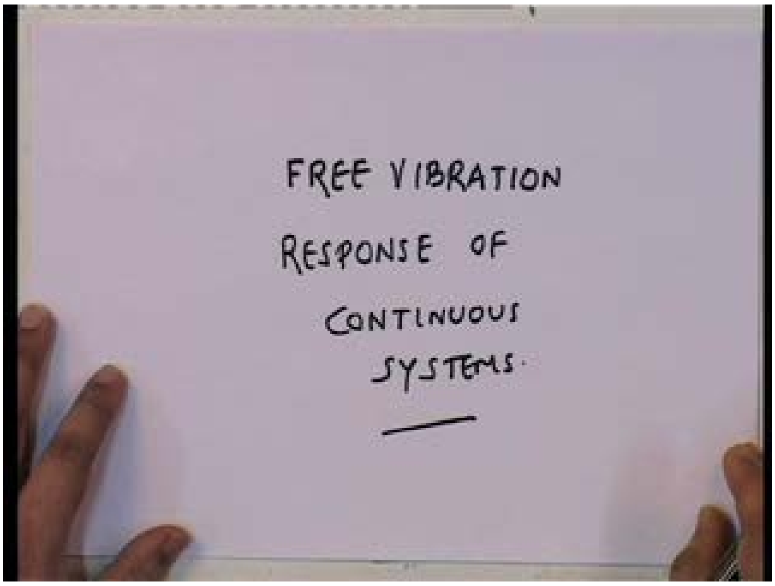
Structural Dynamics
Prof. P. Banerji
Department of Civil Engineering
Indian Institute of Technology, Bombay

Lecture - 35
Free Vibration Response of Continuous Systems

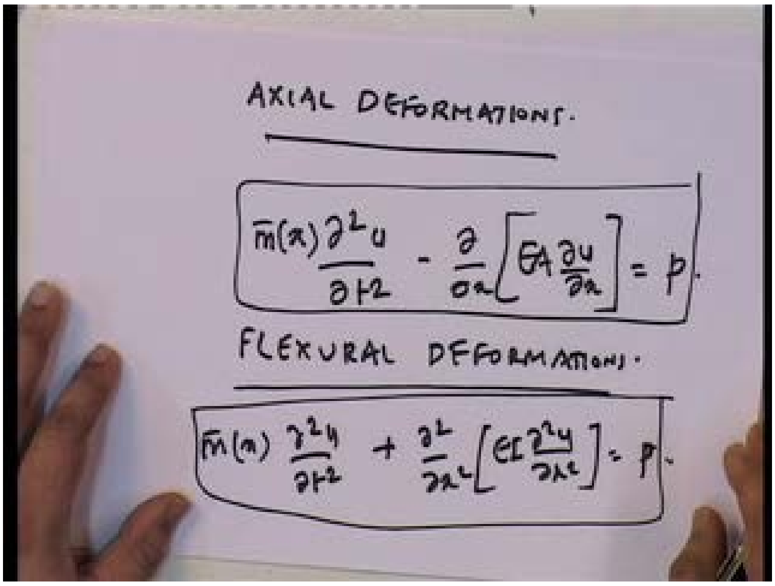
(Refer Slide Time: 00:26)



Hello there, today we are going to be looking at continuing to look at a free; we are going to be looking at a Free Vibration Response of Continuous Systems. Just to refresh our memories on what we had done, we had seen that if you look at the response the equation of motion for a simple beam axial deformation.

So, axial deformations, the equation becomes this way m bar x del square u by del t square minus del upon del x EA del u by del x equal to p and for flexural deformations. This is for axial deformations, for flexural deformations these were the equations of motion, this is for a flexural deformations of a simple beam the reason. I am calling a simple beam is that only flexural deformations are considered and only linear inertia is concerned.

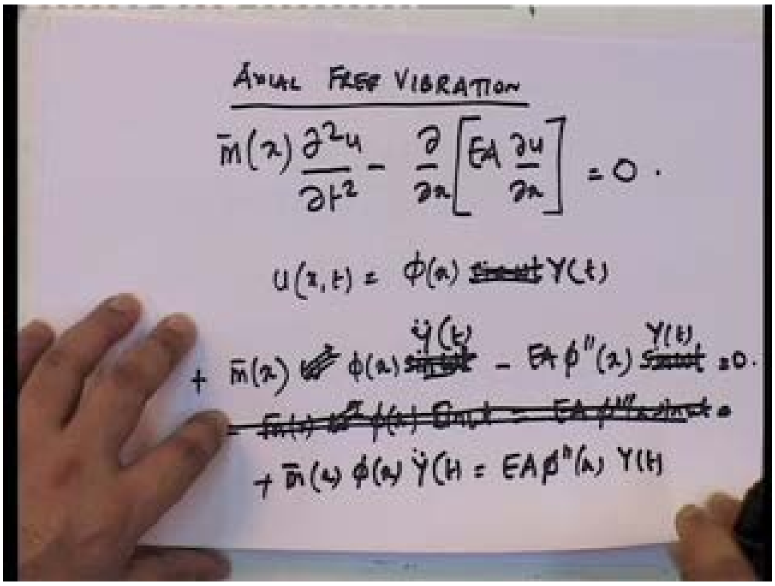
(Refer Slide Time: 01:18)



Later on in this course, towards the end of the course I am going to be talking about what is what we call it as Timoshenko beam. Timoshenko beam not only do you consider flexural deformation, you also consider shear deformations. And in addition to the linear inertia you also take rotatory inertia of a beam, that is know as the Timoshenko beam. So, we will look at the dynamics of the Timoshenko beam later on, but right now dynamics of the simple beam in axial deformation, gives this equation and in flexural deformation gives this equation.

Of course, the axial deformations you only have the simple beam, you do not have because you only have one directional motion and that is axial and linear. So, because there is no other motion, in flexural deformations what happens is in addition to flexural deformations, you have shear deformations and plus because it is moving like this, in addition to the linear mass you may have rotatory mass. Simple beam, only considered linear mass does not consider the rotational inertia of a cross section and it neglects shear deformation. So, if you neglect the shear deformation this is what you get?

