Structural Dynamics
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Lecture - 38
Examples for Dynamic Response of Continuous Systems

Hello, this is going to be my last lecture in this course on structural dynamics and what all we ending up with is looking at a couple of example problems. One dealing with flexural deformations and one dealing with axial deformations. Although I started my course of continuous systems by looking at axial deformations, I will look at the example of the axial deformations a little bit later. I will start off with the flexural deformations problem primarily, because I want to highlight some very specific thing about the axial deformations problem.

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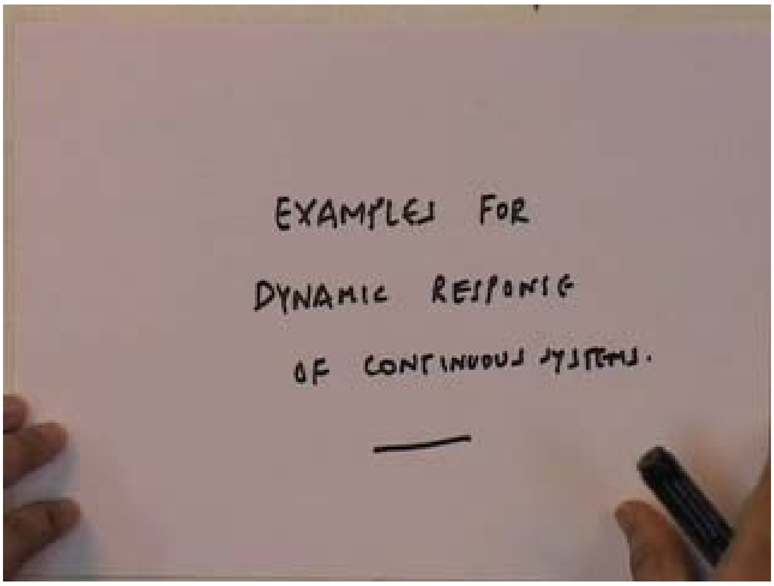

So let us start of… So this particular lecture is final lecture of this course is going to be on examples for dynamic response of continuous systems, and in this, let us start off with looking at a example for flexural deformations and in this particular case, there is no need for me to look at the equation that I developed in the last lecture which was the beam; the team of sango beam where you consider flexure, sheer as well as rotatory inertia, and of course always lineal inertia.

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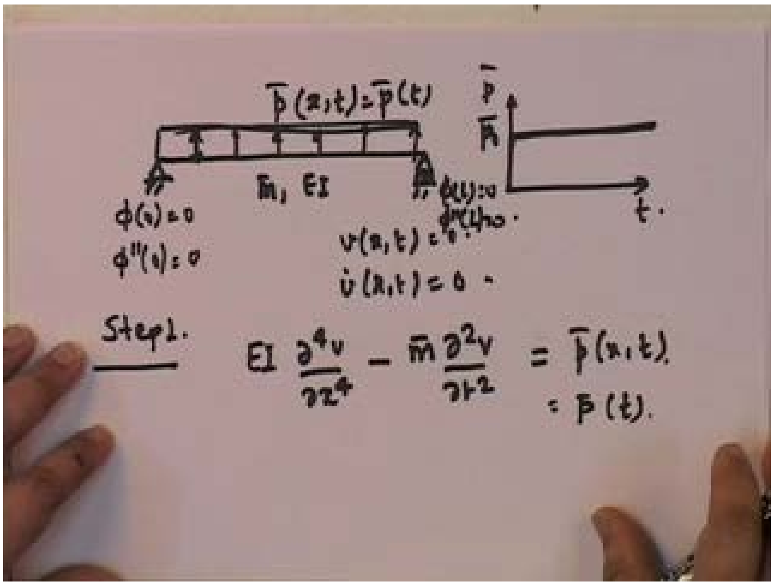

So I will solve a very very simple problem and that is, that of a simple beam. Constant M bar, EI, no rotatory inertia, no shed of distortion. Only flexural deformation and point mass for the cross section and this is subjected to a where, so this is a uniform load where, p bar of t is looks like this. So, it is a suddenly applied load. So this is a suddenly applied load on this particular one, and of course you know, initial conditions are 0.

So, you have 0 and v dot and x bar is equal to 0. So first and foremost, we have to write down the equations of motion and what are the boundary conditions? This since is the simple beam. The boundary condition is phi 0 is equal to 0 phi double prime is equal to 0 why because moment is 0 here. Similarly phi L is equal to 0 phi double prime L is equal to zero. So we written down the equations of motion and p bar x of t is nothing but p bar of t. So it is not a function of x. Now, so this is the first step.

So this is step one. We have done step one .What is step two? Lets I am going to do this in a very algorithmic way. Step two, free vibration. Now I am not go through this in this particular case, this is a simply supported beam simple beam. We know what the solution. We have already evaluated the solution. It is sin n pi x over L and omega n is equal to n square pi square EI upon m bar L fourth and for n going from one to infinity. So, we have done the free vibration.

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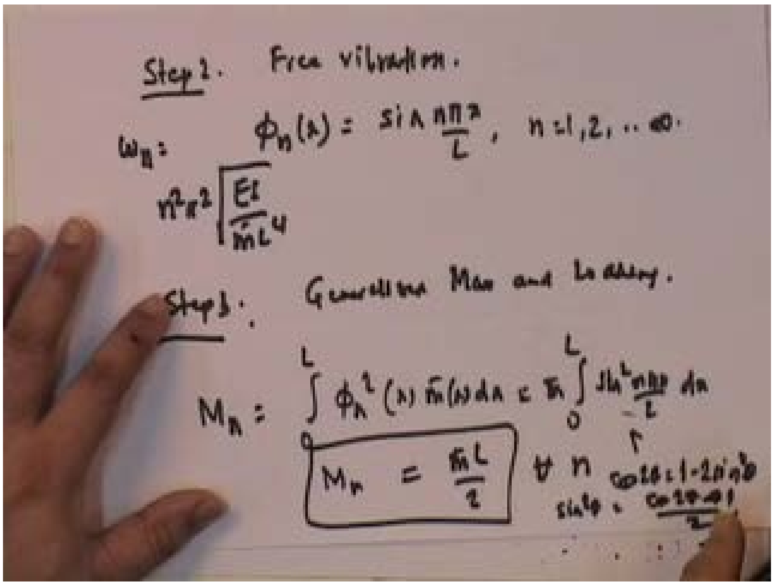

