Structural Dynamics
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Lecture - 34
Introduction to Dynamics of Continuous Systems

Hello there, we have already looked at over the last many lectures, looked at dynamics of single degree of freedom systems. Then we looked at dynamics of generalized single degree of freedom systems, and finally we were looking at dynamics of multi degree of freedom systems.

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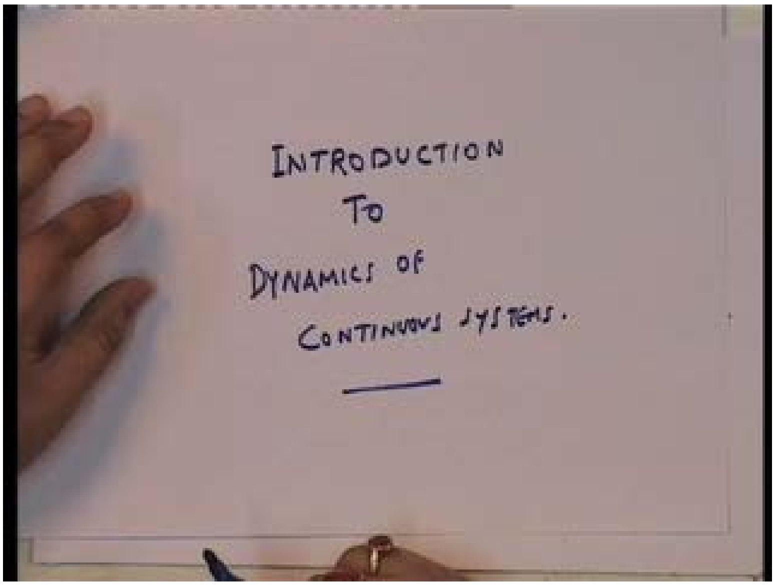

And now I am going to come to the last part of my course and that is going to be an introduction really. This I would just call it introduction, because this field is actually a very large field, so you are going to be Introduction to Dynamics of Continuous Systems. See up till now, we been looking at discrete systems, single degree of freedom, one degree of freedom, multi degree of freedom, n degree of freedom or generalized single. Generalized single degree of freedom was the closest that we came to looking at a distributed flexibility kind of a system. But of course, there also we said that, look we are going to assume that, it is subjected to a particular kind of shape function. Now, this time we are going to start looking at actually...

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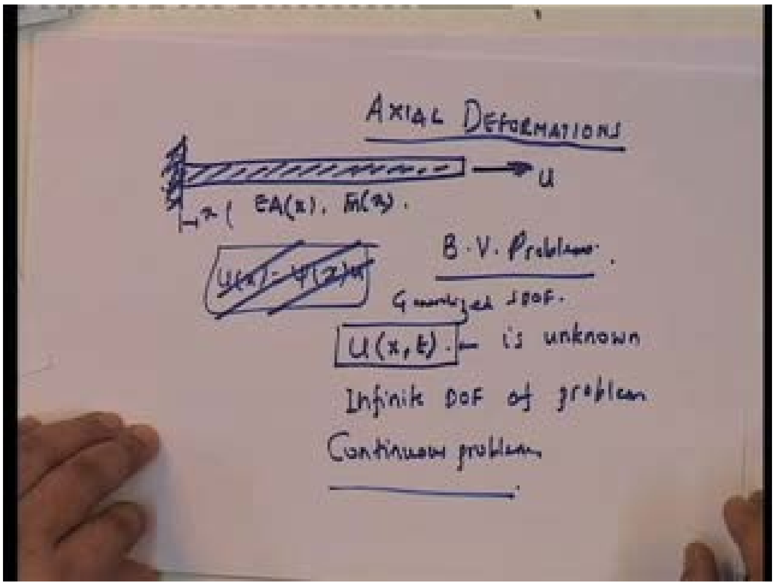

Let us look at a situation and there is a bar, a bar which I will first look at, what are known as axial deformations. So, axial deformations are this motion, the reason I am starting up with the axial deformations is that, it is actually a very simple problem to solve. And then we will see, we will look at flexural, because we have look at flexural in generalizing single degree of freedom problem, right now we will just look at axial. So now, the point here is that, if I were to deal with this see, so this one has, this is the thing.

And I do not know, let us just define for axial, the axial rigid is given in this form and I have m bar x d x, these are kind of standard things. And the question then becomes, we do not want to give, we will talk about the boundary conditions later, these we will see that, these are actually a class of boundary value problems. So, anyway, so this is the thing and if I were to look at it from the perspective of single degree of freedom, what would I say.

I would say that, look this is my u and I would say that, look u of x is given by some psi of x of u and then write the entire equation in terms of psi x and just it will become a generalized single degree of freedom. Now, I shall not do that, I will not make this assumption, I will say that look, u x of t which is displacement at any point, let me start x from here, u x of t is unknown. Now, if u x of t becomes an unknown, you see what happens is then you have a situation, where what you are saying is that, look displacement at every point is an unknown.