(Refer Slide Time: 04:09)

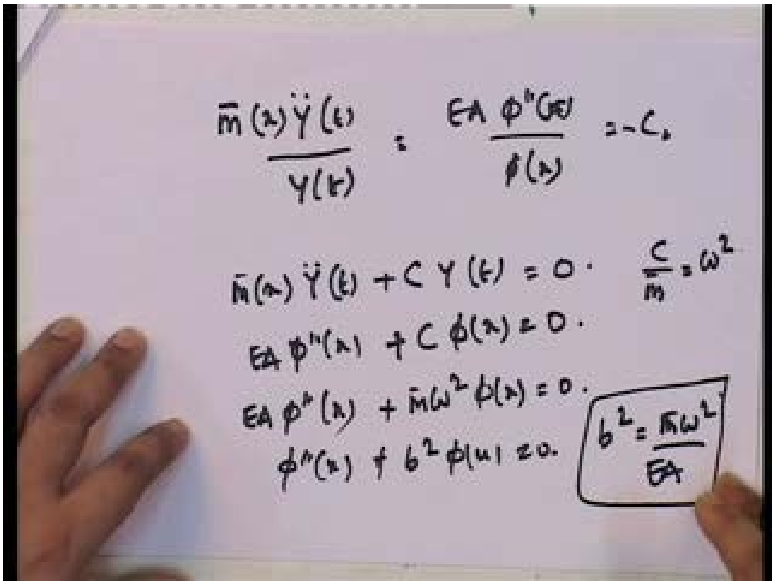


And last time we also look at the free vibration response, we fundamentally looked at the free vibration response of for axial deformations which was. So, this is axial free vibration and assuming that u of x t is of the form phi x sin omega t right I am just reviewing this we have already done this, but am just reviewing because I want to move on to the next step. And then once you substitute that, what you get you get this equal to m bar x into omega square phi x sin omega t, that is this one.

And this one if you look at it, it becomes equal to phi double prime, note that phi double prime of x essentially in where is the second derivative of phi x. And now, if we divide throughout by phi x and sin omega t, so I am going to divide throughout by phi x sin omega t, what you get is the following you get it equal to minus I am going to actually divide throughout by m bar EA.

So, you get m bar x omega square minus E, well let me just put it this way, let me put this phi x sin omega t is equal to ES phi prime x sin omega t I am going to just put these two together. And then what we get is the following, that minus m bar x is equal to the following right I am going to say that this is, I am going to divide throughout sorry made a mistake here, not sin omega t this is y of t right. So, if you look at this one is y double dot of t and this one is y of t and so sorry here, what we get is plus m x phi x y double dot of t is equal to EA phi double prime of x y of t. So, if we divided throughout by phi x y of t then we get the following.

(Refer Slide Time: 07:53)

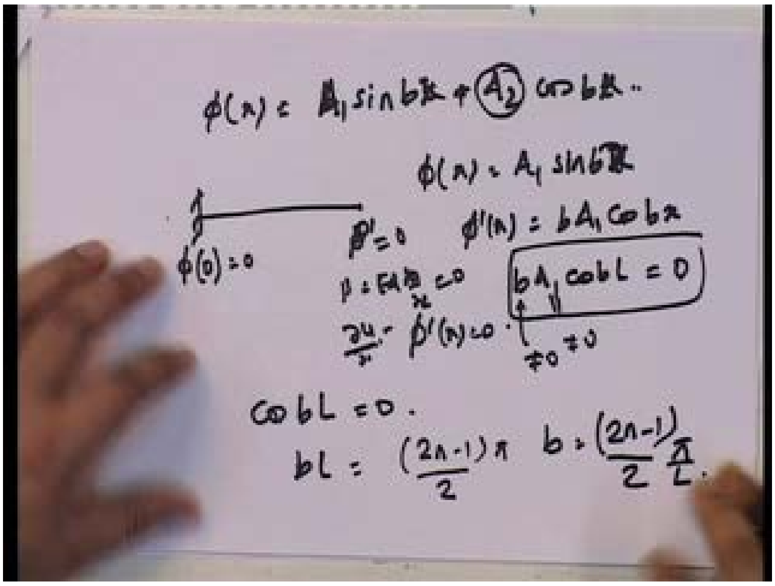


We get m bar x y double dot of t up on y of t is equal to EA phi double prime x up on phi x. And note that if this is equal to this which is the function only of x and this is only a function of t this is c right. So, once we have that I will call that as minus c for obvious reasons and then what we get is, we get the following equation m bar x y double of t plus c y of t is equal to 0 and the other equation becomes EA phi double prime plus c phi of x equal to 0, these are this is what we get.

Now, note that if you look at this c upon m bar is; obviously, equal to omega square because that is what you get from this equation. So, therefore, this one then becomes equal to EA phi i prime plus now, since this is omega c equal to m omega bar square, this becomes m omega bar square phi x equal to 0. In other words I can rewrite this as phi prime plus and I will call this b square phi x equal to 0, where b square is equal to m bar omega square up on EA and having done this.

Then your phi of x becomes equal to b sin sorry A 1 sin b of t plus A 2 cosine b of t and note this is going to b subjected to boundary conditions. So, if I have this situation where it is not allowed to move and it is this thing, so then this becomes phi and 0 equal to 0. And over here, since you are you know it is basically actually equal to you know p is equal to 0 and p is equal to EA del u by del x equal to 0 and so basically del u by del x.

(Refer Slide Time: 09:55)

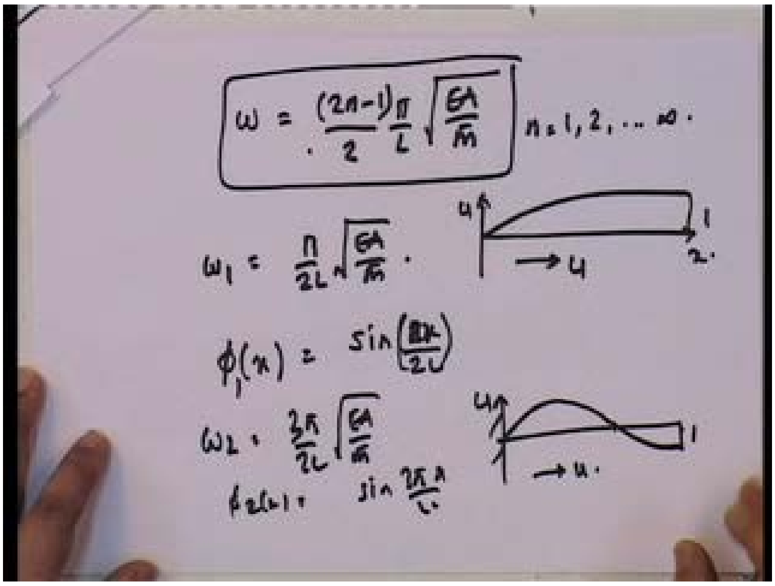


So; that means, in this particular case free phi x equal to 0. If we plug these in into this, what do we get the first one phi 0 is equal to 0 implies that A 2 is equal to 0. So; that means, phi 1 x is equal to A 1 sin b x sorry this is all x b x, now if you put phi prime, so this basically becomes what, phi prime is equal to b A 1 cosine b x. So; that means, b A 1 into cosine b L is equal to 0.