I have already done it before. So there is no need. Step three. What is step three? Step three was generalized mass and loading. How do I find that out? Well, let me find out M n is equal to 0 to L phi n square x m bar x d x. In this particular case, its m bar into 0 to L sin square n pi x upon L d x. Now note that how do I solve this sin square? Well, you recollect that cosine of 2 n pi x upon L. If you have sin square theta, let us say cosine 2 theta is equal to 1 minus 2 sine square theta and utilizing that, you can plug that in and you will get this equal to for all n, it is equal to m bar L by 2. So, M n is equal to m bar L by 2 for all n. Note that you understand that here, this is solved by the fact that cosine 2 theta is equal to 1 minus 2 sine square theta. So this one you just make it into this. So, it becomes that sine square theta is equal to cosine 2 theta minus 1 upon 2. So, this is what you use to solve this. I am not goanna solve that.

Next one is p n. I am still on step 3, finding out the generalized mass and the generalized loading. p n is equal to phi n x p bar x over t d x. So, this is equal to p bar sin n pi x d x and this if you look at it, it is equal to p bar L upon n pi this minus 0 to L cosine n pi x upon L. So, this basically becomes equal to minus p bar n pi cosine n pi minus 1 and if you look at this cosine n pi, cosine n pi is equal to plus 1 when n is even and cosine n pi is equal to minus 1, when n is odd.

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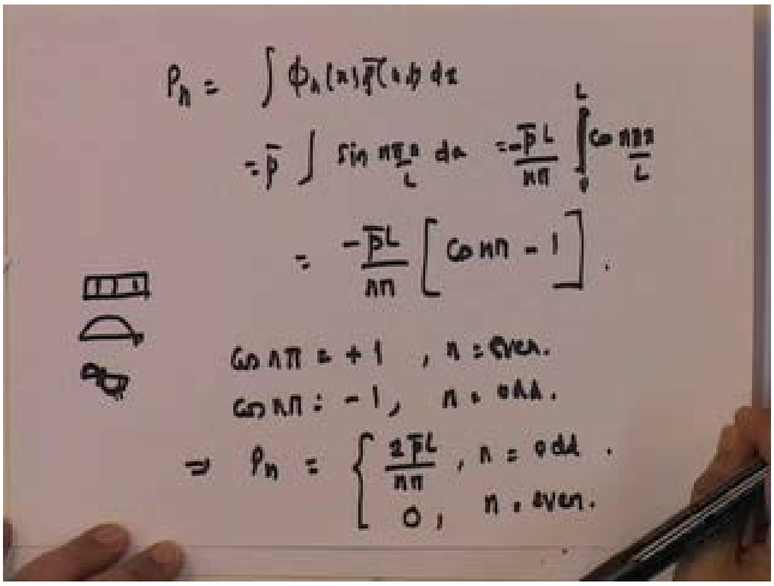

So therefore, what we get is p bar n is equal to 2 bar p L upon n pi for n odd and 0 for n equal to even. This makes sense because if you have see this is UDL if you look at your first one, this one into this is gives your product. If you look at the second one, it is like this half way. So, if this is positive, this is negative and the area under that curve is 0.But anyway, we solve all of that. So it is even and so this is p bar n.

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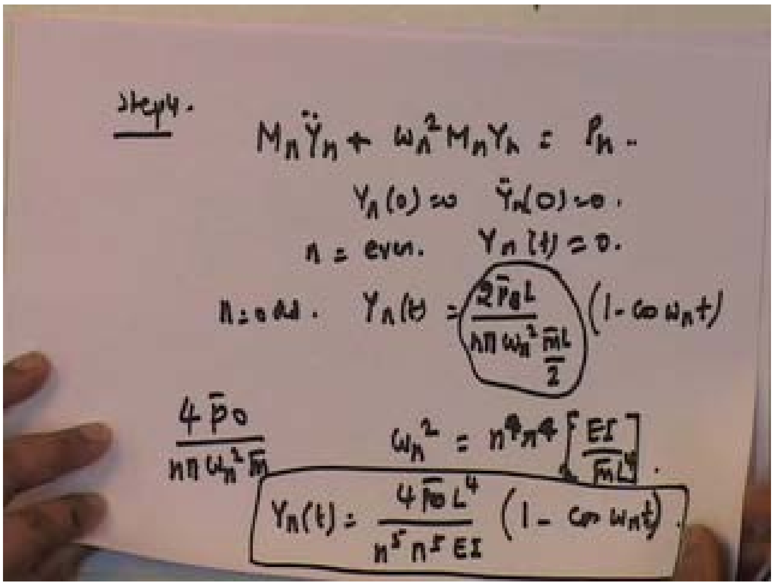

So therefore, now what we have is, we need to step four is to solve M bar Y n plus omega n square M n Y n is equal to p n. So therefore, now obviously, with what? Y n 0 equal to 0 Y n dot 0 equal to 0 because if you plug in, remember we could get Y n in terms of this v x of t by taking integral 0 to n p n x v x of t t x upon n m. So, we can do that and if that v x of t is equal to 0, automatically Y n and y dot n are going to be equal to 0.

