Structural Dynamics
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Lecture - 36
Free Vibration and Dynamic Response of Continuous Systems

Hello there, we been looking at a free vibration response, and what I am going to be doing today is I am going to be looking at finishing off the free vibration response of a single degree of freedom structure. And then we will quickly look at the solution of a dynamic equations.

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So first and then dynamic I will introduce the concept of dynamic response of continuous systems. Now, last time when we stopped the lecture I had tried to solve, this the solution for a cantilever beam, a cantilever beam with constant m bar and EI. And what we saw was after incorporating these boundary conditions, we got a situation where phi x was equal to A 1 into sin ax minus sin hyperbolic x and plus A 2 cosine ax minus cosine hyperbolic ax.

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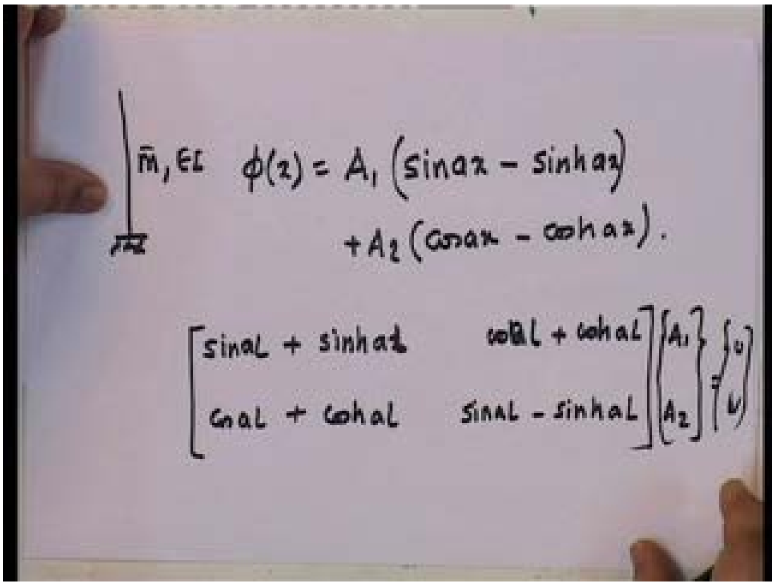

And then we saw that when we substituted the fact that the movement, and shear at these points, where equal to 0 we got a equation of this form, which was sin a L plus sin a L sorry cosine a L plus cosine hyperbolic a L cosine a L plus cosine hyperbolic a L and then sin a L minus sin hyperbolic a L, A 1 A 2 equal to 0. And we saw that this determinant has to be equal to 0 for this to exist, and therefore we got to the situation where you have the determinant. And the determinant was equal to what?

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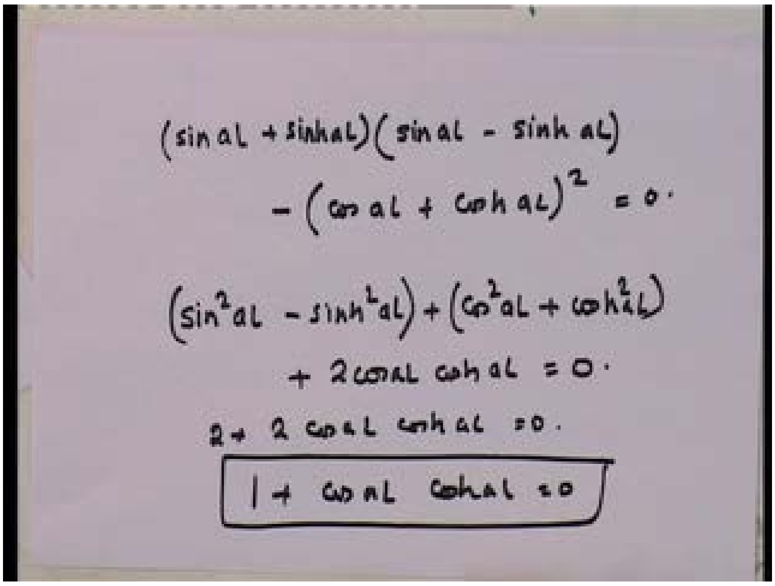

They were equal to sin a L plus sin hyperbolic a L into sin a L minus sin hyperbolic a L minus cosine a L plus cosine hyperbolic a L the whole square is equal to 0. And so if this becomes the case we get sin square a L minus sin hyperbolic square a L plus cosine square a L plus cosine hyperbolic square a L plus 2 cosine a L cosine hyperbolic a L is equal to 0. Now, if you look at this becomes sin square plus cosine square a L is equal to what is equal to 1, so what we have here is this plus this is equal to 1, and look at cosine hyperbolic square minus sin hyperbolic square is also equal to 1. So, what we have is 2 plus 2 cosine a L cosine hyperbolic a L is equal to 0 and so therefore, the equation actually becomes 1 plus cosine a L cosine hyperbolic a L is equal to 0. This is the frequency equation which has to be solved for a L, this is like a transcendental equation and the only way that you can solve this equation is through iterative procedures.