So, if displacement every point is unknown, this becomes an infinite degree of freedom problem and this is actually a continuous problem. So, let us look at this, so I am looking at axial deformations, so I have a situation, in which I have this bar and it has E A x and m bar x, E A x has the axial rigidity and m bar x has it is mass per unit length.

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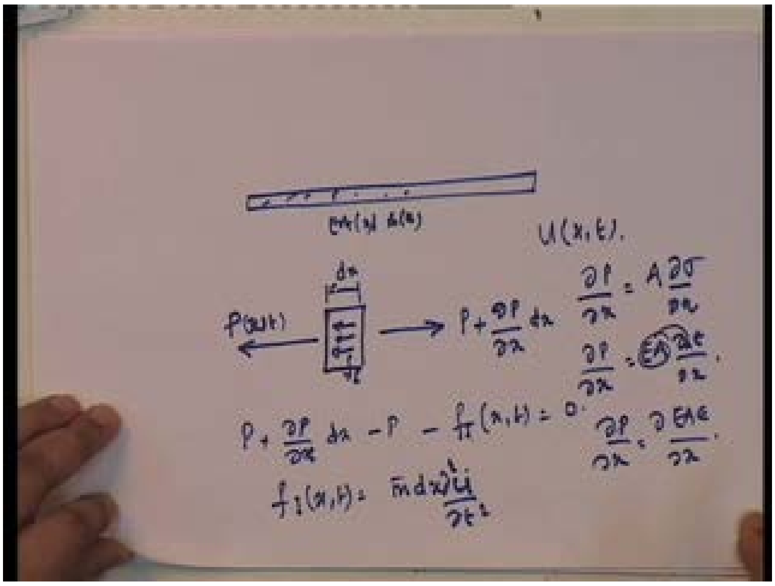

So, if I take an infinitesimal length, what do I have, I have axial force P, I have this is P plus del P by del x why, because P is a function of time. So obviously, this is function of x and time, so it is del P upon del x into d x and this is an elemental d x, these are the forces and on top of that, you have inertia forces, these are what, inertia forces. So, if I look at the equilibrium of this, what do I get, I get the following, P plus del P x minus P minus f I x of t is equal to 0, this is what I get my function as.

Now, let us look at, what f I is going to be equal to, now what is del P upon del x equal to. Let us see, what P, how can I put it in terms of u x of t, so if you look at it, by definition you have del P by del x is equal to A into del sigma by del x, because it suppose to be uniformly distributed. Now, del sigma is equal to E A del epsilon by del x, actually you know, because E A can also be a function, so actually these go inside this. So, it becomes actually del P upon del x is equal to del E A epsilon by x, now the trick is the following.

And that is that, and what is f I, f I x t if you look at it, is equal to m bar d x into u double dot, which is actually del squared u by del t squared. So, that is f I and here this one cancels out, this one comes out this way, so what you have is, it becomes del P x.

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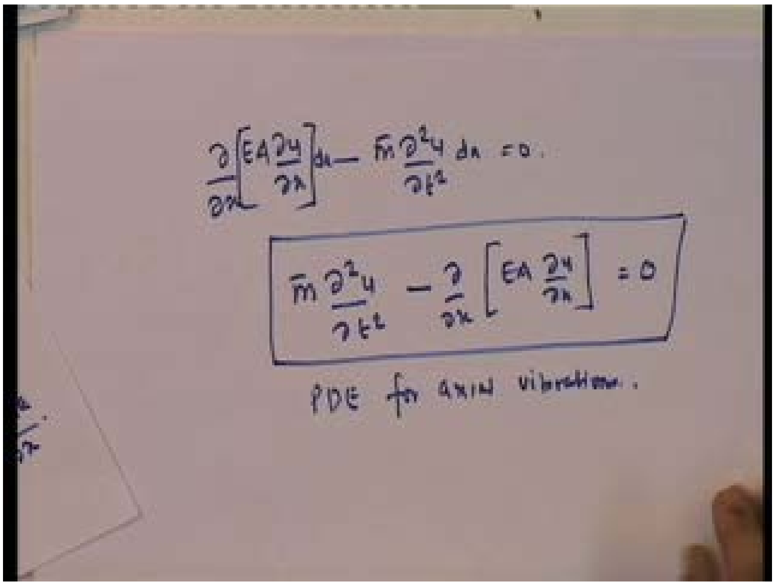

So, it becomes del of del x E A del u by del x, because epsilon is del u by del x then you have the f I I here. So, this becomes what, what it becomes, it becomes then your minus m bar del u by del x t squared, this is u into d x by the way, into d x equal to 0, the d x d x exists in both the places and d x cannot be a function. So, we can write this equation in the following format, m del squared u by del t squared minus del by del x of E A del u by del x equal to 0. So, this become the partial differential equation for axial vibrations, now let us try to see, how do we solve this problem.

This problem has to be solved in using two things, one is we need boundary conditions, we need initial conditions. What are boundary conditions, we need to know u at 0 of t or u dot at 0 of t and u at L of t, that is the boundary or u dot at L of t. So, these are the boundary conditions and what are the initial conditions, the initial conditions are u at x equal to 0 and u dot at x equal to 0. These are the initial conditions, time t equal to 0, boundary conditions at x equal to 0 and x equal to L or any other point. So, let us take this particular thing, so this becomes m bar del squared u by del t squared minus del by del x E A del u by del x equals 0, so this becomes the vibration problem.