So, if you put that in for n equal to even, Y n of t is equal to 0 for n equal to odd, Y n of t is equal to well, let us see what it is? You have p n is 2 p bar 0 L upon n pi. This is the load pi. So, that is p n and that upon omega n square m n. So, omega n square and M n is what? M n is equal to m bar L by 2 into 1 minus cosine omega n t. So, let us see what this turns out to be equal to? Let us have a look at what this term turns out to be equal to? The L L cancels. So, what we are left with is four p bar 0 upon n pi omega n square m bar.

Now let us see what Omega n square is equal to? Omega n square is equal to n square pi square into n fourth pi square into EI upon m bar m. So, if I plug this in, what do I get? I get that Y n of t is equal to four p naught. Now look at this omega n square is n fourth pi fourth. So what you have is, n to the power of 5, pi to the power of five EI. So, what you have over here is so this is what you get and p 4 upon EI into this is p I m bar L fourth. So, there is there is an L fourth term that comes over here which goes on the top. So, it becomes L fourth over here. So, Y n turns out to be this. So, if I were to look at Y n, if I were to look at Y n, what does my Y n? Y 1 looks like.

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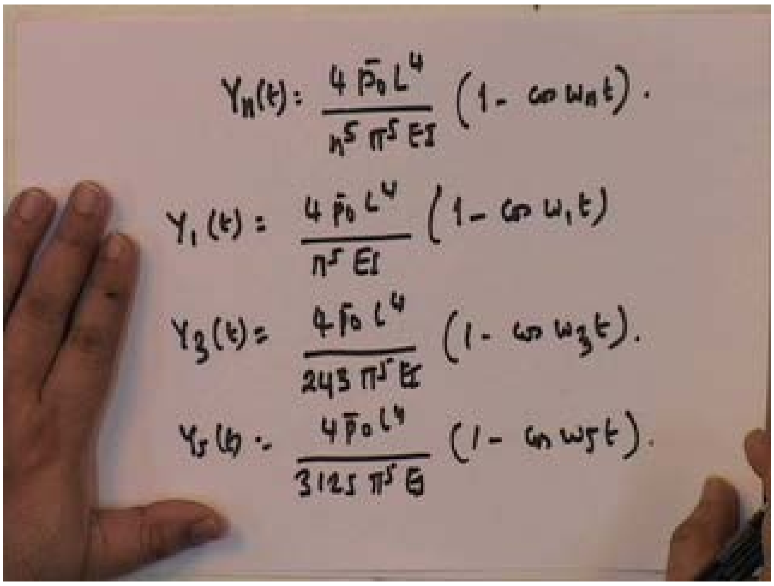

Let us look at y one. See I am going to rewrite this Y n of t is equal to four L fourth upon n to the power of 5 pi to the power of 5 one upon this into 1 minus cosine omega n t. So, if I look at y 1 of t, what does Y n of t? look at if I look at Y n of t, what I get is 4 p 0 L fourth upon pi fifth upon 1 minus cosine omega 1 t. y 2 is of course is 0.y 3 of t is only the odd ones that stay. So, its 4 p 0 L forth now n is 3 to the power of 5 that is 81.It is 243 pi fifth upon 1 minus cosine omega t and so on and so forth. Let us look at y fifth. You get 4 p naught bar L upon. Now look 5 to the power of 5 is 625 into 5 which is 3125 pi to the power of 5 EI 1 minus 3 cosine 5 L. So now if you look at this so these are my y 1 y 3 y 5.

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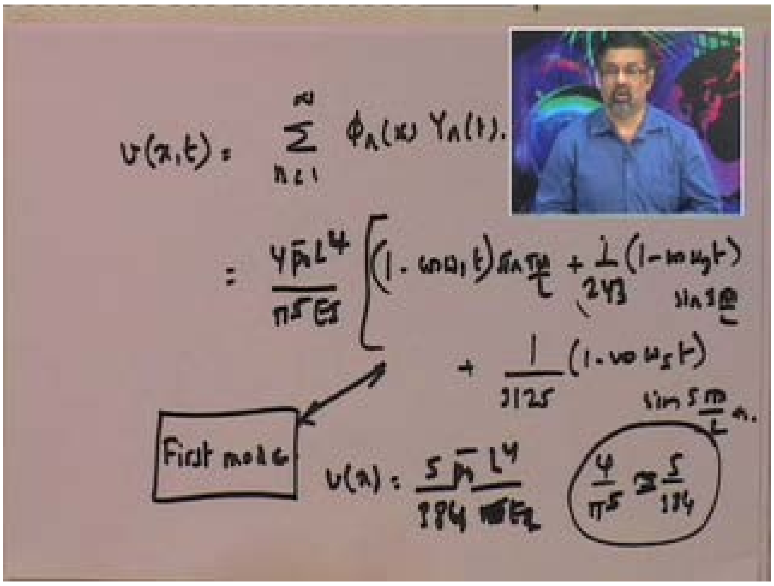

So now if I look at my v of t, my v x of t is equal to summation n going from one to infinity phi n x into Y n of t. If you look at it, this is equal to if I going to just plug this, this would be four p naught L fourth by pi 5 EI. I am goanna put it outside. Now you will see if you look at, this becomes actually like 5 by 384 its very close to 5 by 384 into the first one is 1 minus cosine omega 1 t into sin pi x by L then plus now think about it. This is 1 minus. So, there is 243, 1 upon 243 into 1 minus cosine omega 3 t into sin 3 pi x upon L plus 1 upon 3125 1 minus cosine omega 5 sin plus.

So, fourth note something very very interesting. Look at these terms for all practical purposes. This term, even these terms are almost negligible and if you look at this, the first mode, if you going to look at it, you know if you look at static v x, what is it going to be? 5 by 384 p naught bar L fourth by 5 pi EI. This is what you get. So if you look at this, if you look at this particular term and you look at this, you see that these two are almost identical. That is very different very little difference between 5 upon 384 and four upon pi to the power of 5. In other words, what we are saying is, we will see that four pi is almost equal to 5 by 384 and this, if you look at it is you know like pi square I will just explain to you pi square will be about ten this will be about 100 3.84 into 100.

