looked at already the un-damped. So, now we going to look at damped structure response to harmonic loading. So, what is that the how does the mathematical equation look trivial. We have seen this enough number of times by now I am just going to write this is the equation of motion I have said right this is the equation of motion how do we solve this well the homogenous path the particular path we keep doing this what is the homogenous path? Look like I have already solved it what is that e to the power of minus xi omega t into C 1 sin omega d t plus C 2 cosine omega d t that is my harmonic solution.

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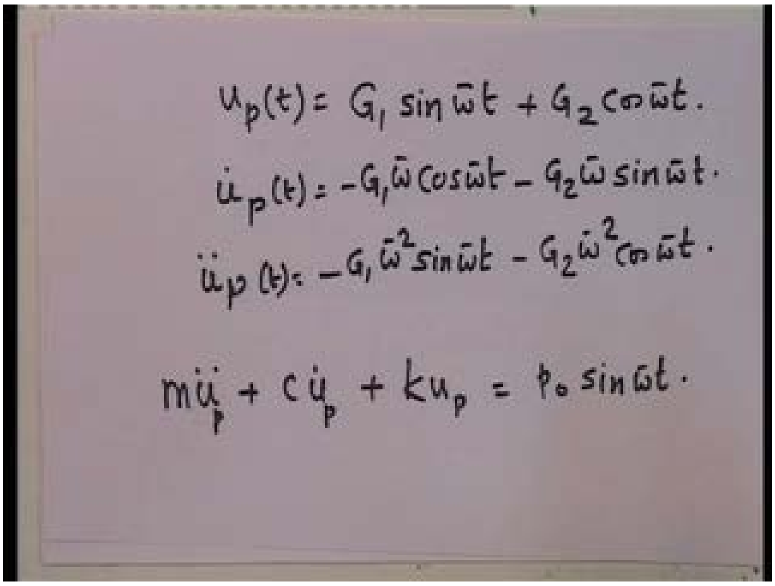

Now, let us see what the particular solution looks like well the particular solution for all kinds of harmonic load, we know is of the form u of p G 1 sin omega t plus G 2 cosine omega bar t. So, this is of the form. So, now, let me just substitute this becomes again you know go through the steps u double dot p of t is minus G 1 omega bar square sin omega t minus G 2 omega bar square cosine omega bar t. So, these are this, and now we substitute this we would have plug all of these into this now let us see if you look at this particular situation what happens is m u double dot becomes minus m into G 1 omega bar square sin omega bar t minus m G 2 omega bar 2 cosine omega bar 2 you know then C will become C into G.

So, substituting all of these and then collecting all the different parameters into one solution becomes what let us look at what it becomes, and I am going to write this what I am doing? I am substituting all of these I am substituting these into this and then collecting terms collecting all the sin omega bar terms, and cosine omega bar t terms together. So, if I connect collect all of them this is what I get? I get.

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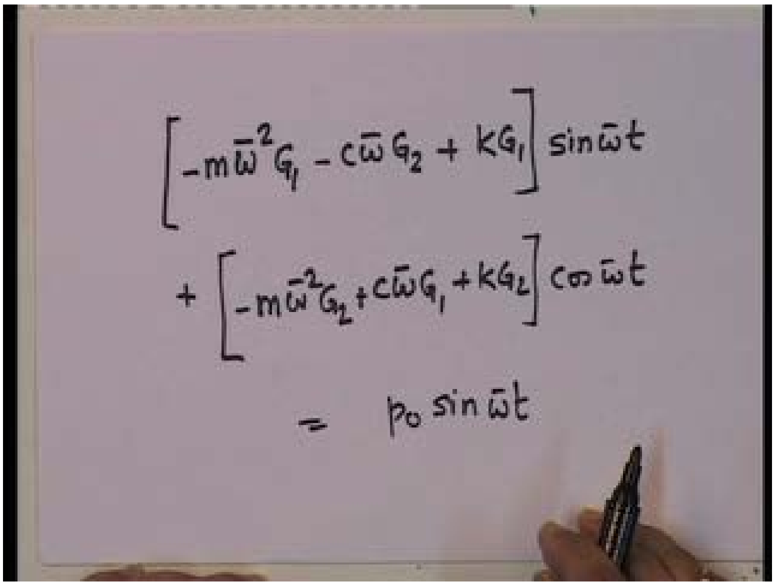

See m into minus G. So, it is minus m into omega bar square G 1 that is the sin omega bar t term I know. So, now, let us look at what is the sin omega bar t comes from the C term that is minus C omega bar G 2 that is the sin plus k into G 1. This is my sin omega bar t terms put together then I put in all the cosine omega bar t functions together. So, then what will I get is m omega bar G 2 minus C omega bar I am. So, sorry I made a mistake here I want to go back and correct that this is G 1 sin omega bar t plus cosine. So, it is actually plus c omega bar G 1, and then plus k G 2 cosine omega bar t is equal to p naught sin omega bar t.