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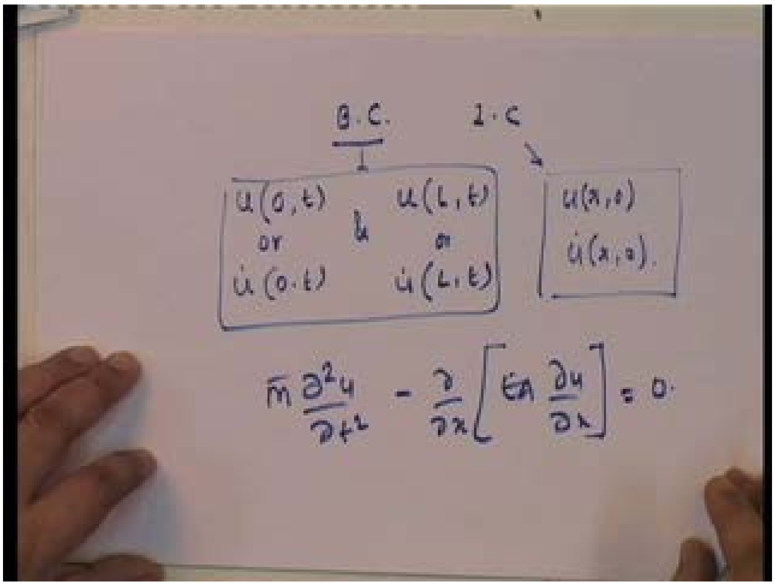

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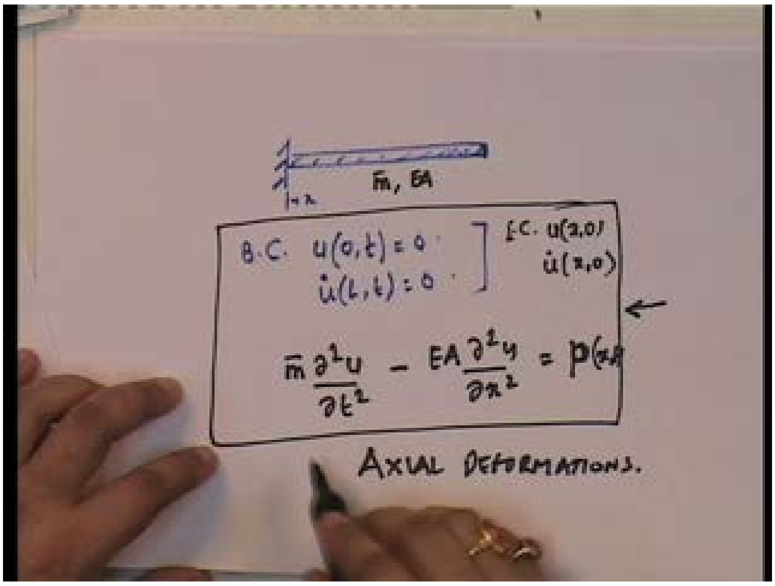

So, now let us look at this particular thing and let us have a situation, where let us put it in that, I have this bar, this is the bar which is fixed at one end and free to go at the other end. So, what do I have then my boundary conditions are the following and I am going to take x from here. So, my boundary conditions are the following, u at 0 t is equal to 0 and look over here, the axial force is equal to 0 I mean, this strain is equal to 0, because there can be no strain at the free end, so the other one is u dot L of t is equal to 0.

So, this becomes the specific problem and what you have is these are the boundary condition, forget about the initial conditions for now. We are not interested in the initial conditions and what happens then is that, you have the situation, let us see and I am going to say that, this is m bar constant, E A constant. So, it is a uniform beam with uniform mass distribution and of course, you know E A is a constant, because of it.

So then this equation becomes the following, m bar del u by del x squared minus E A del squared u by del x squared is equal to 0. So now, I have a situation that, this becomes my problem to solve, so this is for axial deformations, now how we solve these will come to later, right now I am just doing equations of motion. So, boundary condition of course, I forgot initial conditions, initial conditions is u x 0 and u dot x equal to 0. And of course, what we can do is, if we have any load on that, we can say that, look this is equal to some P x of t, some loading, that is neither here nor there. So, that becomes your axial motion kind of a system, now let us look at the case of flexural, I have looked at axial, now I will look at flexural.