Now, note b cannot is not 0 A 1 cannot be 0 this implies that cos b l equal to 0. And therefore, you get the situation that b L is equal to 2 n minus 1 pi by 2 that is for this particular, situation and in this particular situation you get the situation that this b L is equal to and once you get this becomes what, if you look at; that means, b is equal to 2 n minus 1 up on 2 pi by L. And once we substitute the fact that b square is equal to m bar omega square. So, therefore, we have the situation that omega is equal to EA upon m bar square root into b and so the therefore, your omega in this particular case, becomes equal to the following.

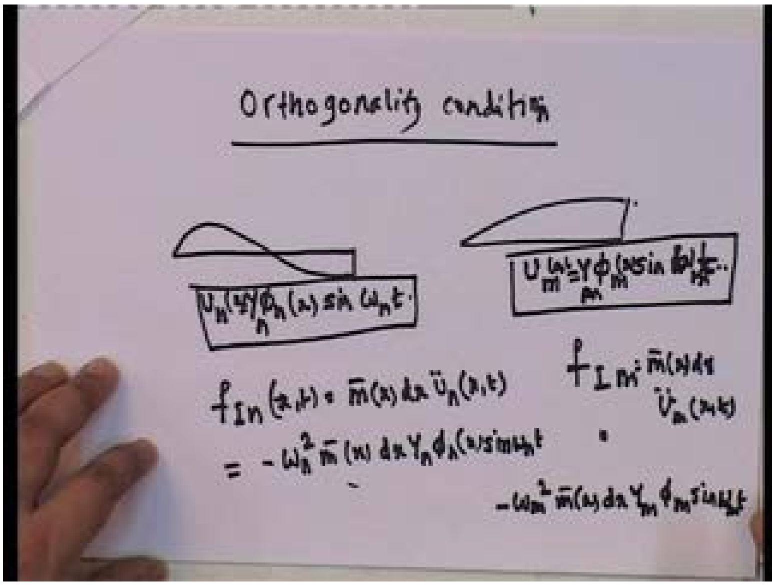
Omega equal to 2 pi n up on 2 pi over L into square root of EA by m bar. So, that is for axial that becomes the omega and for each omega and so therefore, if you put n equal to this is n equal to 1, 2 to infinity. And therefore, omega 1 is equal to pi by 2 L root EA by m and therefore, if you look at the corresponding f phi x it is equal to sin pi x by 2 L that is phi 1 of x and same way we have already looked at that. So, basically phi 1 is this way, where all the displacements are in this direction.

(Refer Slide Time: 12:46)



I am actually plotting u with x that is all I am plotting actually there are in this direction, so this is the first mode. The second mode where omega 2 is equal to 3 pi upon 2 L EA by m bar then phi 2 x is equal to sin of 3 pi x upon 2 L, this thing becomes where this is 1 and this is u, u is in this direction actually. So, similar way you can find this out, now one important point, so this is the omega in phi and note that this is almost identical to what we have generated, even for multi-stored structures.

(Refer Slide Time: 15:06)

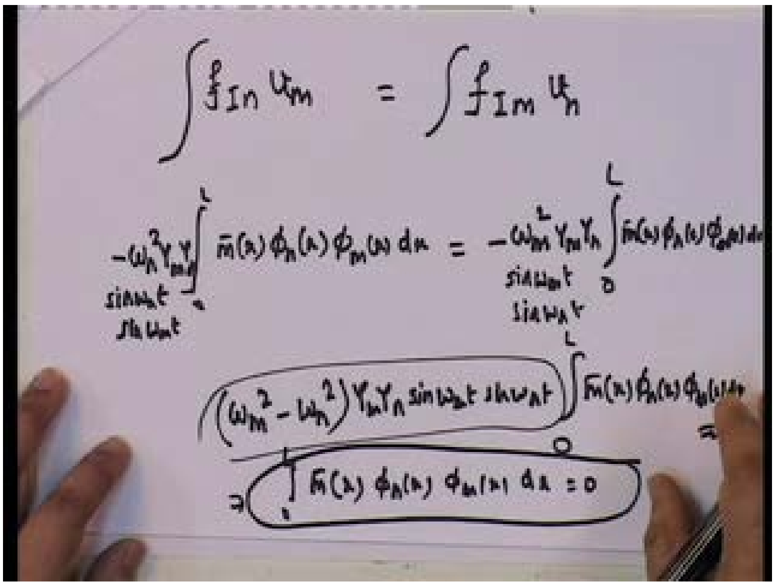


One important point to note is that, I will only show the mass orthogonality because in this particular case the stiffness orthogonality becomes little bit tricky. And you know I do not want to go into that it is it is actually fairly easy to do it, but I do not want to go into it is easy to look at a kind of a situations. So, let us have this situation, let us say that well you know let us take this, so I will call this v m, v m if we look at it is equal to phi m into sin b 1 x.

And similarly, if I call this v n it is equal to phi n and this going to be a y sin omega 1 t phi m x and this is going to be phi n x y n, this is y m phi n x y n of t sin omega n of t this ultimately is my displacement in the n’th mode and this is the n’th mode. So, if you get this, if we get a inertial force due to this load and call it what, we will call it f I n and an inertial force due to this load, due to this displacement I will call it has f I m.

What would phi I m be equal to, if you look at it would be equal to this is v n x of t this is v m x of t, this is also phi in x of t, this is at a particular point it is going to be equal to m bar d x that is the mass into v n x of t double dot. And similarly, this would be equal to m bar x d x into v m x of t because this is the acceleration. So, if I look at the acceleration this is going to be equal to what, it is going to be equal to v n is actually, if we look at v n is given in this form. So, if you look at double differential it will become minus omega n square m bar x d x y n phi n x sin omega m t. And similarly, this one would be equal to minus omega m square m bar x d x y m phi m sin omega m t.