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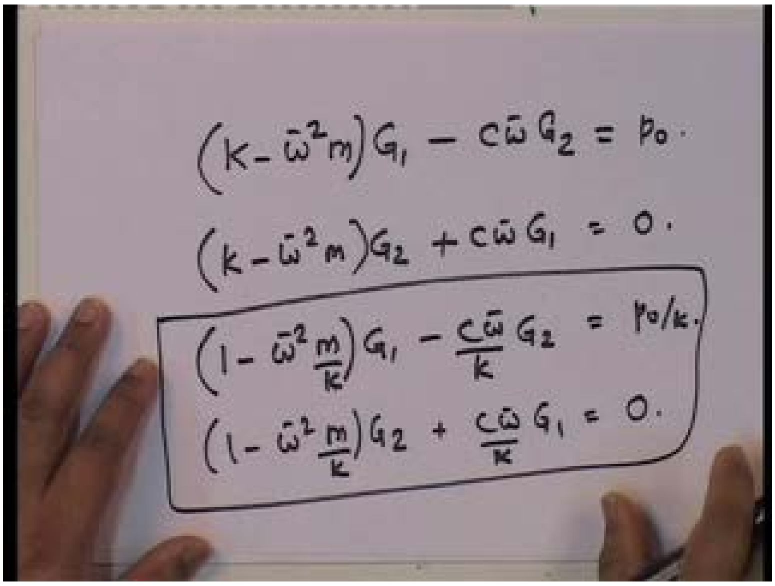

So, what I have done is I have substituted u p u dot p u dot p directly into the left hand side and then that is equal to p naught sin omega bar t that is what I have done here. So, if you look at this then what we get. K minus omega bar m into G 1 minus C omega bar G 2 is equal to p naught and we get k minus omega bar G 2 plus C omega bar G 1 is equal to 0, because note that the sin omega bar t this is equal to this, and this is equal to 0. So, that is all I have that as all I am just writing down those terms over here, now what I am going do is I am going to simplify this by dividing both sides of the equation with k, if I substitute if I do that then what I get is 1 minus omega bar square k upon m G 1 minus C omega bar upon k G 2 is equal to p naught upon K, and we get 1 minus omega bar m upon k G 2 plus G 1 is equal to 0 right.

Now the question sorry upon k right these are the two terms that you get now from these two you can solve for G 1 and G 2. So, but before we do that lets simplify some of the terms, if you look at this we know that K upon m is equal to omega square and we have already defined the fact that.

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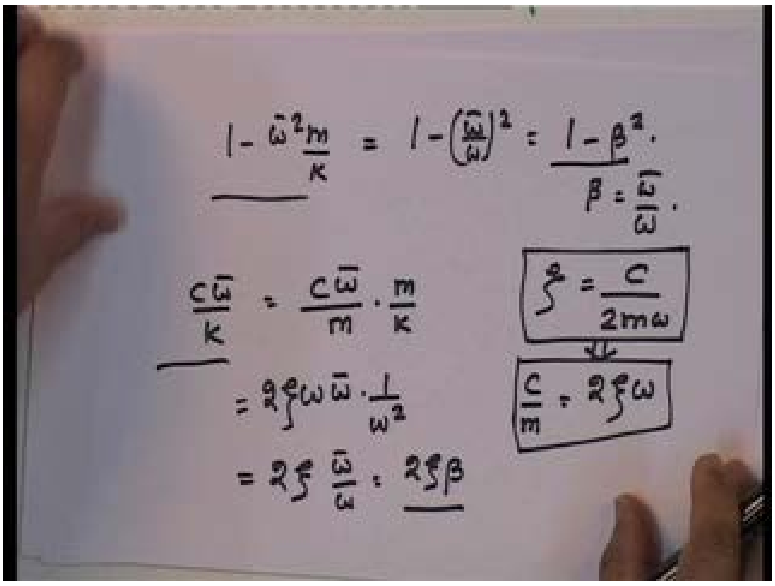

One minus omega bar square m upon k is equal to one minus omega bar upon omega the whole square, which is equal to 1 minus beta square, which we have already defined the term beta remember right. Now, let us look at the other term now I am going to write this term in a slightly different way, I am going to write it as C omega bar upon m into m upon k it is I am just you know multiplying the top and the bottom by m. So, that is. So, you know this is the same, but you put this in what happens now look at it by definition what is zeta equal to zeta is equal to C upon 2 m omega remember the damping ratio definition, which we are defined earlier right C upon omega.