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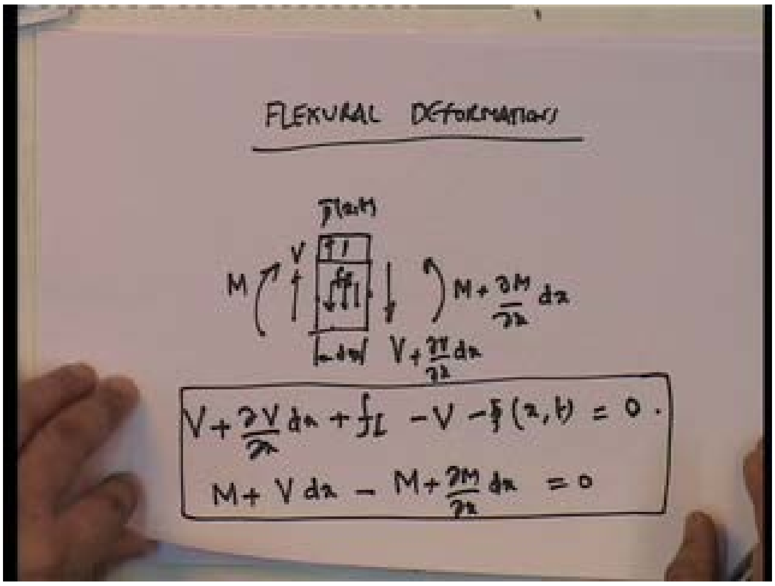

So, we will look at flexural deformation and if I take any d x what do I have, I have M M plus del M by del x d x then I have V then I have V plus del V by del x d x. Then I have load P bar x of t and on top of that, since this is moving up in this direction, I also have f I. So, if I were to look at the two equations, what do I get, I get two equations, there is a force equation in this direction. So, if you look at the force equation, what I get is, V plus del V by del x d x plus f I minus V minus P bar x t is equal to 0, that is the this. And then if I take moments what do I get, suppose I take moments about this point, no let us take moment about this point. So, what we get is, M plus V d x minus M plus del M del x d x is equal to 0, these are the two equations that I get. Let us look at the bottom equation first, it is a easier equation to look at.

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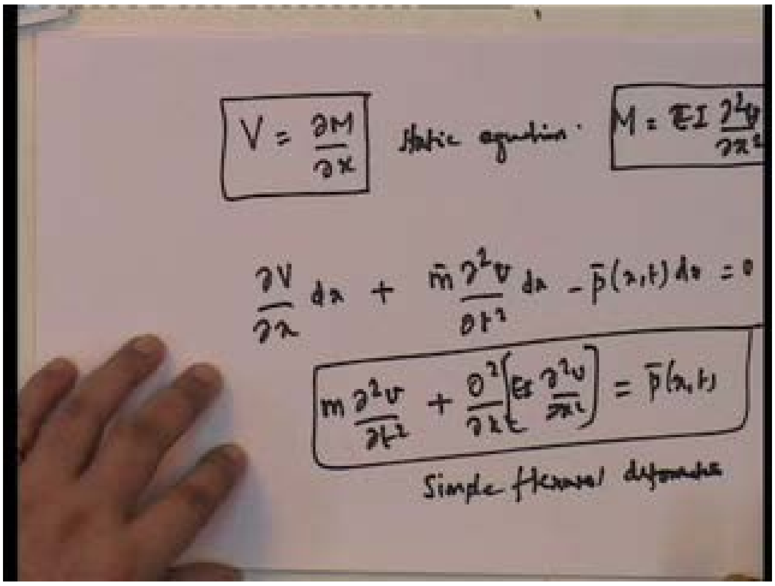

And what it says is, if you look at it that, the bottom equation, the second equation says, V is equal to d M by d x, we give static equation. Now, we also know that, M is equal to E I into del squared u by del x squared, we know this. Now, the top equation gives us what, it gives us that, del V by del x into d x plus f I and what is f I equal to, f I is equal to M bar into del squared V by del t squared, because V is in this direction, u is axial deformation, this is lateral deformation.

So, it is M into d x then minus P bar x of t d x, this is also P bar, because as the total load, is equal to 0, so d x exist in all of them. And so I can eliminate that and ultimately what I get is equal to M into del squared V by del t squared plus, now you look del V by del x that means, this is equal to del squared x of M. So, this becomes del squared x of E I del squared V by del x squared equal to P bar x of t, this is the simple flexural deformations.

Now, note something very interesting and then I did not mention that and that is that, since you have in t squared, a double differential, there has to be two initial conditions.

Since you have a double you know differential in space x squared, you have to have two boundary conditions that is, De Rigueur.

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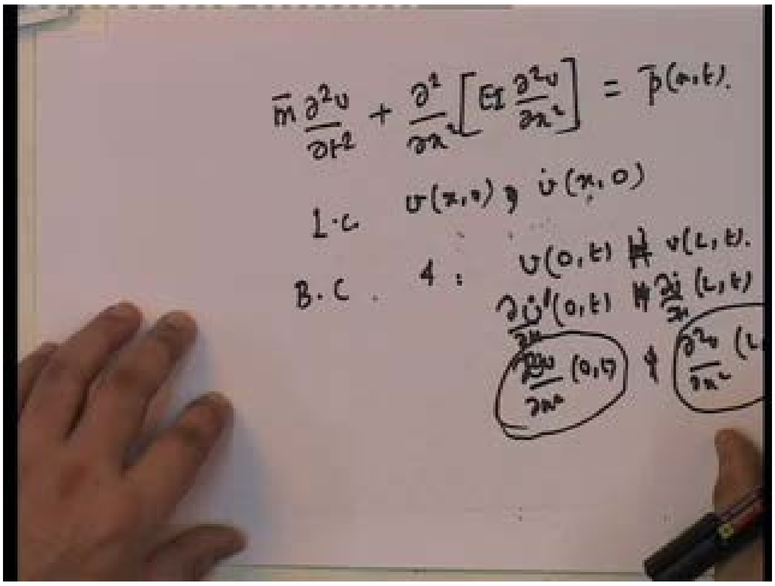