(Refer Slide Time: 18:51)



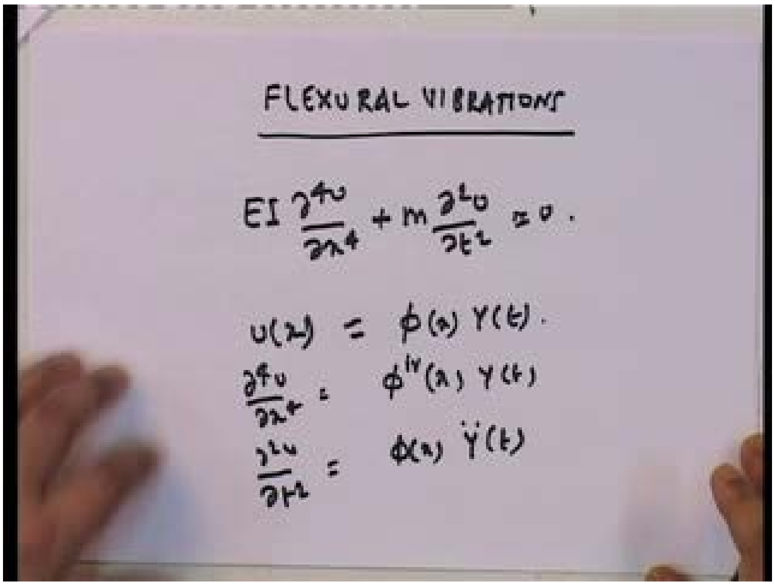
So, these are the you know, now what does Betty's law say. Betty's law says that if you have f in undergo v m over the whole length, this is equal to over the whole length phi I m is equal to v n. In fact, betty's law says the work done by the inertial forces, given by this undergoing these displacements is equal to the work done by, these forces undergoing these displacements that is what Betty's law says and so that is what we have done.

So, if I look at this if I plug it in what do I get, I get if I and I am going to substitute and v m I am going to substitute. So, I am going to get equal to minus omega n square y m y n because both of them y m comes from this, y n comes from this omega n square come from this, it is going to be equal to m bar x phi n x into of course, in this particular case we have sin omega m t and sin omega n t. So, here sin omega n t and sin omega m t that will come outside, it is going to be equal to phi n x phi m x d x right.

And then the other one is equal to, minus and here we get to omega m square because this is the inertia omega m square and we get y m y n, m coming from here, n coming from here sin omega m t sin omega n t integral from 0 to L 0 to L m bar x phi n x phi m x d x. So, if I put these to all these terms are the same, so if I put it in I get the following I get, omega m square minus omega n square into all of these y m y n sin omega m t sin omega n t 0 to L m bar x phi n x phi m x d x is equal to 0 these cannot be 0 automatically this implies that 0 to L m x phi n x phi m x d x equal to 0 this is the mass orthogonality condition that we have for the this thing.

So, this is what you have that you have this particular thing. So, this in a sense is orthogonality condition, this is a mass orthogonality condition that I have derived in same way we can derive, but it just gets a little bit more complex and you know, there is no need to all I say is that we know that the, orthogonal the mode shapes are orthogonal in this particular case. So, much for the your what shall we call it, your axial deformations we have derived this. Last time we had looked at all of this the only thing that we had not looked at was the orthogonality condition, last time I stopped by looking at the equation that you are going to get, if you have flexural vibrations.

(Refer Slide Time: 23:05)

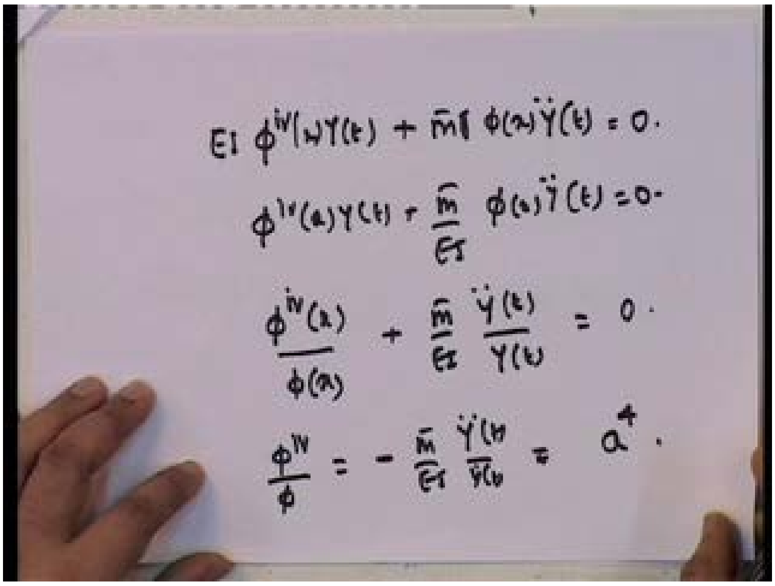


So, now I am going to start looking at flexural vibrations, free vibrations flexural free vibrations, we are going to look at free vibrations. And in this particular case what we will do is, we are going to be looking at specifically how to derive it, we will derive it from first principles and then we will start seeing a different boundary conditions how they become differently. So, let us look at it, what is the equation look like, it looks like and I am going to take it for uniform case, so uniform case you have this plus m del square v by d in actually it is d square, but since e I is a constant.

So, we can take the it is inside, so this is the equation and v of x which is a lateral displacement, we take it equal to phi of x and y of t. So, if you look at it the fourth order derivative here, would be equal to the fourth differentiation of this into y of t and del square v by del t square is equal to phi x into y double dot of t.

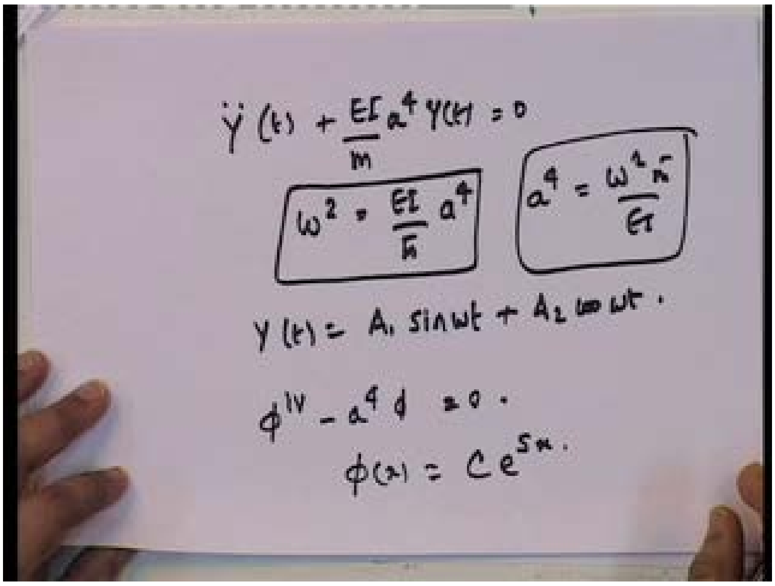
So, putting in these here what do I get, I get it equal to the following and this becomes E I phi fourth x y of t plus m bar. Now, this one I am not going to put is a x I am going to leave it as m bar its uniform beam, so it is m bar into phi x into y double dot t is equal to 0. So, I can put it this as the following I can say that look, it is phi fourth x y of t plus m bar up on E I phi x y double dot of t is equal to 0.