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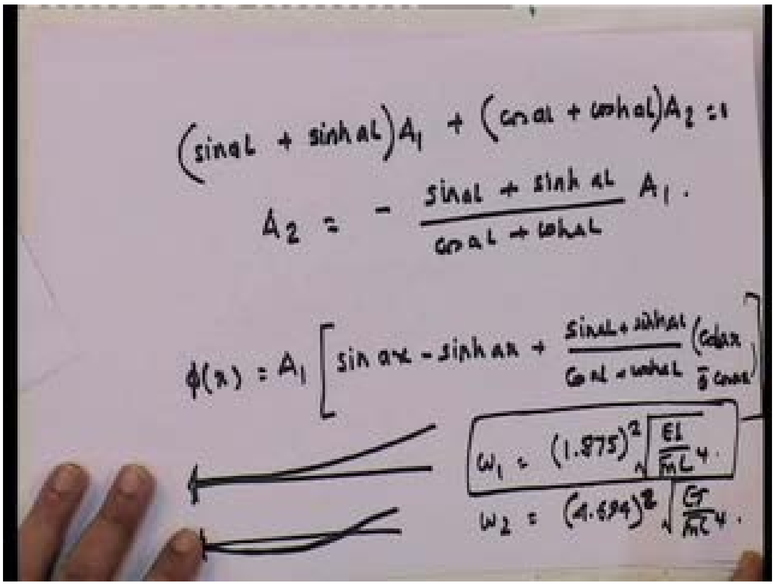

So, you can solve iterative procedures, and once you have solved this we get a situation that from the first equation we get that sin a L plus sin hyperbolic a L into A 1 plus cosine a L plus cosine hyperbolic a L into A 2 is equal to 0. This implies that A 2 is actually equal to well, let us see what it is minus sin a L plus sin hyperbolic a L upon cosine a L plus cosine hyperbolic a L into a 1.

So, once we have that if you look at it phi of x becomes equal to A 1 sin ax minus sin hyperbolic ax plus sin a L plus sin hyperbolic a L all upon cosine a L plus cosine hyperbolic a L into cosine ax plus, it is just the opposite it is cosine hyperbolic minus cosine ax. This becomes your phi x, where a is as per obtained from, this equation which we have got which is cosine a L cosine hyperbolic a L.

Having solved this and substituted this what we get is that first equation looks like this and omega 1 is equal to 1.875 square into EI upon m bar L fourth, in other words the value of a L is actually minus is 1.875, that is the first solution of all that is the solution. For this the second one omega 2 is equal to 4.694 that is the value of a L and therefore, this is what we get as. So, in other words this is the way you can and the second mode is of course, something like this not something interesting remember we had solved this equation for the generalized equation.

And we had taken this as one minus cosine well this is nothing like one minus cosine right, fairly complex equation by substituting of course, a is equal to I mean sorry a is equal to 1.875 x up on L. So, this you know is how you solve it is a level higher then the you know the generalize single degree of freedom problem, but this has the advantage that is exact. So, I am done with solving the equations for where we have homogeneous equations remember one of the homogeneous equations well you know when we looked at axial deformation. We looked at the situation, where you have either you have phi at this point is equal to 0, and the actual force at this point is equal to 0 that is what you used as your boundary conditions.

For axial for simply supported flexural deformations well, we took displacement and moment at both ends are equal to 0 and we solved those equations. So, all of them were 0 in other words we got phi equal to 0 phi double prime equal to 0 or phi L equal to 0 phi double prime L is equal to 0 for the simply supported or you know if you have a cantilever we got phi 0 is equal to 0, phi prime 0 equal to 0. And the other one was phi double prime L is equal to 0 and phi triple prime L is equal to 0 that was moment and shear at the ends. So, these are all known as homogeneous boundary conditions, now I am going to introduce you to a concept where you have a situation where you do not have homogeneous boundary conditions for example, let us take an axial deformation.

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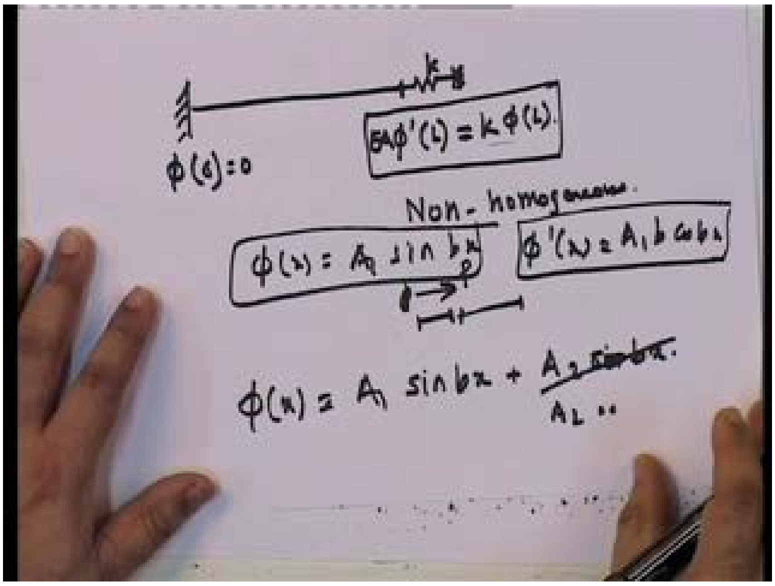

Let us take the axial deformation and let us take a situation, where you have this and instead of it being a free end what you have is you have a k and x here, so note that what is happening here is the following that although this phi is equal to 0. This one is equal to what? Well let us see you have a situation, where if you goes this way this presses this way and so you need a force this way, so what you have is that phi EA phi prime L. That what is this one this is the force, the force is in this direction and since it is going in this direction this is pressing back on it is sending back.