So, then C upon m is equal to 2 xi omega right. Obvious, it is just naturally follows from here. So, we get. So, C upon m and therefore, this term becomes 2 psi omega into omega bar upon now m k is one upon omega square. So, if you look at this becomes 2 psi omega bar upon omega. So, this becomes 2 xi beta. So, now, you know having substituted this is equal to this, and this is equal to this if I substitute that into those equations, what we get is the following this term is 1 minus beta square and this term is 2 zeta omega 2 zeta beta sorry 2 zeta beta.

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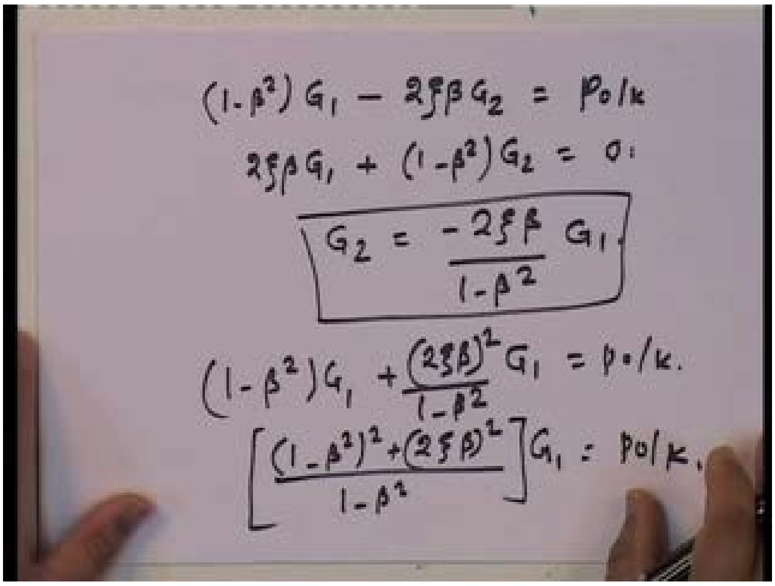

So, if I substitute that in what I get is 1 minus beta G 1 minus 2 zeta beta G 2 is equal to p naught by k, and the other one becomes 2 zeta beta G 1 plus 1 minus beta square G 2 is equal to 0. Now you know I can actually solve this simultaneously, but I will I will go through these steps here very easily, because what I can do is from this equation I can see that G 2 is equal to minus 2 zeta beta upon 1 minus beta square G 1 its obvious right. So, if I substitute that into the top equation what I get is 1 minus beta square G 1 minus and the minus becomes plus. So, this becomes plus 2 zeta beta the whole square upon 1 minus beta square G 1 is equal to p naught upon k that is what I get. So, now, if I the both are G 1.

So, I can take out G 1 and note that this I can write as both sides. So, then I will get it as 1 minus beta square the whole square plus 2 zeta beta the whole square all upon 1 minus beta square into G 1 is equal to p naught upon K fairly complex competition, but ultimately what we get.

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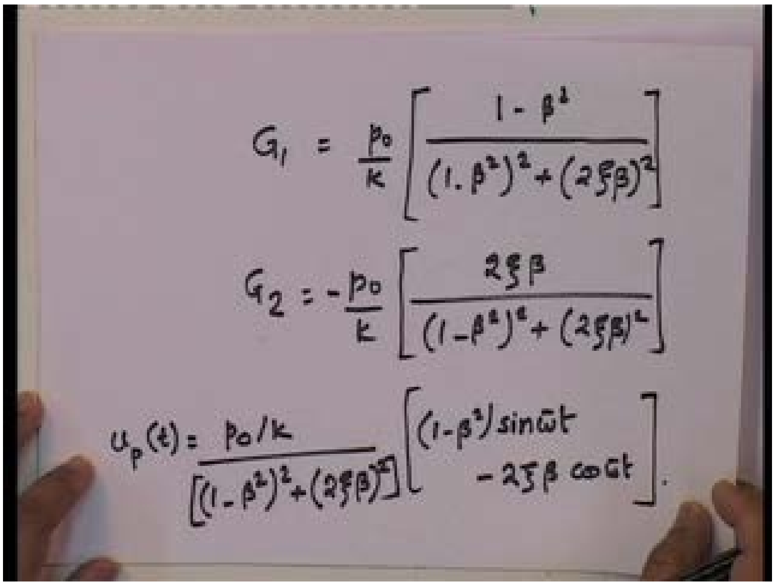