This basically becomes like 314. So, 4 upon pi 5 is approximately 5 upon 384. It becomes its very close to each other and for all practical purposes, we can see that the first mode dominates the response. Remember even in multi-stored things, I have pointed out that displacement is essentially governed by first mode. In this particular case, you see very clearly that the first mode if you look at the coefficients, these coefficients are so small that for all practical purposes even though is it supposed to be infinite series. If you take the first term, you almost get it equal to the value. So, that in a sense is displacement response. Now let us look at, we have got the displacement response. So now, let us look at the force response. Let me look at the bending movement.

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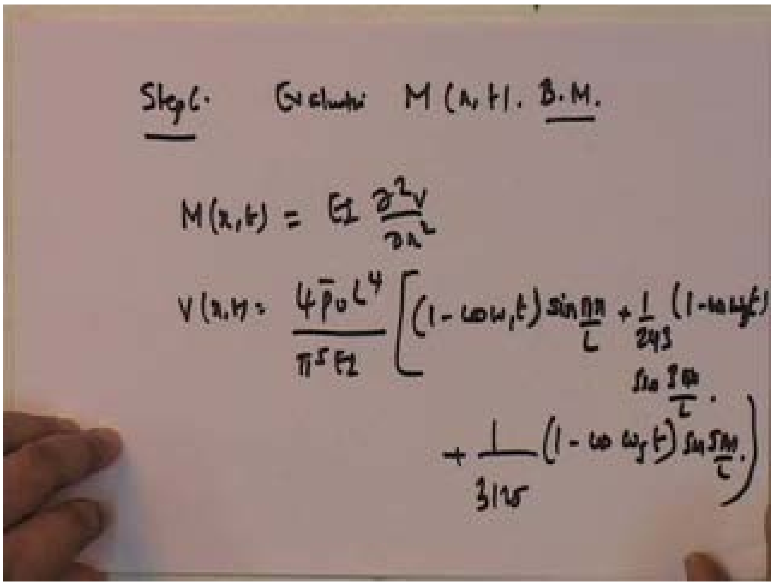

So I will look at. So this was step 5. Now step 6. Evaluate M x of t. Let us look at M x of t. What is M x of t, M bending movement? Bending movement at any instant is equal to EI del square v by del x square. So, what do we have? If you look at it, what is v x of t is equal to? v x of t was equal to 4 p L fourth upon EI into 1 minus cosine omega t into sin pi x upon L plus 1 upon 243 1 minus cosine omega 3 t sin 3 pi x upon L plus 1 upon 3125 into cosine omega 5 into sin 5 pi x upon L.

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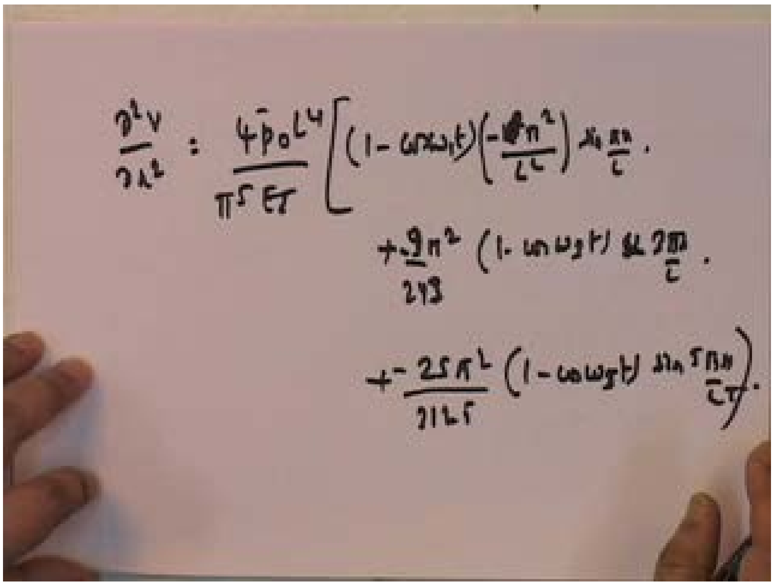

So, differentiate this. If I differentiate this, what do I get? What is del square v by del x square? del square v by del x square is equal to the following. 4 p naught L fourth upon pi 5 EI 1 minus cosine omega 1 t. This one does not differentiate, but this one differentiates and this one differentiates to what? It differentiates to minus n square pi square.

So, this is like pi squared upon L squared into sin pi x upon L plus 1 upon 243 and this when you differentiate sin 3 pi, what do you get? You get minus 9 pi square upon 243 into 1 minus cosine omega 3 into sin 3 pi x upon L plus. Now if I differentiate becomes 25 of course, minus again upon 3125 1 minus cosine omega 5 into sin 5 pi x upon L plus, but let us look at, this is very interesting and that is that what does this term turn out to be?

So if I write down the moment, I get m x of t equals four p naught L fourth upon pi 3 EI and there is a L square in every, in every term. So, the L squared becomes p naught L square and inside we get 1 minus cosine omega bar t into I have taken L by this way. So, this is minus because the minus all minus go out. So, this become sin pi x by L then plus. Now if you look at this 9 upon 243 81 and 27, so this becomes 1 upon 27 1 minus cosine omega 3 t sin three pi x upon L and now if you look at this, if you look at 25 becomes 125.