So, what we have is this is equal to k into phi L you see this, let us look at this again what you have is what is the boundary condition here. The boundary condition at this point is look p is no longer equal to 0 p if I put p in this direction, positive in this direction, now if it does this then phi is also in this direction and because of this blocks in this direction, and so what you have is that is equal to k into phi L.

So, what you have on this side is no longer a homogeneous boundary condition, but you have a mixed boundary conditions, non homogeneous boundary condition, so what does this do let us look at this equation again. So, what did we have, we had a situation where you have phi of x is equal to A 1 sin b x plus A 2 sin b x sorry, cosine b x. So, now obviously, phi equal to 0 implies that A 2 is equal to 0, but A 1 sin phi; that means, p is equal to A 1 sin b x, and so therefore, phi prime x is equal to A 1 b cosine b x. Now, when we had the situation, where there was free end here what did we get we got cos b L is equal to 0, but in this particular case note that it is no longer is equal to 0, what we get in this particular case is the following you get the right hand side left hand side still remains the same.

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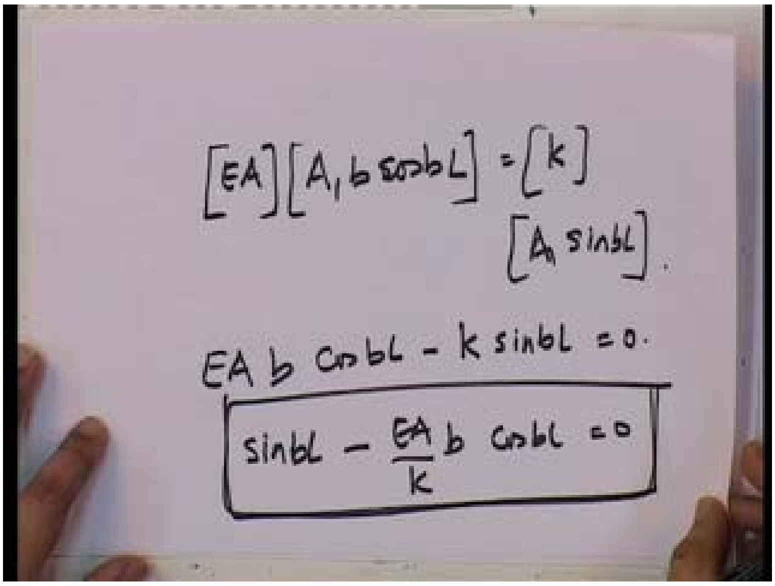

The right hand side becomes what, it becomes EA into A 1 b cosine b L that's e a into phi prime is equal to k into phi l, so this is k into phi L is equal to A 1 sin b l. So, what we get is we get ea into b cosine b L minus k sin b L is equal to 0. So, if we rewrite this we get it equal to sin b L minus ea by k into b cosine b L equal to 0, note that the neat little equation that we got last time when we had homogeneous boundary conditions.

Now, we see that even for the simple case where we got cosine b L is equal to 0 and we got b is equal to 2 n minus 1 n pointy pie we have to solve a transcendental equation to get the solutions. And let me just take this forward to the next kind of situation I am just showing you I am note that I am just illustrating to you, the concept of homogeneous non homogeneous boundary conditions, non homogeneous boundary conditions are where you have a variety of situations that come in.

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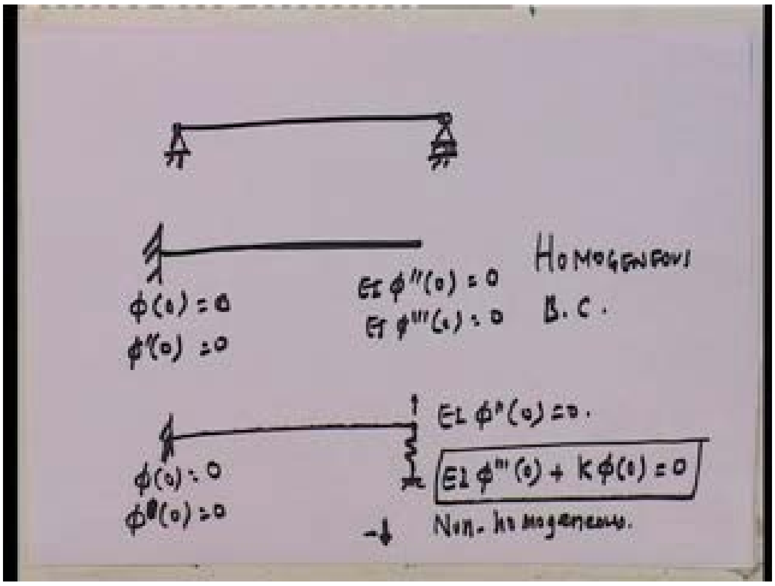

For example, now let us look at this particular case, this one is homogeneous, now let us have a situation ,well in this particular case it is difficult, I mean you know let us just look at this case this is a easier case to look at. What did we have for this what was the boundary conditions phi equal to 0 phi prime is equal to 0, still homogeneous boundary conditions, and on this one phi double prime. Actually, it is EI phi double prime and EI phi triple prime that are equal to 0, but ultimately the EI you can drop off and you get again homogeneous boundary conditions.