(Refer Slide Time: 24:49)



And then if you plug this in what do we get, we get it equal to the following I am going to divide throughout by phi x y. So, this becomes the following phi fourth x upon phi x plus m bar upon E I y double dot is equal to y of t is equal to I am going is equal to 0. So, then if you look at this upon, this is equal to minus of m bar upon E I and; obviously, we say that this is equal to a term which I am going to say is equal to a to the power of 4 I am going to put it in this fashion.

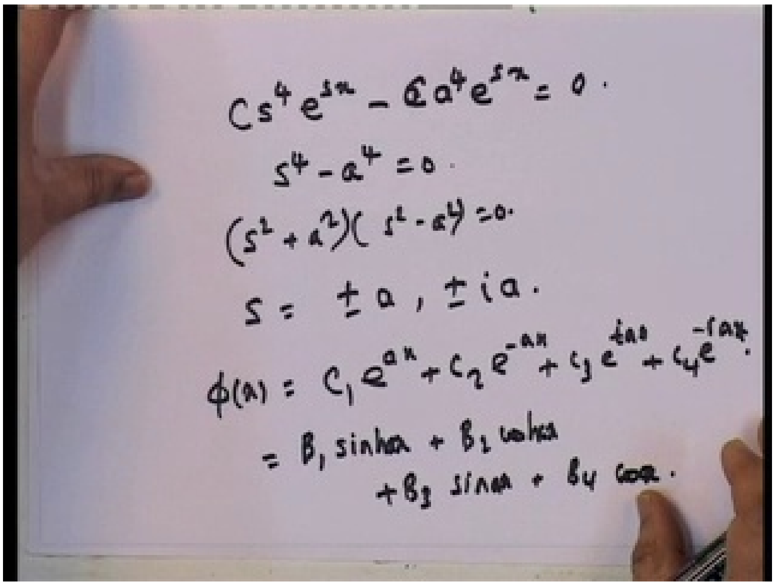
(Refer Slide Time: 26:59)



So, if I put it in this fashion let us see what happens, what I get is the following that one equation makes it the following that is that y double dot plus E I upon m bar a 4’th right, if you look at this is equal to I will take E I over here. So, it becomes E I m 4’Th into y dot of t, so omega square is this, so if you look at is equal to 0. And so; obviously, you have a situation where, omega square is equal to E I upon m bar a 4’th or we can say that a 4’th is equal to you know I mean either way I mean let us not go let us just go with this or we can say the other way.

So, if we do it this way; obviously, y of t becomes equal to A 1 sin omega bar t plus A 2 cosine omega bar t and this A 1 and A 2 can be found out from the initial conditions. Let us look at the other one, the other one says what it is equal to phi 4’TH minus a 4’TH phi is equal to 0. Now, let us substitute a situation where phi of x is equal to c to the power of e s x, if I substitute this into this equation note that phi 4’TH become just s 4’TH. So, this basically becomes the following.

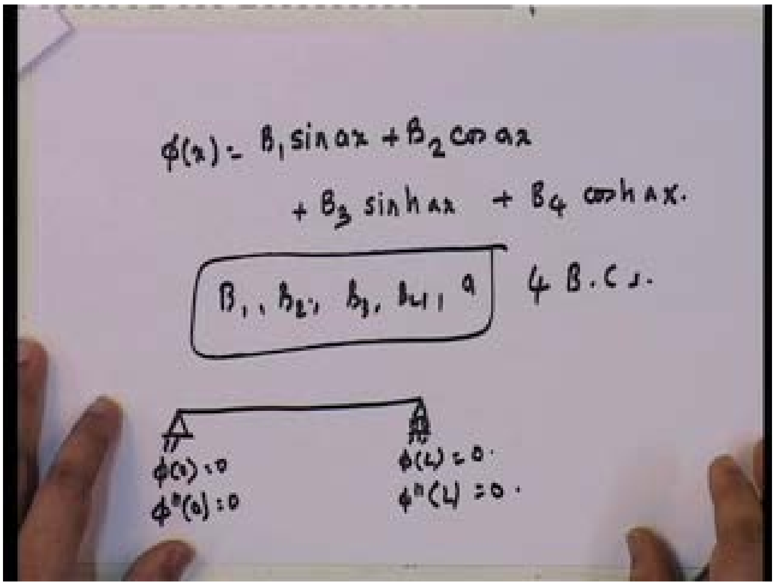
(Refer Slide Time: 29:11)



It becomes C into s 4’TH e to the power of s x minus C a 4’TH e to the power of s x is equal to 0. So, what we have is we have the situation just C to the power of x which is in both cases cannot be equal to 0, it implies that s 4’TH minus a 4’TH is equal to 0, this implies s square plus a square into s square minus a square is equal to 0. So, s is equal to plus minus a and plus minus i of a these are the four routes that you have, so essentially phi of x becomes equal to c 1 e to the power of a x plus c 2 e to the power of minus a x plus c 3 e to the power of i a x plus c 4 e to the power of i ax.

Note that, you know minus i ax, note that we have already seen that this can be since, this is the real quantity this has to be a combination of sin and cosine. Similarly, this one we can write in terms of sin hyperbolic and cosine hyperbolic, where sin hyperbolic sorry cosine hyperbolic is e to the power of x plus e to the power of minus a x by 2. And sin hyperbolic is equal to e to the power of a x minus e to the power of minus a x by 2. So, those are the sin hyperbolic functions and cosine hyperbolic functions and if we write it this way, this becomes the following B 1 a x sorry these are all a x. So, essentially that implies that for the flexural vibrations the phi of x is of the form.

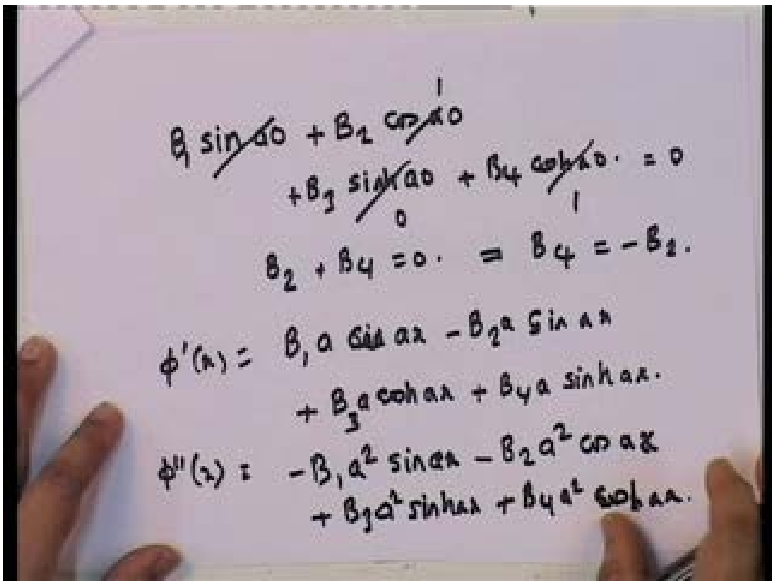
(Refer Slide Time: 31:44)



Let me write it down properly again, it is equal to B 1 sin a x plus B 2 cosine a x plus B 3 sin hyperbolic a x B 4 cosine hyperbolic a x. So, now, if you look at this, this becomes the following so; that means, how do I get B1, B2, B3, B4, and a well what happens is you have 4 boundary conditions. And for the four boundary conditions, you can find out three of these, in terms of one and you can find out the frequency question, which is the a you will get a.