So, here this particular problem, if I were to rewrite it, I will put it down properly it becomes the following. It becomes m bar del squared v by del t squared plus del squared x E I del squared v by del x squared is equal to P bar x of t, initial conditions, v x at 0 and v dot x at 0 have to be specified and boundary conditions. Now, note that, there is a fourth order, fourth order means, we need four boundary conditions to be able to solve it.

And the four boundary conditions that you typically have are given in terms of v 0 t or v L of t and the other boundary condition becomes v dot, which is the slope at 0 of t or v dot L of t. You know it is all I mean, it could be any one of them, you have to have all told there are four, so it is this, this then you have finally, this is slope. So, it is del v by del x del v by del x and del squared v by del x squared at 0 of t and del squared v by del x squared at L over t.

Now, note that, these are the six, what are these corresponding to, these actually correspond to the moment boundary conditions. They are the moment boundary conditions, because similar to the one that you had over here, you had what is the boundary conditions you have, the boundary condition was this. Why is this, because this is actually, this not strain, it is really the load and there is no load here, so the strain is equal to 0.

So, similarly this is really moment, because curvatures related to E I into moment, so if moment is known then you know moment and out of these six, any four will have to be known. For example, let me take a situation, for what kind of boundary conditions do I get, if I have a boundary condition for this particular case, remember I showed you that, if you have this then boundary conditions given in this way. Suppose, this was what is the boundary condition in this particular case be, the boundary conditions would be u 0 of t equal to 0 u L of t is equal to 0, that is how the boundary condition would be. So, similarly let us look at some typical kind of problems that we have and you know, we can actually look at this specific form of the boundary conditions.

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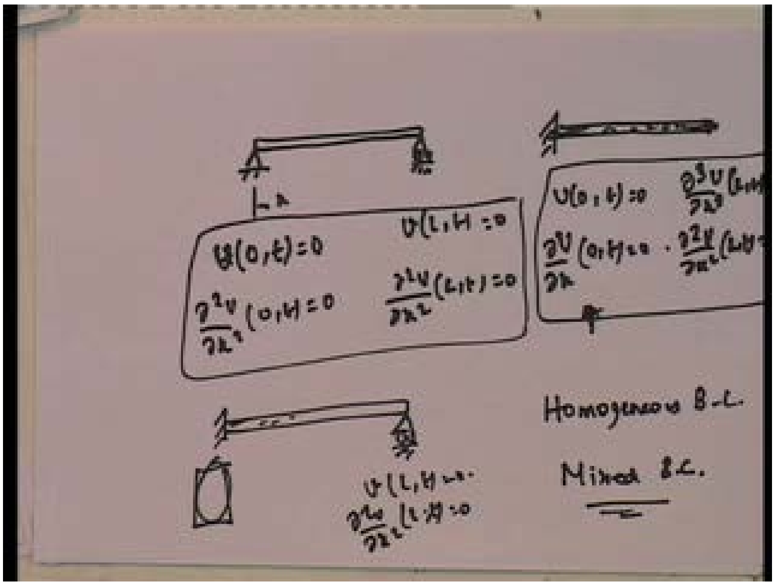

For example, let us take simply supported, what kind of boundary condition do I have in this particular case, let take x from here. If I look at this, what I have is boundary condition here is that, v of 0 t is equal to 0, I mean slope I have nothing to do about, but I know moment is equal to 0. So, del squared v by del x squared 0 of t is equal to 0, here similarly v of L t equal to 0 del squared v upon del x squared L t is equal to 0, because moment and deflection are 0 in this particular case.

Suppose, I have this, what do I get over here, so these are the four boundary conditions for this, here what are the four boundary conditions. Let us see, v 0 t is equal to 0, del v by del x no slope at 0 t equal to 0, here I know nothing about a displacement, displacement can occur, I know nothing about slope, slope can occur, but there can be no shear. So, d cubed v by d x cubed at L upon t is equal to 0, similarly moment is equal to 0, so d squared L upon t equal to 0.

So, here you have kinematic boundary conditions, you have force boundary conditions, here you have both kinematic boundary conditions and forced boundary conditions at both ends. So, this in essence, you always have to find out, what your boundary conditions are in a particular case. So, you will always have to have for example, let us say, suppose you have something like this, you have this boundary condition, what are this boundary condition be.