Now, let us take this situation, what happens in this particular case, this is a pin what happens. In this particular case we have a situation where this continues to be homogeneous this one since it is pinned here, EI phi double prime is equal to 0. However, if you look at this particular case what you have is you have a case where if you look at positive v is in this direction, so if you have positive v in this direction if it is been pulled up v is positive in this direction and now note that if you pull this up this is also be in this direction.

So, you get a situation where phi I phi triple prime 0 plus k into phi 0 is equal to 0, you see this completely this is homogeneous, this is non homogeneous boundary condition. Of course, so the this one gives you still that A 1 sorry b 1 is equal to minus b 3 and b 2 is equal to minus b 4 all of those kinds of things and then we get the following things, this is not phi prime, this is phi prime this is this is just slope. So, we get that and then from this side, we got the transcendental equation which was one plus cosine a L into cosine hyperbolic a L is equal to 0.

This one is going to give a completely different I am not even going to go about trying to solve it, but this is what happens? Let us look at another boundary conditions. See the point that am trying to bring together is that ultimately it is the boundary conditions, that determine the frequency equation and the mode shapes etcetera, we saw that. So, now, let me look at another situation, and then I am going to end the free vibration problems in this in let us look at it very, very simply.

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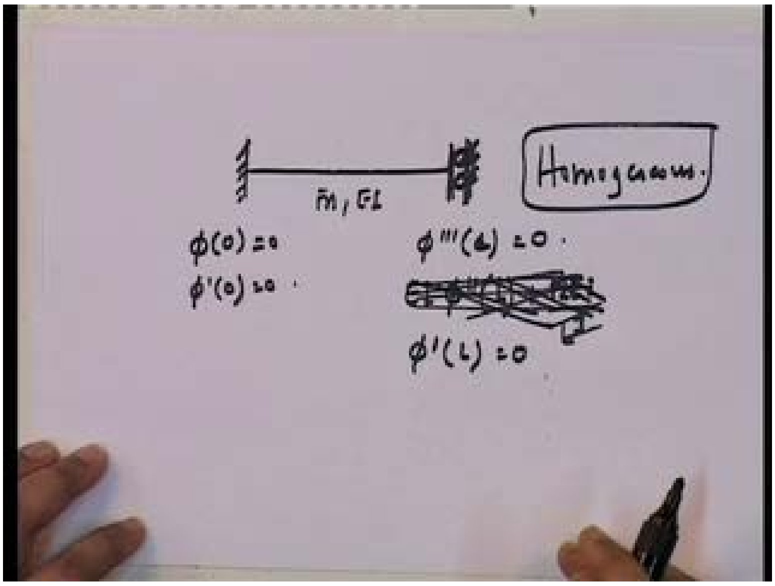

Let me look at this situation, where I have noise I have this and this is uniform, what are the boundary conditions that you have over here very, very interesting, let us see what happens over here. And that is that if I have a boundary condition like this noise note that we know over here that if there is delta over here the moment over here is equal to 12 EI upon L cubed. So, therefore, in this particular case the boundary conditions are the following, again on the left hand side I have homogeneous boundary conditions here what about the shear, shear is 0.

So, phi 3 L is equal to 0 over here, this is not 0 this is L equal to 0 and these are L, these are L, so please about that this is phi 3 L, and note that what they have is you know moment is positive in this direction. So, what we have is the moment is EI phi double prime L and note that, due to this motion you get this kind of motion, so what you have is that this is phi l, so you have plus twelve EI upon L cubed. Now, note that the movement equation no longer holds true in this particular case what is 0 if here shear 0 what else, so therefore here because of this boundary condition it is still homogeneous noise, but completely different boundary condition.

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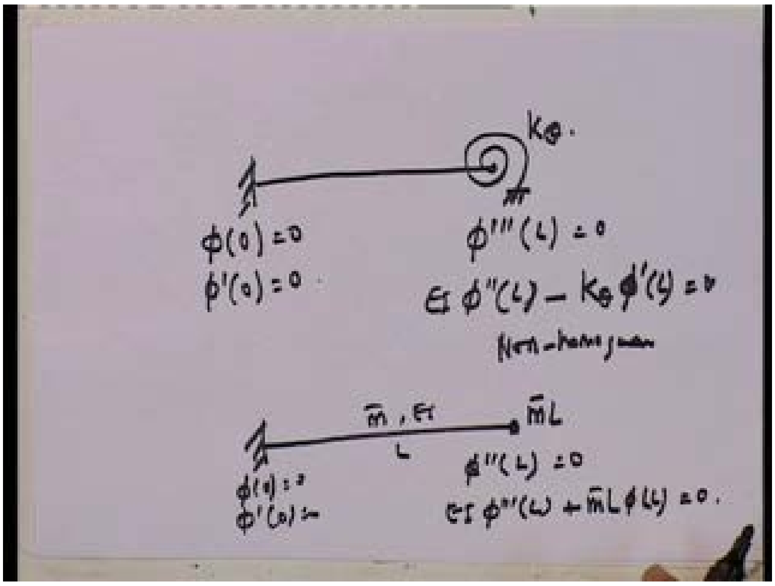

Now, if I have a situation where instead of this kind of situation I have let us look at this what happens, if you put a load in this direction this will put it in this direction, and what is theatre positive, theatre positive is in this direction. So, when you do this you are opening out the equation, so what we have here is phi equal to 0 phi prime equal to 0 note that there is no shear, so phi triple prime L equal to 0 noise, but phi I double prime EI that is the moment.