Now let me take, so this boundary condition depends on what kind of a system. Let us take a basic simply supported beam, what are the bond conditions, well I know that phi at 0 is equal to 0, phi at L equal to 0, I also know that since moment is equal to 0 that curvature here is 0 and I also know that curvature here is equal to 0, these are my boundary conditions that I have. So, I am going substitute these boundary conditions into this equation. So, let us see what happens, well let us see the first one, the first one becomes the following we get a situation where, since you know this is moment actually E I phi I prime, but does not matter it is phi I prime. And ultimately let me just put this next page on here, I am sorry I already have a page vocalized having put a page.

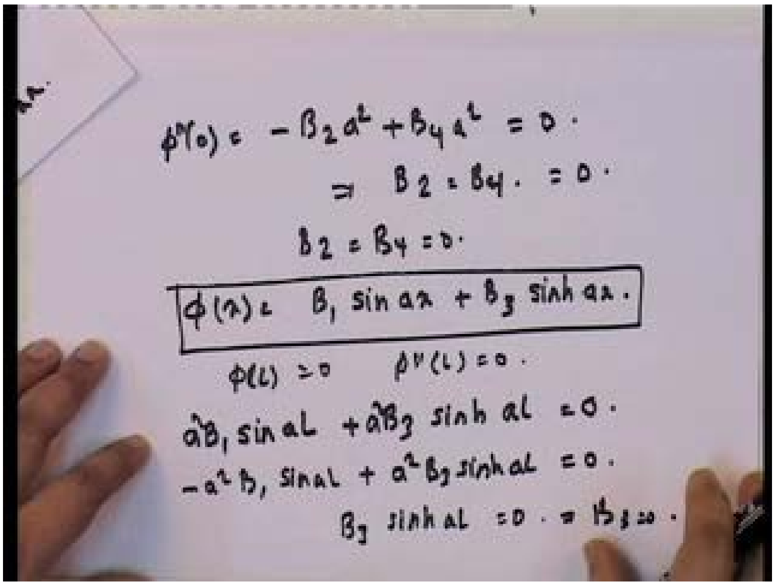
(Refer Slide Time: 34:11)



So, what do we get having put phi equal to 0 this implies that B 1 sin a 0 plus B 2 cosine a 0 plus B 3 sin hyperbolic into 0 plus b 4'TH cosine hyperbolic a 0 is equal to 0 this we know, from the first boundary condition which is phi 0 equal to 0. Now, note that this is equal to 0, this is equal to 1, this is equal to 0 and this is equal to 1. So, basically what we get from this that B 2 plus B 4 is equal to 0 and this implies that B 4 is equal to minus B 2.

Now, let us put the following let us put now, differentiate this if you differentiate this p of phi prime of x is equal to B 1 into a into sin a x minus B 2 a into sorry this is cosine a x and this is into sin a x the other one B 3 a cosine hyperbolic a x plus sorry this is going to be equal to plus B 4 a into sin hyperbolic a x that is plus, this is not minus. And then phi double prime x is equal to minus B 1 into a square sin a x and minus B 2 a square cosine a x and this one becomes B 3 a square sin hyperbolic a x plus B 4 a square cosine hyperbolic a x. Again phi double prime is equal to 0.

(Refer Slide Time: 37:08)

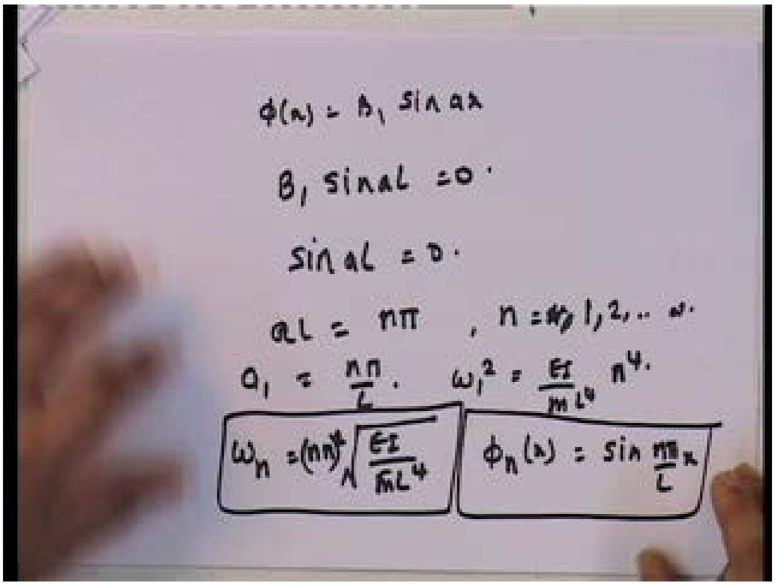


So, this implies the following that phi double prime 0 means, minus this B 1 disappears. So, it is going to be minus B 2 a square then this sin hyperbolic disappears, so you have B 4 a square is equal to 0, this implies that B 2 is equal to B 4. Now, the only way that both B 2 is equal to minus B 4 and B 2 is equal to B 4 is if they are equal to 0, this implies that these two boundary conditions imply that both B 2 and B 4 are 0 that is what this implies.

So, now, let us plug in, so we have got this, so essentially both B 2 and B 4 are not there and so phi x is of the form B 1 sin ax plus B 3 sin hyperbolic a x, this is what it becomes. And now, let a substitute the two which is what, phi of L equal to 0 and phi of double prime L is equal to 0. So, if I put this in phi of L all it says is that if we look at it, it becomes equal to phi 1 L becomes B 1 sin a l plus B 3 sin hyperbolic a l is equal to 0.

Now, the phi double prime note that when I this, this becomes minus a square B 1 sin a l and then plus a square B 3 sin hyperbolic a l is equal to 0. So, this is what we get, so if I now put a square and a square here, all we get is if you add this, this disappears and what I get is that B 3 sin hyperbolic al is equal to 0. Now, note that sin hyperbolic a l is not equal to 0 by definition e to the power of a l minus e to the power of minus a l upon two cannot be equal to 0. So, therefore, this implies that B 3 is equal to 0.