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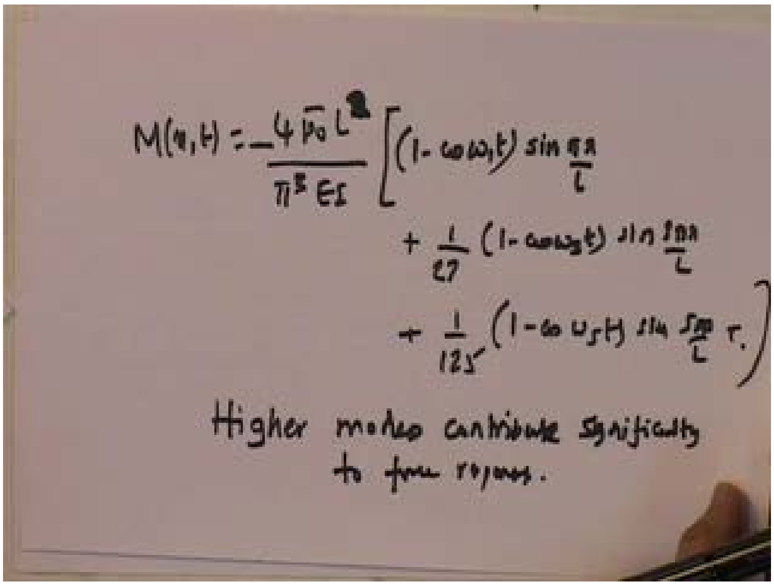

So, this becomes one upon 125 1 minus cosine 5 t sin 5 pi x upon L plus blah bla bla. But note one thing that whereas for displacement, these terms where one upon 243 upon 325; that means, the higher order terms were negligible. But if you look at this, you see one 1 upon 27 1 upon 125, the higher modes contribute much more and this is something that higher modes contribute significantly to force. This is again something that, we have already said. But here, we see this particular case very very clearly now suppose, I were to find out you know m at L upon 2, which is the mid span movement under UDL mid span movement.

So, we have to find the mid span movement. What would I get? I would get M at L upon 2 mid span, mid span movement. Mid span movement would be equal to minus 4 p naught L square upon pi cube EI and look at sin pi x upon L. This would become pi upon 2. So, this would become equal to 1 minus cosine omega 1 t and then sin pi x upon L x is equal to L upon 2. This becomes pi upon 2 pi upon 2 is 1. So, its 1 upon cosine, then this 1 sin 3 pi by 2 .What is 3 pi by 2? 3 pi by 2 is again minus 1.

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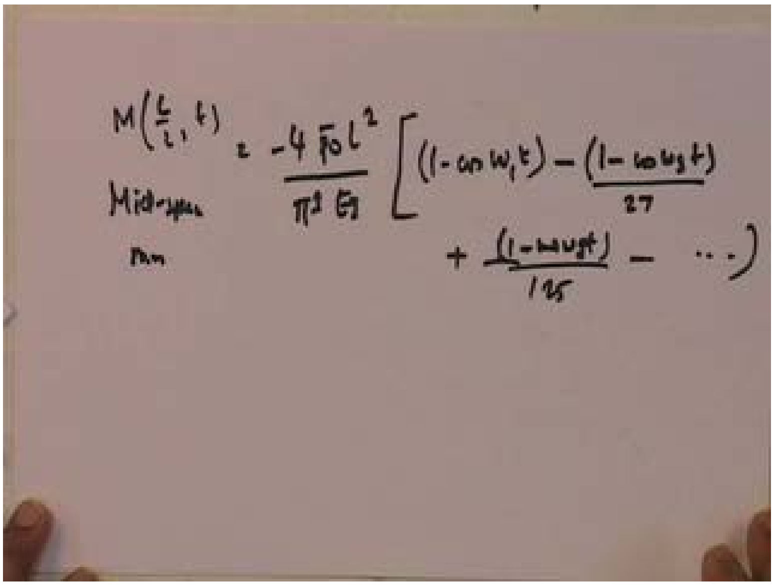

So, this becomes minus. So, this becomes minus 1 minus cosine omega 3 t and this of course is 27 plus 1 minus cosine omega 5 t upon 125 minus. In this fashion it continues that it becomes plus minus plus minus this way it continues. So, this is the mid span movement and you know, this is fairly easy to evaluate that is the point that I am trying to make and this is the fairly easy equation to develop. In fact you know, we can we can easily do that. So, this is how you would solve a dynamic problem. This is of course, for a step loading suddenly applied load for a flexural vibration.

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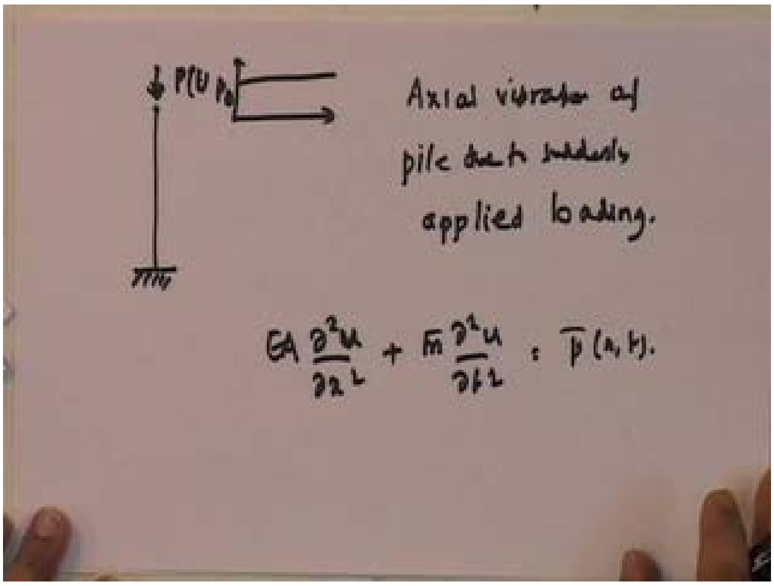

Now let us look at a situation, which is a fairly common problem. Let us look at a pile with a suddenly applied load and this suddenly applied load is also of the same form p naught. It is a suddenly applied load and this load is applied at the tip. So, that is my loading. So, I want to find out and this is for axial vibration. I want to find out axial vibration of pile, due to suddenly applied loading.