The moment is positive in this direction, and note that if you move it in this direction this will give in this direction, so what you have is you have minus because this is positive and this one is negative. So, this will become k theta into now, what is the rotation? Rotation is phi prime L is equal to 0 non homogeneous, so we have seen that you have non-homogeneous boundary conditions, when you put springs, do you have any other kind of a situation, where you have non-homogeneous boundary conditions?

Well yes, let us take this particular this, and put a point mass with m, let us call it actually lets this thing call it m bar L, where m bar is here EI and L is total length if we have it here then what do we have, well let us look at this phi 0 is equal to 0, phi prime is equal to 0. Well let us look at it will the moment, we equal to 0, sure there is no moment you know it is free to you know rotate, so it is not going to give rise to a moment.

So, in fact, phi double prime L is equal to 0, but let us see if you have this motion, so it is moving in this direction and you know when you move this in this direction, sure in this direction. And there is also a m bar L into phi L, which is acting in this direction note that this is phi so the this thing is the same. So, what you get is EI phi 3 L plus m bar L phi L is equal to 0, so you see that even if you have a mass, you get a situation where you have a specific non-homogeneous boundary condition, springs mass all these kind of things bring in homogeneity in the boundary condition.

So, that is so much and of course, once you do I am not going ahead with it you just have to be able to incorporate that, and just what happens is whenever you have a non-homogeneous boundary conditions. Even if you got a simple equation for a solution of the procedure, in this particular case you do not get a solution, you always can the transcendental equation you do not get a simple equation. So, non-homogeneous boundary conditions always make the free vibrations solution stifle more complicated in this sense is that well you get a transcendental equation.

So, you require reiterative solutions to get each frequency, and you can always use root finding techniques to get those solutions so much for free vibration. So, what have we done till now in continuous systems in continuous systems, we found out the equations of motion for a axial deformation and flexural deformations of a simple beam, where only flexure and linear mass are important.

So, we found out those equations of motion and then based on these equations of motions, we saw how to solve free vibration equations how to find out free vibration response of a for both axial deformations and flexural deformations of a simple beam. So, now, what I am going to do for the rest of today is going to introduce to you to the concept of dynamic response for continuous systems, how do we find out dynamic response for continuous systems well it is really simple.

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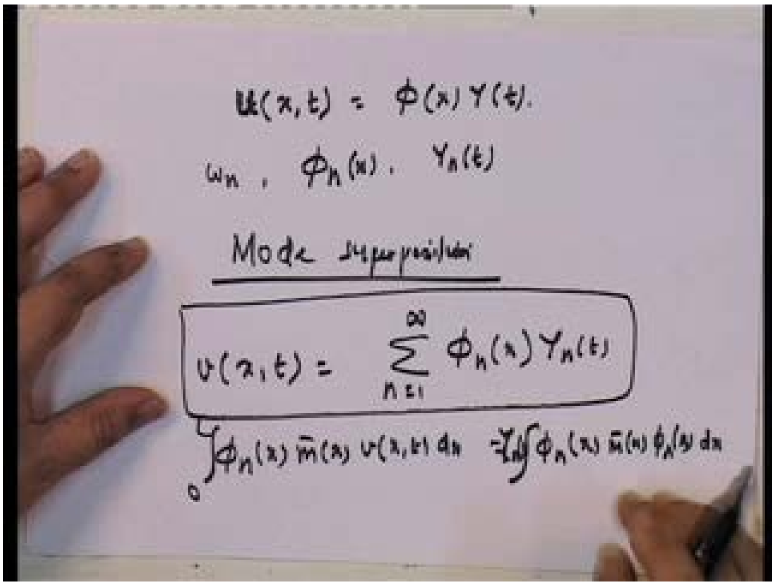

You see what do we have? We have a situation, where we say that look whether it is axial deformations axial is u x of t is equal to phi x y of t, and what we found out was that look you have this solution only the existing for certain phi n and corresponding Y n of t, so this is omega n phi n y n. So, therefore, this is the modal amplitude this is the mode shape right this is the frequency and therefore, using mode superposition, if you take the assumption that v x of t is a superposition of all the modes that you have.

So, you have now note it is infinite indeed, because this is a continuous system it has infinite modes, so noise this is the solution that you get. And this is getting v x of t from Y n of t, now how do you get Y n of t from v x n of t, well pre-multiply by phi n x m x m bar x into v x of t d x an integral over the whole length. And on this side you do this what we get is that everything else when you pre-multiply it by phi n x the only thing that remains is phi n x only and so that integrated over the whole length.

And of course, on this side you have this, so this basically becomes Y n of t the only one that remains because of mass orthogonality all the other terms which are not m which is not equal to n all disappears, what we are left with is the following, which, is Y n of t is equal to 0 to L phi n x m bar x v x of t d x all upon 0 to L phi n x square m bar x d x.