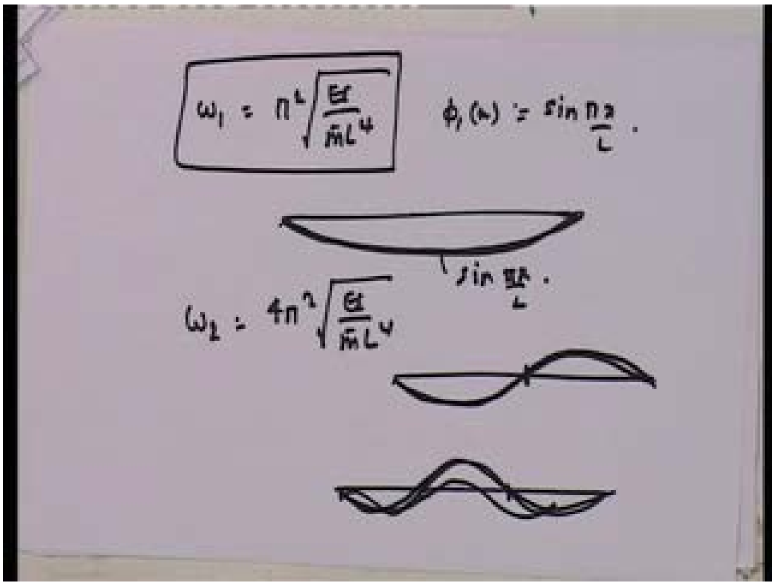
(Refer Slide Time: 40:14)



So, from the third set of boundary conditions what I get is that look, the only form of phi x is equal to B 1 sin a x and then; obviously, that phi L tells us that look it implies that B 1 into sin a L is equal to 0. Now, since B 1 cannot be equal to 0 because if B 1 is 0 this becomes a trivial solution, so therefore, it has to satisfy that sin a L is equal to 0 and the way sin a L is equal to 0 is if it is equal to n pi. So, if it is n pi then you have the situation that where n is equal to 0, 1, 2, n of course, the 0'Th term does not exist. So, we always start from 1 n has to be 1.

And, so therefore, you have a situation where a 1 is equal to n pi by L and note that we had written earlier, going back to where we had defined this, if you look back we had written that a 4'TH is equal to omega, omega square is equal to E I upon m 4'TH. So, if I write that down, then it becomes omega 1 square is equal to E I upon m L 4 and what we get over here, is n sorry n is already done pi 4'TH. So, what we get is that omega n is equal to square root E I upon m bar l 4'TH and we have n pi square. So, that is I mean sorry n pi the whole square that becomes omega n and the corresponding phi n of x is equal to sin n pi l x that is the corresponding one. So, this for a simply supported beam and if you put that equal, then what we get we have to put the first one.

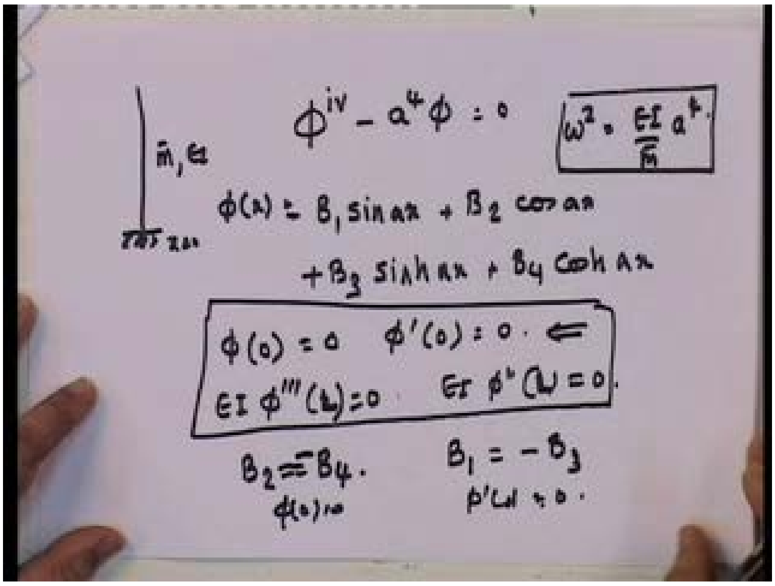
(Refer Slide Time: 43:08)



The first one is equal to omega 1 is equal to pi square, square root E I m bar l 4’TH and phi 1 is equal to sin pi x upon L. So, if I plot this, this is where this is of the form sin pi x by L omega 2 is equal to 4 pi square E I upon m bar L 4’TH and if you look at it this is equal to, so in this way what we can do is, we can find out and I am not going into the next one it is 9 pi and it will have 3. But, note again one fundamental point that I made in the last time, first mode lowest fane energy no node, second mode slightly higher strain energy one node. Similarly, if you look at the third second one it will be like this, sorry two nodes higher mode strain energy mode. And so the point remains that the fundamental mode has the lowest strain energy mode shape, and as you go higher the strain energy goes up and so on and so forth. So, this is for a simply supported beam.

And note that let us not forget the fact that omega square is equal to E I upon m bar into a 4’th this equation remains firm, this equation also B 1 sin a x plus B 2 cosine a x plus B 3 sin hyperbolic a x plus B 4 cosine hyperbolic a x, this is the you know the shape for a general simple beam vibration, flexural vibrations. The only thing that is different in each case is that the boundary conditions are different, in this particular case what is the boundary conditions at x equal to 0, we have phi equal to 0 equal to 0.

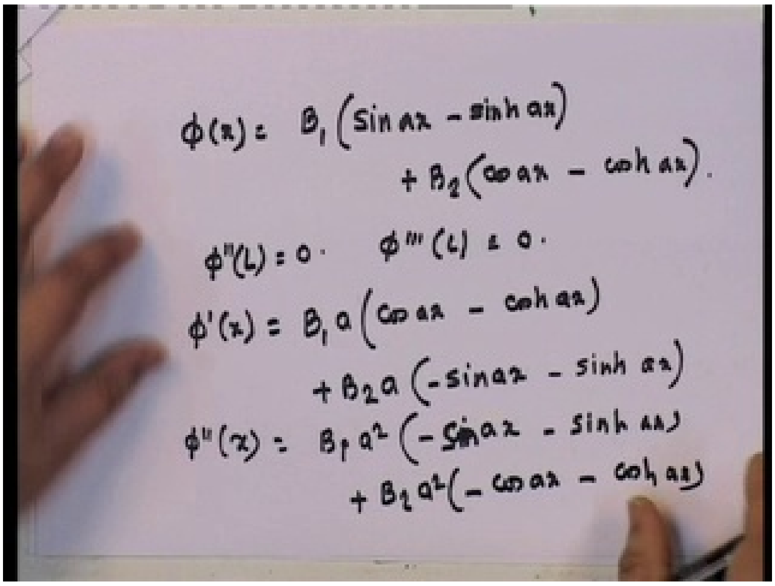
(Refer Slide Time: 45:31)