So, we are looking at an axial vibration. So, what are the equations look like? Well the equation looks of the type. EA is u. As a first step we boundary conditions we know, we have already done this. Let us look at the second step. The second step is something that we know which is the free vibration, free vibration mode shapes and frequencies I know the mode shapes and frequencies in this particular case, we have already derived it.

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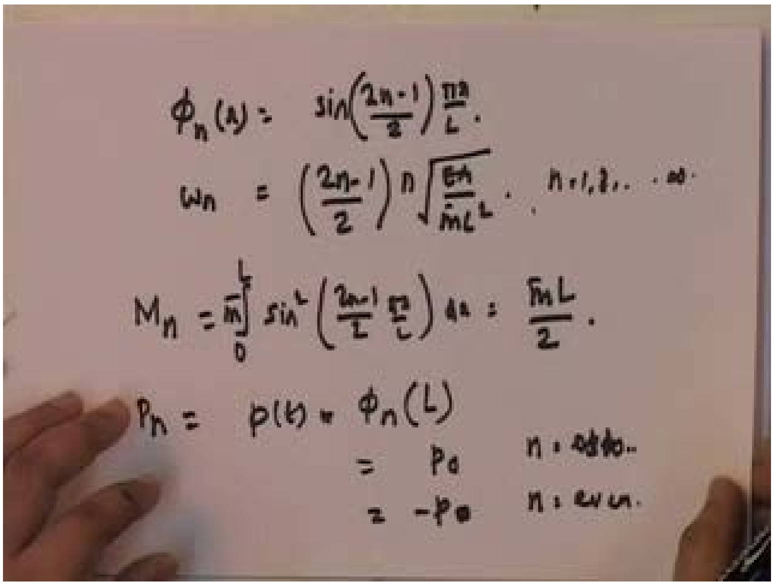

The phi n of x is equal to sin 2 n minus 1 upon 2 pi x upon L and omega n are equal to 2 n minus 1 by 2 pi EA upon m bar L squared and this for n equal to 1 2 to infinity. So, now, now the thing is to find out the generalized mass. M n is equal to 0 to L m bar outside sine square 2 n minus 2 pi x upon L the same thing that we did last time and we get that it is equal to m bar L upon two exactly the same way that we have done it last time and p n note that since this is a point load what is going to be over here when we apply this and so therefore, p n is equal to p of t into phi n at L. If you look at that, what happens is, this becomes p naught when n is even. For example, put n equal to 1. If you put n equal to one, what do we get? We get this is two n.

So, this becomes pi x upon L and pi if you put this sin p upon 2, you get this to be equal to 1. So therefore, this becomes this is odd and this is equal to minus p naught and n is equal to even. Put it equal to 2. When you put it equal to 2, you get three by 2 sin of three pi by 2 is minus one. So, that that is how it happens. So, this is where we get odd and even we get this. So, now, let us go forward by one step and that is a third step we found out our generalized mass and loading.

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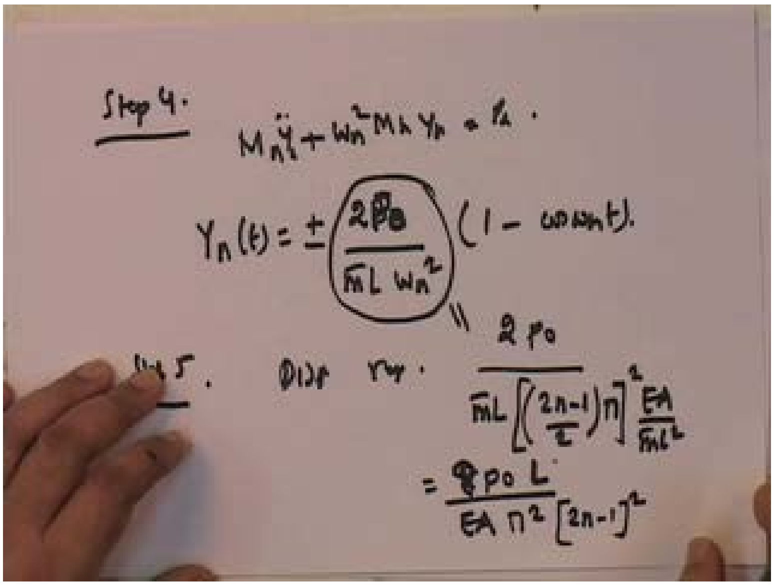

What is the next step? The next step which is step 4 is to solve the model coordinate equation. So M n plus omega n square M n Y n double dot plus Y n is equal to p n and note that we already know what Y n of t is going to be? It is going to be equal to plus minus 2 p naught sorry this is not no longer p naught, but this is actually p naught upon m bar L by 2 omega n square one minus cosine omega n t.

So, let us look at the next step, step number 5. Let us look at displacement response. Displacement response turns out to be equal to the following. Now note that, if you look at omega n, what do you get? You get, let us just put in this. Let us first calculate this. This becomes two p naught upon m bar L and omega n square is equal to 2 n minus 1 into pi so this whole square into EA into m bar L square so m bar m bar cancels L upon. So this becomes equal to 2 p naught L upon EA into.

Now, let us see this becomes 4. So, this becomes 8 and so this becomes pi square and I am going to take 2 upon n inside 2 n minus 1 inside 2 n minus 1 the whole squared. I am going to take that inside. So, if I take it inside, of course, this one then becomes what? 2 n minus 1 the whole square, that is that is what I get the 4 goes up.