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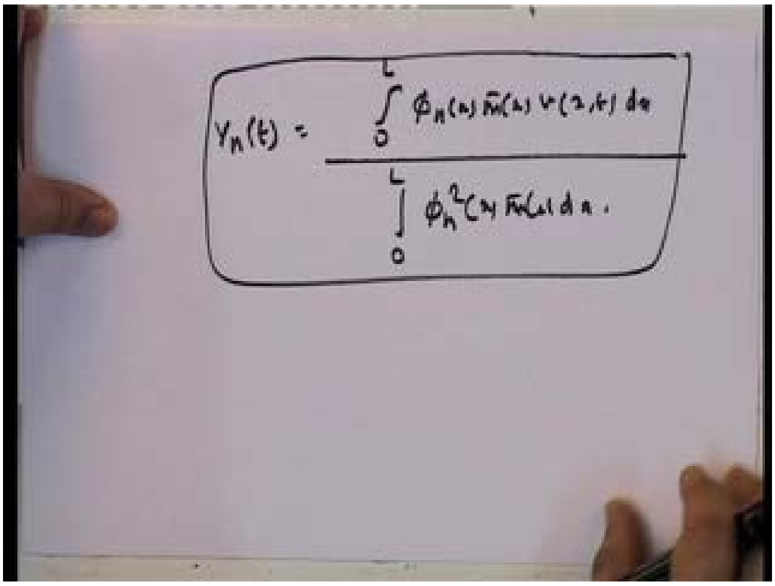

Note that this is identical similar in nature, so this is how you get Y n from v x of t and this is how you get v x of t from y n, so once you have got this you see the only difference that happens that we get is that you have a situation, where mode superposition is valid. Now, you know phi x and phi x of t is infinite sum of phi n x Y n x is valid, again because it is a continuous system, so the infinite the infinite domain it is valid mode superposition. And therefore, if mode superposition is valid then you see what do we get a situation like this that let me look at axial force I will look at axial situation, and axial deformation a response.

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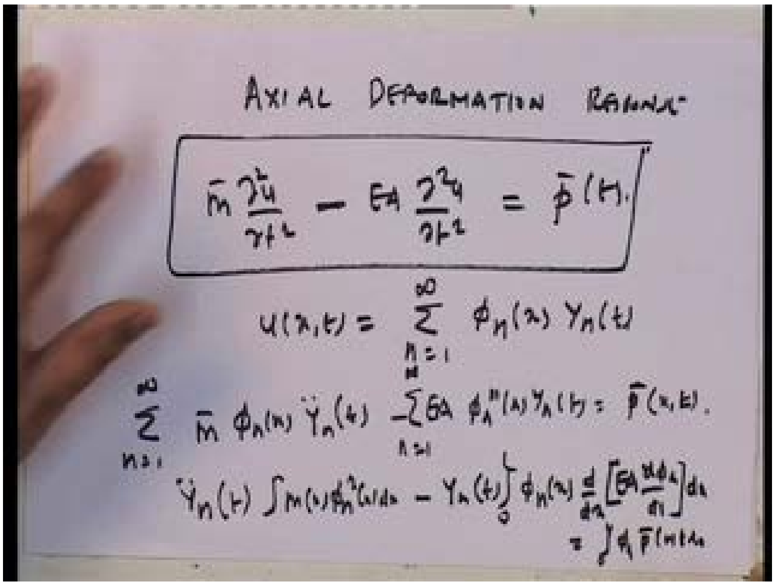

So, let us look at this what you have is m bar del u d t square minus EA del square d t square is equal to p bar of t this is the equation of motion, now what do we have in this particular case we have a situation where we can look at it in this fashion. That look u x of t is equal to n going from one to infinity phi n x Y n of t, so now, if you plug this in this one becomes what, m bar x if you look at this del u.

So, I get summation n going from 1 to infinity m bar phi n x Y n double dot t minus EA whole summation n going from 1 to infinity EA into phi n double prime Y n of t is equal to p bar x of t, I have just substitute this in. Now, what I am going to do is I am going to pre multiply through with this and so without having to put down very much what we can see is that going to be equal to Y n double dot t into m x phi n square x d x. And we have minus Y n of t and you get 0 L pre multiplied by phi n, so I get phi n x EA into d phi n by d x, and note that this is another thing that we can show is equal to this thing is equal to 0 to L phi n into p bar t d x and that is all we have.

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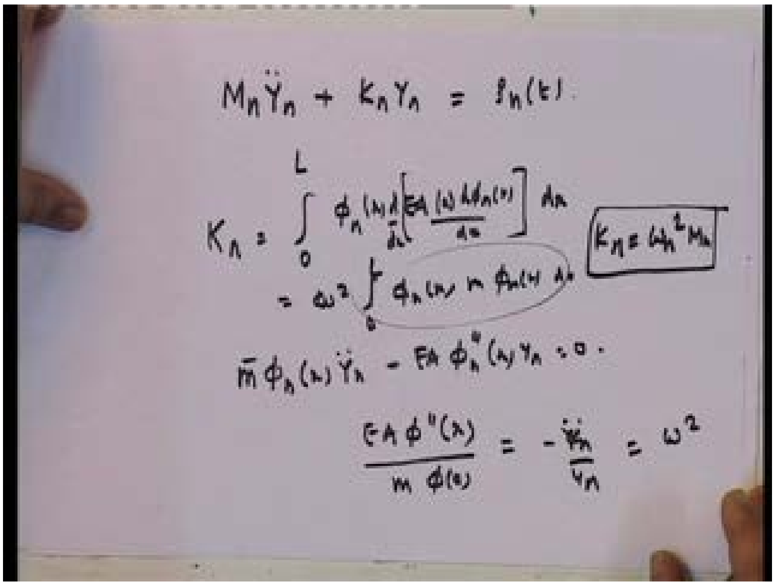

So, therefore, if I look this what we get is the following that I can write this as M n Y n double dot plus k n Y n dot is equal to p n of t, and note that I can show that k n if you look at k n which is equal to 0 to L phi n x. And I call this sorry d by d x E x d phi n e x by d x d x, now if I look at this I have the following, let us look at it m bar I have m bar phi n x Y n double dot minus EA phi n double prime x Y n equal to 0.