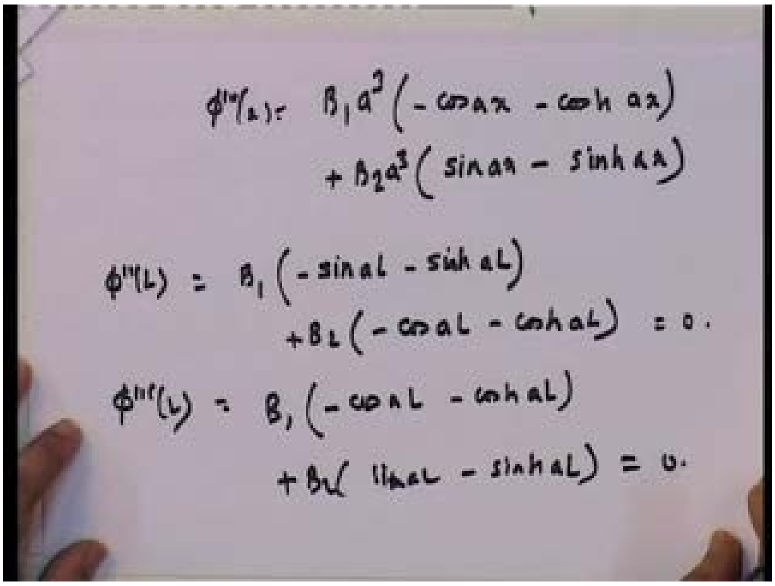
Similarly, phi prime is equal to 0 and at this end what we have, is E I phi 3 is equal to 0 an E I both the moment and shear at that and 0. So, these are the four boundary conditions, at L and at L these are the four boundary conditions. The first boundary condition, we already know gives us the fact that B 2 is equal to B 4 and the other boundary condition which is phi prime, we have already seen that phi prime is this becomes cosine and this becomes sin and this becomes cosine.So, what we have is that B 1 sorry B 2 is not equal to his minus B 4 and from this we get B 1 plus B 3 is equal to 0. So, B 1 is equal to minus b 3, these two boundary conditions this is given by this and this is given by this. So, if we plug these in our equation becomes the following.

Now, the equation becomes phi of x B 1 sin a x minus sin hyperbolic a x plus B 2 cosine hyperbolic a x minus this is what we get I mean you know I am of course, this is the overall equation that we get. So, once we get this equation then we substitute what, the fact that, so if you look at these equations. Now, you look at this is phi x now, if I look at it phi prime of x is equal to B 1 a, this will become cosine a, this will become cosine hyperbolic a plus B 2 a, this will become minus sin a x and this will become minus sin hyperbolic a x is equal to B 1 a square minus cosine a x sorry minus sin a x minus sin a x plus B 2 a square minus cosine a x minus cosine hyperbolic a x.

(Refer Slide Time: 48:45)



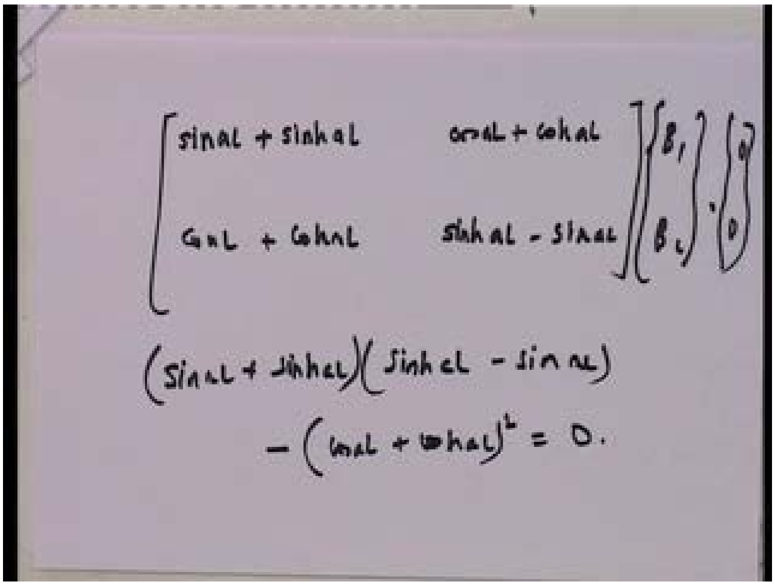
(Refer Slide Time: 51:22)



And similarly, we get that phi 3 becomes equal to B 1 a cubed and the sin becomes cosine a x minus cosine hyperbolic a x plus B 2 a cubed this one becomes, sin hyperbolic a x minus sin hyperbolic a x. So, having done that what do we get, we get the following substituting phi double two, I get an L this one a square I can neglect right because a square is in these in both of them. So, what I ultimately get is B 1 into minus sin a L this is phi double prime L is equal to this minus sin hyperbolic a L plus B 2 minus cosine a minus cosine hyperbolic a L and phi 3 L is equal to again both of them have a 3 and note that this is equal to 0.

And similarly, that is why a 2 and this case a cubed disappear and what we have is minus cosine a L minus cosine hyperbolic a L plus B 2 sin a L minus sin hyperbolic a L is equal to 0. So, this you see is two equations is B 1 and B 2 and I can solve this in this following manner, I can say that look, note that both of these have minus in them. So, I can take them in the other side and what I have, is the following the equation looks like this.

(Refer Slide Time: 53:40)



The first term sin a L plus sin hyperbolic a L, the second term cosine a L plus cosine hyperbolic a L, this one if you look at it, again this is what happens that this one becomes plus plus and this becomes plus and this becomes minus. So, what we get is cosine a L plus cosine hyperbolic a L and this one becomes, sin hyperbolic a L minus sin a L B 1 B 2 is equal to 0 0. Now, the only way this can happen is if this determinant is equal to 0.

So, this is becomes what is known as the frequency equation, the frequency equation is the following sin a L plus sin hyperbolic a L plus sin hyperbolic a L minus sin a L minus cosine a L plus cosine hyperbolic a L square is equal to 0, this is the frequency equation that we have. And the next class I am going to show you, how this equation can be solved to for getting the cantilever, you know free vibration frequencies and motions.

Thank you very much, bye, see you next time.