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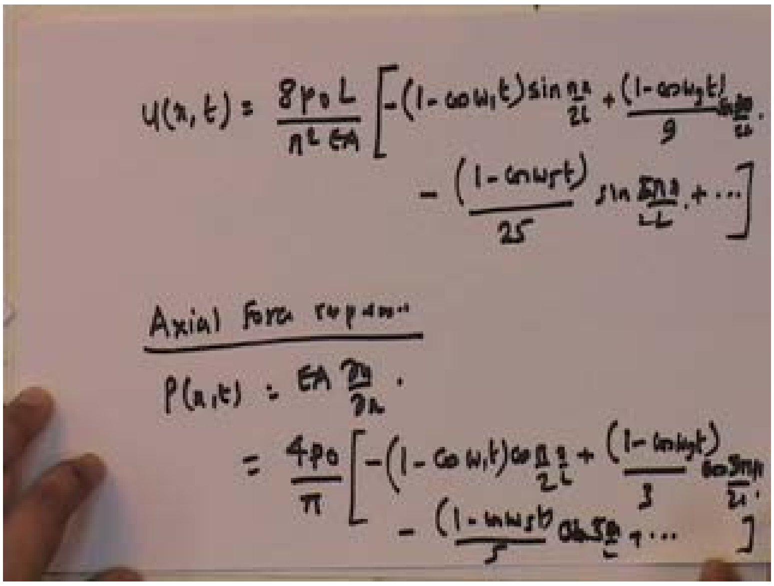

So, this becomes the. So, my displacement response becomes the following. It becomes equal to u x of t is equal to 8 p naught L upon pi square EA into now let us see I have a situation where in the first one, it becomes one minus cosine omega one L t into sin. This is the downward force. So this becomes since it is a downward force, its minus into sin pi x upon two L, then we have a plus 1 minus cosine omega three upon. Now this becomes, nine into sin three pi x upon 2 L minus 1 upon cosine omega 5 t upon 25 into sin 5 pi x upon L so on and So forth.

So these are my terms. So, I can rewrite this in the following format. So now, this is u x of t and now suppose I want to find out my axial force, axial force response. If you look at p x of t is equal to EA del square u by del x square. So all that happens is that EA this EA cancels del square u all that happens is, this becomes cosine pi over 2 and therefore, the following thing happens. This becomes equal to 4 p 0 upon pi. This one pi comes out since the pi upon 2 comes out. This becomes four the L also comes out at the bottom. It becomes pi upon 2 L. In every case, pi upon two L comes out. So it becomes what? pi upon 2 L L cancels. This cancels as this becomes four p naught EA is already gone and the pi pi cancels.

So, this becomes four p naught upon pi into now the minus disappears. It becomes equal to minus 1 minus cosine omega t cosine pi x upon L. Again plus 1 minus I get now here. If you look at it, this becomes three. So, this becomes one upon 3. So it is 1 minus cosine omega three t upon 3 cosine 3 pi x upon 2 L and then minus 1 minus cosine omega 5 upon 5 cosine 5 pi x upon L plus. Now note that suppose I want to find out the axial force at 0, what do I get? Well, this cosine pi x upon L becomes one. This becomes three pi becomes minus 1.

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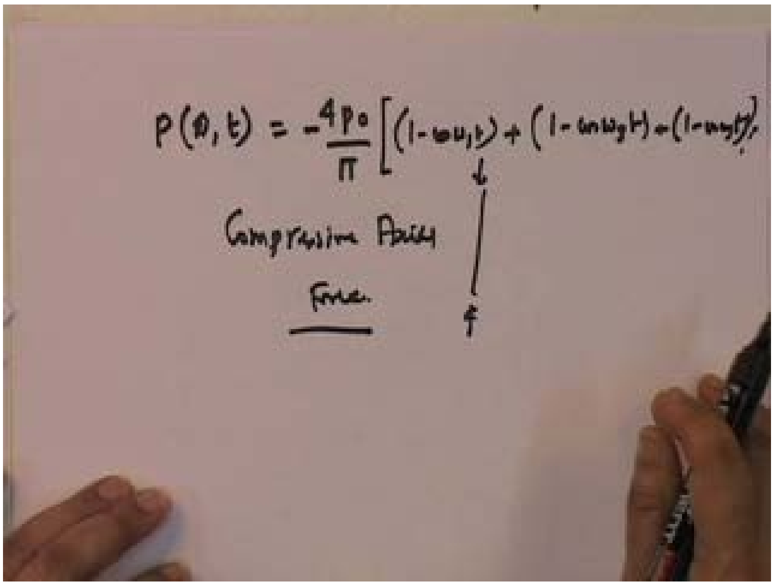

So, the entire thing becomes this and so what you get is p at 0 t is equal to 4 p pi. The whole thing becomes just minus. So, this becomes minus outside and what we have inside is 1 minus cosine omega 1 t plus 1 minus cosine three omega t plus 1 minus cosine omega 5 t plus. So, this in other words is my, why is it minus? Because, it is compressive force. Note that if you have this is this way, the compressive force and the minus signifies that it is a compressive axial force. Note something very interesting and that is that you know this load is a pile that actually expands over here. What is the reason behind it?

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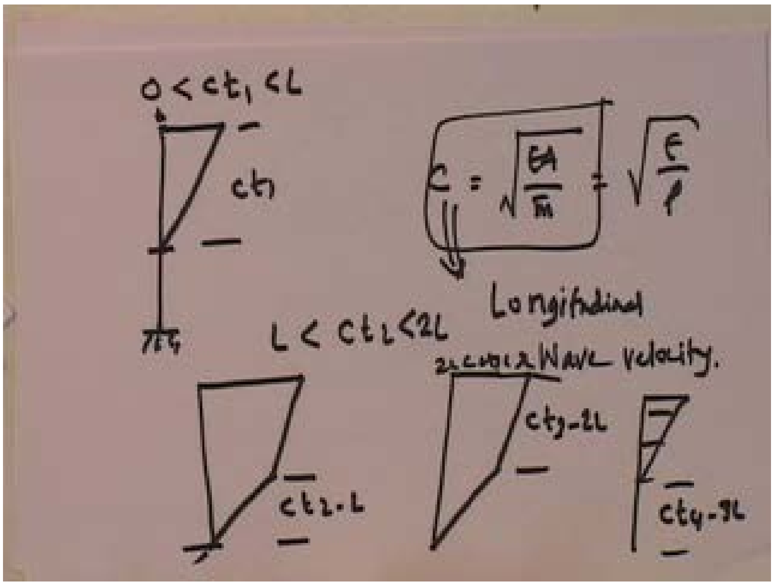