Now, here this over here will be the same as this, this one is exactly the same as this one and what will be it here over here. So, at every place, at every point, you will always have two boundary conditions, so here you have two and here what are the two, you cannot say anything about shear, but you can definitely say the displacement. So, v at L of t is equal to 0 and what else, you know the moment is 0, so you know del squared v by del x squared at L equal to t is equal to 0.

So, this is the way of course, these are known as, what are known as homogenous boundary conditions. We might have mixed boundary conditions too, I will talk about those later, but the point then becomes is that, you have a situation, where whether you have axial deformations or simple flexural, right now we are considering simple flexural. You can always write down the equation for the axial deformation, it is given by m bar del squared u by del t squared minus del squared by del x squared del by del x of E A del u by del x equal to P bar x of t, where P bar is the axial loading, that is for axial vibrations.

For the simple beam case, the equation turns out to be equal to m bar del squared u by del t squared plus del squared upon del x squared E I del squared v by del x squared is equal to P bar x by t. And since you have every time, you have a double differential in time, so you require two initial conditions. So, that is u at x equal to 0 and u dot at x equal to 0, these are the initial conditions. And typically, you know we always start off with saying that, initial conditions are 0 at stress, so they are 0 at all times, so that is the initial conditions.

For axial deformations, since you have del by del x E A del u by del x, you have double differential, so you require two boundary conditions. So, the two boundary conditions, if they are homogenous boundary conditions, one will be at 0 t and one will be at L over t, you can always find out these. So, you always find out boundary conditions, given the particular boundary that you have and for flexure, since it is a fourth order equation, you require four boundary conditions.

And I have given you some typical boundary conditions, homogenous boundary conditions, that you are likely to have in this particular case. So now, the question becomes, how to we include, this was simple case, where there was no rotatory inertia, there was no shear deformation anything. So, in this particular case, how do we solve the problem, so in this particular case, let us look at this, let us come back. This kind of a beam where you consider shear deformation and rotary inertia is known as Timoshenko beam.

So, one we have simple beam, which you have looked at, where inertia is only linear and we neglect shear deformations, we are only considering flexural deformations. And then you have the Timoshenko beam, where I will come to the Timoshenko beam a little bit later, right now let us not complicate the issue. So now, let us look at, so these are the equations of motion, I have got a simple flexure and a simple axial, one bar boundary conditions we know, we know how to solve the problem.

So now, how do we do the dynamic analysis, let us look back, for dynamic analysis, before dynamic analysis, even for single degree of freedom, dynamic analysis dependent on the free vibration characteristics. Multi degree of freedom again we saw that, the entire fuse board super position, the entire solution process requires free vibration analysis. And here, what we are going to do is, we are going to look at a free vibration analysis.

So, I have looked at equations of motion, simple beam, in both cases I have looked at simple beam, I have looked at axial deformation and I have looked at flexural deformation. The Timoshenko beam which is more complex, where you consider shear deformation, you consider rotatory inertia, that the most complex kind of beams solution that you can get, we will look at that later.

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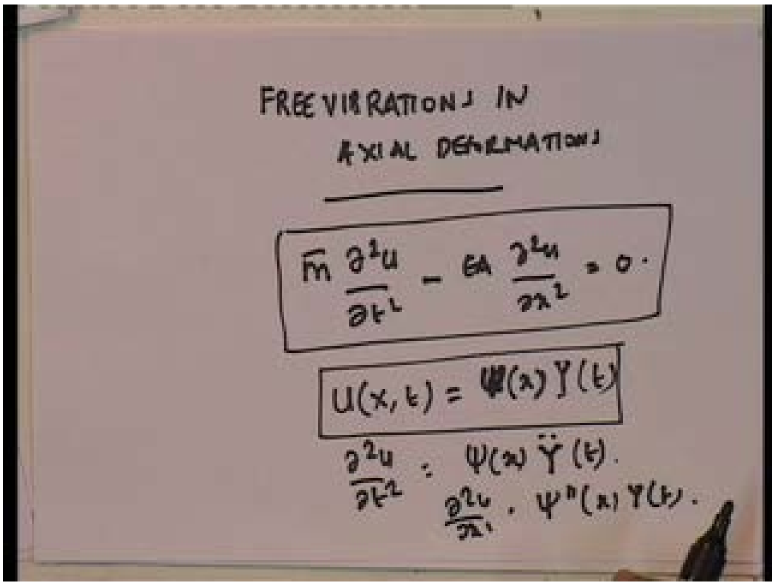

So, right now let us look at the kind of situation, where we have free vibration, we look at free vibration. And to begin with, I am going to start off with free vibration in axial deformation, a start off with that. So, let us see what the free vibration, when you have free vibration, this is the equation that you have and I am considering that, E A is a constant, let us look at this particular form. So now, to solve this, in a partial deferential form, we have to make the assumption that, u x of t is equal to some, let me call this psi x, now this psi x is not known, psi x into Y of t.

So, what I am assuming is that, this now note that, this is completely different from what we have done in the generalized single degree of freedom. In the generalized single degree of freedom, we had started with the assumption that we knew this, here this is not known, all we are saying is that, we start off with the, is saying that, assume that the response is separable. If the response is separable then what do I have, I have this kind of a situation. If I want to look at it, del u by del t squared becomes what, psi x into because this is the only one and del squared u by del x squared is psi double prime Y x of t. So, if I were to put this equation into this, what do I get, I get the following, let us put that down.