That G 1 is equal to p naught upon K into 1 minus beta square upon 1 minus beta square the whole square plus 2 zeta beta the whole square that is G 1 similarly substituting the fact that G 2 is this we get G 2 is equal to minus 2 zeta beta 1 minus beta square plus 2 zeta beta whole square, and you must have forgotten what these G 1 and G 2 were. So, let us go back to where what G 1 and G 2 were they were the particular solution while particular solution is G 1 sin omega bar t plus G 2 cosine omega bar t. Now we have these. So, therefore, we get u of p as equal to p naught divided by k.

And I will put the rest of it outside 1 minus beta square the whole square plus 2 zeta beta the whole square this one stays outside and inside becomes 1 minus beta square sin omega bar t minus 2 zeta beta cosine omega bar t. So, this becomes my particular solution if this is the particular solution then now. So, what is my total solution? You see this is becoming extremely complex. So, what I am going to do is I am going to define no I am not going to define anything, let us just put down the u of t right.

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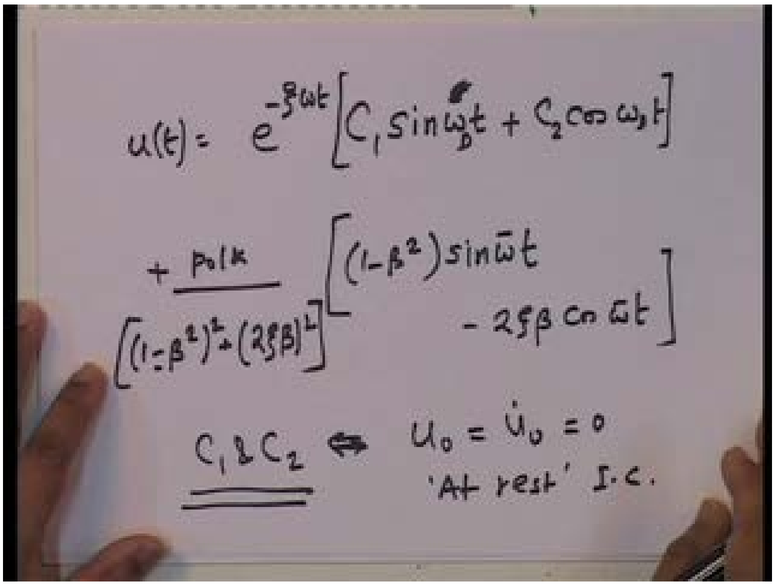

Now, u of t becomes e to the power of minus zeta omega t into C 1 sorry there's no this is sorry there's nothing here sin omega t then absolutely nothing here plus omega d t plus C 2 cosine omega d t plus p naught upon k into 1 minus beta square the whole square plus 2 zeta beta the whole square into 1 minus beta square sin omega bar t minus 2 zeta beta cosine omega bar t, and how can we get C 1 and C 2? C 1 and C 2 we get from u 0 is equal to u dot 0 is equal to 0, which is at rest initial conditions.

So, now you substitute into this you see the basic problem that happens over here is the fact that although you know, we can get C 1 and C 2, you can you know put in here. And you will get some you know you will get some what you will get is you put t equal to 0 this one becomes one this becomes 0 you left with C 2 into 1 plus this one this one becomes 0. So, this one disappears this one becomes 1. So, it basically becomes that C 2 is equal to this into 2 zeta beta.

So, p naught upon k 2 zeta beta 1 minus this that C 2 and C 1 we can obtain by differentiating u dot of t etcetera the point comes down to this that we do not care about C 1 and C 2, you can look up any book, and it will show it will give you C 1 and C 2 with all the mathematics behind it is not relevant the relevant part here is the following that this is the solution, where C 1 and C 2 are known these are known this is the solution complete solution of a single degree of freedom damped system with damping xi damping ratio xi and the excitation frequency to the natural frequency ratio as given by beta, if that is the situation this is what we get, but note that omega d is the damped frequency, which is omega d is equal to omega into 1 minus xi square you have already seen this earlier. So, this is the solution. So, now what we have done in this lecture is the mathematics behind un-damped system and damped system response.

Thank you.