This is the solution of this equation what we had done earlier, so if you look at this if you look at this then this particular case I can plug in n phi x over here this is equal to, if you look at this is EA upon m phi x is equal to minus c we had seen that. And therefore, we saw that look at this particular one EA d a prime which is this one we can replace by m phi x into c, c we can take outside. So, then this becomes equal to c and this is actually if you look the c was equal to the same as Y n double dot upon Y n which was equal to omega square. So, that c is actually omega square. So, if you look at it omega square over here 0 to L phi n transpose m phi n d x, which if you look at it is nothing but n, so therefore, k n is equal to omega n square M n noise that comes out directly to and therefore, if you look at it.

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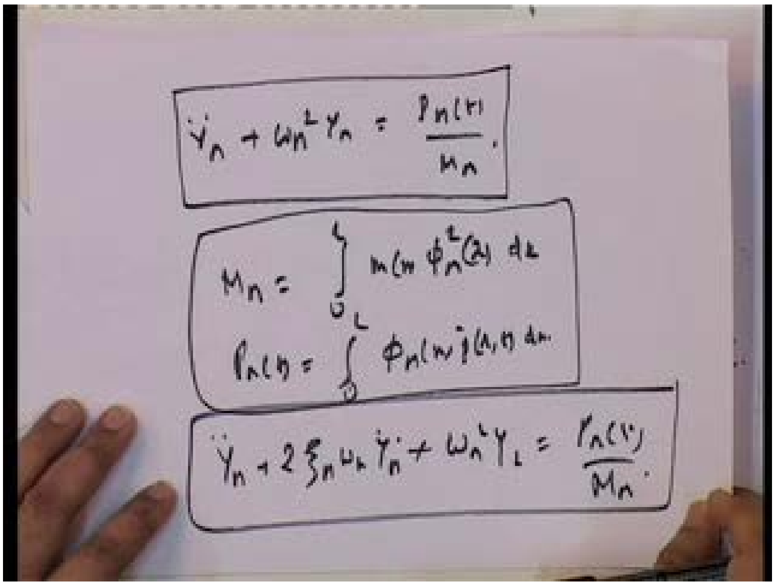

And of course, what you have in this situation is the following that, you get then nothing but Y n double dot plus omega n square Y n is equal to p n of t upon M n, where M n is equal to 0 to L m bar x phi n square x d x and p n of t is 0 to L phi n x p bar x t d x this is p n of t. Once we have this again all that we have is, we have a single degree of freedom system, and of course, this is as I said damping you could include at this level itself by saying well.

So, the entire problem essentially based on to what, the essentially the problem based on to a axial deformation we are looking at the axial deformations only at this particular moment. In axial deformations the entire problem based on to the following, and that is that you essentially solve the free vibration problem and once you solve the free vibration problem, and you have got your omega n and phi then using those phi n and omega n.

And of course, note that these are no longer nontrivial these are integral equations, so you know depending on m bar x and this phi n square x these are nontrivial solutions. Especially, you know if you have a transcendental equation these are not obvious elegant, but the point is that even if you know digitally, you can numerically solve it does not matter you do not have to get it.

The only point that we are saying is that this is a continuous system, and a continuous system this given M n is given this way phi n is given this way and once you have got that you can always put it into any kind of equation. So, now, this is for axial equations of motion, now let us look at the flexural motion, because in flexural motion it is a little bit different, it comes out of the same thing it is still the modal equation it is just that it looks a little bit different, so it might be worthwhile.

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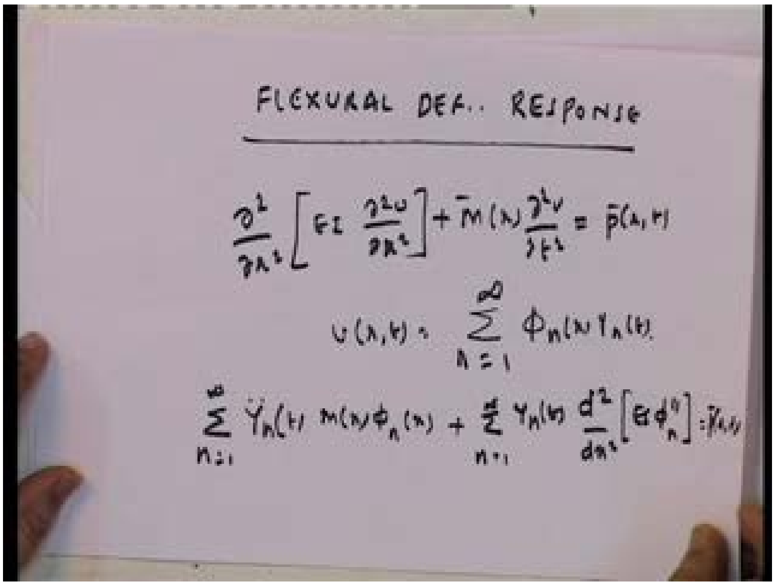