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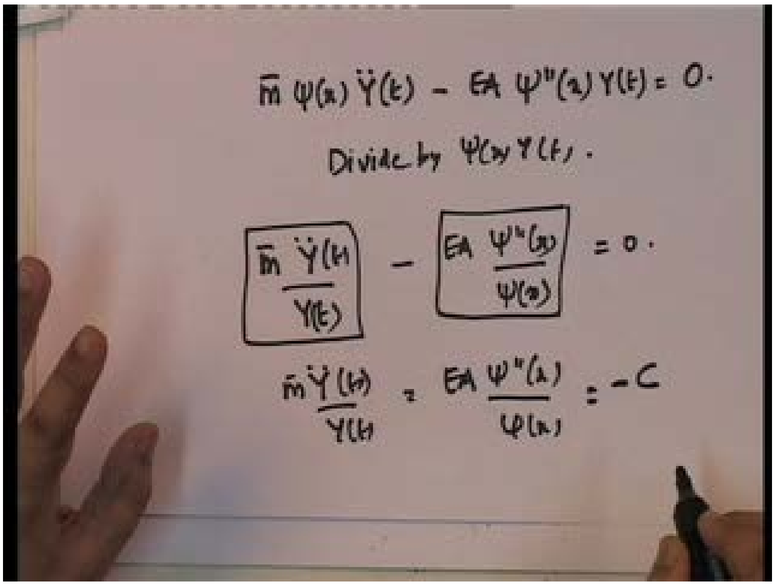

So, I will put this in I get m bar psi x Y double dot of t minus E A and del squared x, so this becomes psi x Y of t is equal to 0. Now, let me put this through by divide throughout by psi x and psi x Y t. So, if I divide by psi x Y t, if I divide that what I get is the following, m bar Y double dot upon Y t minus E A psi I x upon psi x is equal to 0, this is what we get. Now, note then something very interesting, this is only a function of t, this is only a function of x and the only way that you can have a function of t minus of function of x equal to 0 is that, if both of them were equal to a constant.

So, in other words, if I were to put down m bar, this could only happen if you had this equal to E A psi I upon psi x, is equal to some constant C. And for now, I will call that as minus L, explain why I am putting that is minus c, some constant. If we put that then you see this becomes equal to 0 and because this is a function of t, this is a function of x. There is no way, that they could cancel out at all instants of x and t to be equal to 0. The only way is that, they were a constant and the ((Refer Time: 38:02)) I put minus constant will become obvious now, what happens is you get this kind of a situation.

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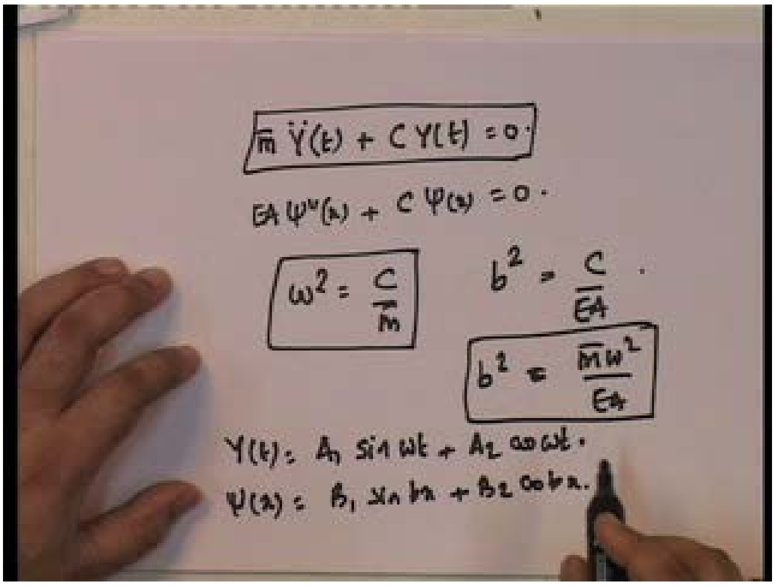

You get the following, you get m bar Y double dot plus C Y bar is equal to 0 and the other equation becomes E A psi prime plus C into psi x equal to 0. So now, if you look at this, what does it give me, it says that look, omega squared is equal to C upon m. And I will say fundamentally, I will call it another term b squared, because this again the same kind of a problem as this, excepting this as a function of x. So, I will say that, b squared another term is equal to C upon E A.

So, then C is equal to m omega squared, so if I put m omega squared here, what I get is b squared is equal to m omega squared upon E A, m bar of course, these all m bar. So, this is what you get, so if you get here, you get Y of t to be A 1 sine omega bar t plus A 2 cosine omega bar t and what A 1 and A 2. you get from the initial conditions. And you get psi x is equal to B 1 sine b x plus B 2 cosine b x and for these, you incorporate the initial conditions. So, if you have a situation, where you have, so now let us look at the particular problem that I was looking at, if you look at that particular problem, you get the following.