Well, let us try to make a plot of this particular. If I make a plot of the displacement and I define a parameter called c, where c is actually equal to EA upon c is equal to square root EA upon m bar, which is actually equal to E upon rho. This c is known as the longitudinal wave velocity. This problem actually you know, if you plot this, this is the displacement wave and this is the stress wave or the force wave whatever you want to call it. If you look at this, if you plot these, you will see this actually behaves in this fashion as long as you have 0 less than c t one is less than L.

In other words, this wave that started here at time t equal to 0. How far does is it travel? If it takes, see if the longitudinal wave velocity is this so if you look at t 1, its L upon c in another words it is not come up to here. If you plot the displacement, you will get something like this where this is equal to c t 1. This is also the displacement wave looks. If you plot it between L and c t 2 between 2 L in other words, it is come here and not gone back this thing well look something like this. Well this is c t 2 minus L. Similarly, you will get a similar kind of thing, where this is c t 3 minus 2 L when 2 L less than c t 3 less than 3 L.

And now, when you go to 4 L, you see this becomes something like this where this is c t 4 minus 3 L. So in other words, this if you look at axial, I suddenly apply the load. This wave travels from here and comes here and gets reflected. So, when it gets reflected what you have is, you have a combination of a wave you know traveling this way and your wave traveling this way. So, what happens is over here, you get a combination of just this way where we comes over here. So, this particular value would have come over here and all it get reflected. It adds up and then it reaches the free end. It actually goes out and so once it goes out, that wave is gone and so now you have a wave, which is coming back again. So, this is it repeats itself every 2 L, every 2 L it repeats itself 2 L. I mean for time 2 L upon c is the periodicity of this wave. So, very interesting and this you can get from this solution.

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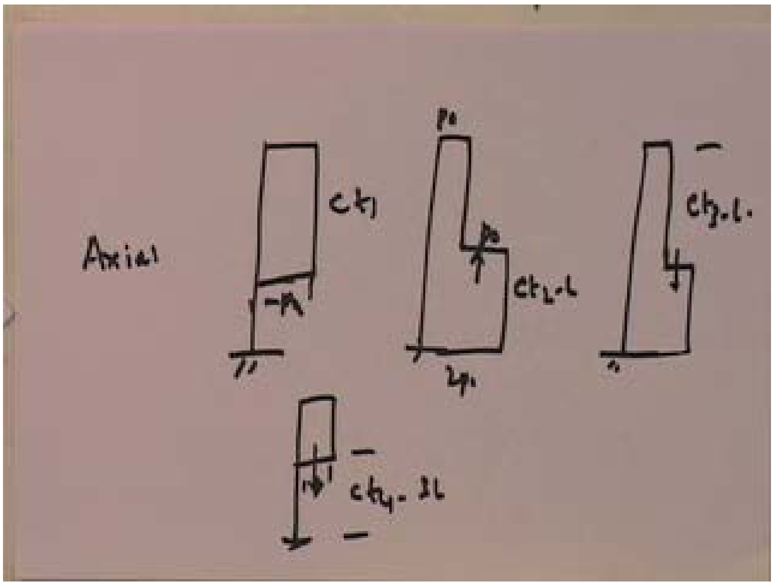

If you look at the stress, the stress wave when I what I call it as the axial force wave. The axial force wave looks like this. The axial force wave is constant. This is c t 1 and the entire thing is p naught. So, the wave that comes is p naught. Then you have a situation where you have p naught and p naught. So, you have 2 p naught where this is c t 2 minus L. As you go further, you get p naught and p naught where this is c t 3 minus L and finally, when you have the last wave, this is p naught where this is c t 4 minus 3 L.

So you have, you have the load applied here on p naught suddenly applied that is sets up axial wave which comes here and as it comes and hits here, it gets reflected and the stress since it is a rigid boundary what happens is, this stress wave comes over here and is reflected back. So, when you see this, this is the reflection that goes here. Then it comes, the reflected wave comes here and then what happens is because of the boundary conditions are free, part of the wave disappears and what you have is, now your wave coming back which is only p naught coming back. Then it comes over here and these two waves destructive and when it becomes instructive p naught and minus p naught it becomes.

So, this wave goes back. So, this becomes unstressed part and this goes here. So, it goes back here and after 4 L at 4 L, you have an unstressed kind of situation. So, suddenly applied it is unstressed. Then the waves are traveling constructive. So, its charge going act as 2 p naught then again because of their free end destructive. So, the 2 p naught is coming back to p naught is getting reflected, but it is its getting slowly you know added then it comes over here this one gets.

Now, the 2 p naught goes back. So, it goes back here 2 p naught and what happens is, you get a compression wave. So, what you have is, this is a compression wave this is a tension wave this is a compression wave. So, this is how the entire process of wave happens. So, this what we what we got over here from those I am actually you for at any time instant of time x and of t, of course for various components and c is square root of e upon rho is a longitudinal wave.

So, this is the longitudinal wave. It is actually travelling along the axis and then going back along the axis coming back going back. This is a wave propagation problem. I am not going to be talking about the propagation in this course. It will be taken up in the later course, but the essence that happens is that, this beauty that you see it comes from this 3 solution. So, that is all I have to say thank you very much. I have really enjoyed myself in this course and I hope that you will enjoy this course too.

Thank you very much, bye.