So, flexural deformation response that is what we are looking at right, now flexural deformation response and in this particular case let us go back d square x by d x square EI del square v by del x square plus m bar x del square v by del t square is equal to p bar x of t. So, this is my equation and I can put this if I put in the fact that v x of t is equal to m going from 1 to infinity phi n of x Y n of t. Then what we get is the following we get

Y n double dot of t m x phi plus double summation going from n going from 1 to t phi n of t and this becomes t square d x square into EI phi double prime n is equal to p bar x of t. So, this is what we get and then again going through the entire process, ultimately the equations become the following you know orthogonality condition.

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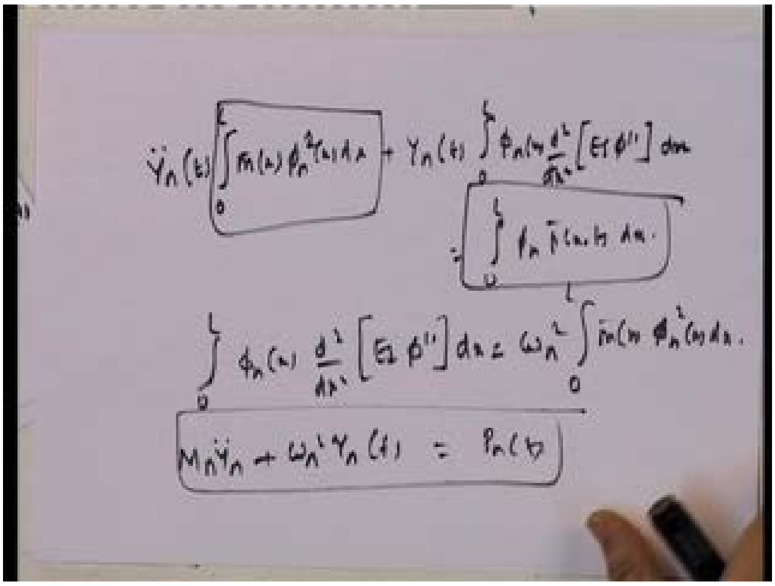

We get Y n double dot into 0 to L m x phi n square x d x plus Y n of t 0 to L phi n x into d square upon d x square EI phi prime d x is equal to 0 to L phi bar n p bar x of t d x. Note, that again in this particular case we can say that this into d square by d x square EI phi double prime d x is equal to omega n square 0 to L m bar x. Again going back to the same kind of thing that we had got earlier, you know you can show this and therefore, this equation again becomes M n Y n of t plus omega n square Y n of t is equal to phi n of t, where phi n is equal to this, and M n is equal to this.

So, this becomes then the solution process, in other words even for a continuous system what we see is that mode superposition, you know we do not have to do look at dynamic response, because the entire thing becomes the only difference that becomes is that it is an infinite series. In mode superposition for this we had n terms because it was n degree of freedom here, in a continuous system it has infinite number of modes and frequencies and therefore, what happens?

Essentially, is that you have the modes position equation looks like v x of t because v is a field x it is a function x, and t is equal to summation n going from 1 to infinity phi n x Y n of t ones we do that you know we get, because of the orthogonality of the mode shapes, we get again infinite here infinite number of single degree of freedom systems. So, the problem is still, but therefore, we have we already know from for modes of position, that for standard kind of load and for most normal loads that we dynamic loads that we have it is only the first few modes that contribute to the equation.

And therefore, you know in this case also we take a continuous system we break it up into m number of modal single degree of freedom systems, find out Y n of t and then v x of t is equal to summation n going from 1 to m, now small m, phi n x Y n of t. This way we get the displacement field and even for continuous system the only thing that happens, please remember is that you have to solve for omega n and phi n. And sometimes what happens is you know, especially when you have to solve transcendental equations omega n is known it is discrete value phi n is given in such a complex form that in reality you have to almost plot it. And so when you plot it does not matter, the integral equation that you have which is m bar x into phi n square it basically just becomes a numerical integration that you can do very easily. So, this in essence is the overall arching idea of and then you know you can always take damping in the mode.

So, I am going to stop here for today with the proviso that we have only investigated how to get v x of t at a particular, but note that once you know v x of t, but say I want to find out the bending moment at a particular point, how do I find it. It is actually trivial, all I need to do is I know v x of t I can find out v double prime x of t very easy second derivative, even if it is numerical you can always find out numerical in the second derivative multiply that by EI and that gives your bending moment of that particular point.

You want to find the shear force of that particular point well what do you do, suppose I want to find out in a cantilever the shear force of will left end due to loading what do I do? Well, all I do is EI into phi double prime 0, and that gives me my moment, so that in a sense is how you go through dynamic response. Thank you very much, I am going to stop now, in the next class we will look at in a little bit more detail of how to with some examples, of how to solve a dynamic response for a continuous system. And in the final lecture of this series I am going to look at wave propagation problems, one-dimensional propagation problems in beams. Thank you very much, bye, bye.