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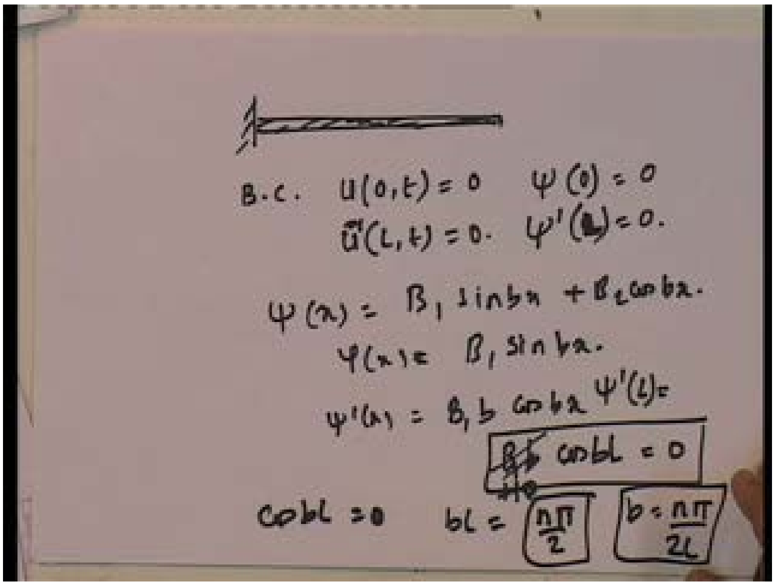

You get a situation, where the boundary conditions tell you that, this is going to be what are the boundary conditions, that u of 0 t is equal to 0 and the other one is that, u dot L of t is equal to 0. So now, let us look at this, now t is not a function, so I can just say, I can separate it out and say, look psi of 0 is equal to 0 and psi, this is prime not dot, psi prime at 0, this is the thing is equal to 0. So, from that, I can find out my, this is B 1 sine b x plus B 2 cosine b x.

Now, if psi is equal to 0, put psi equal to L equal to 0 prime, so psi x put equal to 0, this one turns out to be 0, so B 1 into 0 is 0, plus B 2 into 1 is equal to 0. So, from that, we found out that, B 2 is equal to 0 so that means, psi x is of the form B 1 sine b x. Now, we differentiate this and what do we get, we differentiate this we get B 1 into b cosine b x is psi prime and I am going to put... So, psi prime L is equal to B 1 b cosine b of L is equal to 0.

Now note, that these are not 0, so for this to be satisfied, you have to have a situation, whether cosine b L is equal to 0. And when is cosine b L equal to 0, cosine b L is equal to 0 when b L is equal to n pi over 2 that means, b is equal to n pi upon 2 L. Now, we have already done that, b squared is equal to m bar, so now let us plug that in, because we found out the value of E.

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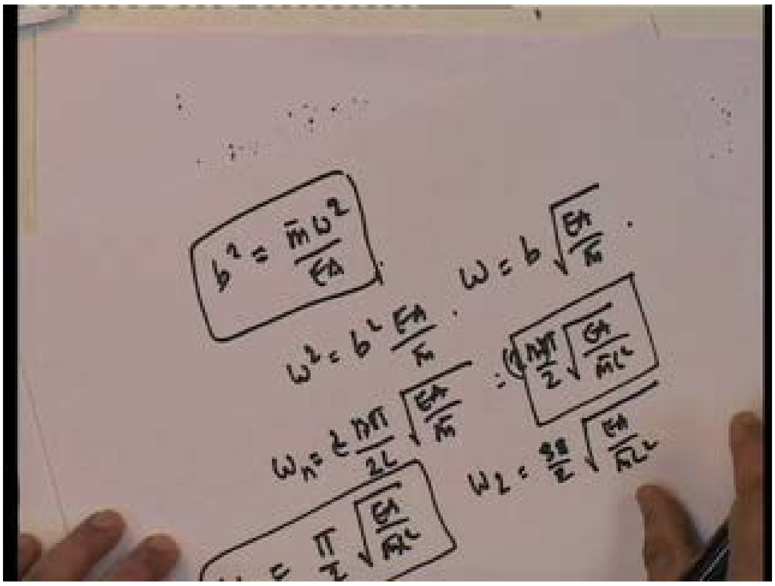

So, let us kind of try to see, what we get, we get see b squared, we know that b squared is equal to m bar omega squared upon E A. And if you look at this, implies that omega squared is equal to b squared E A upon m that means, omega is equal to b square root of E A by m and b, I have seen is equal to n pi by 2 L. So, omega is equal to, so this is equal to where am I, b squared, so I will put this in. So, this will going to be equal to omega n is equal to n pi upon 2 L into square root of E A upon m bar.

Now, this E A is units, so I am going to put L inside, so this becomes n pi by 2 into E A m bar L squared. So, omega 1 is equal to pi by 2 square root of E A m bar L squared, that is omega 1, omega 2 is equal to 3 pi by 2, this is 2 n minus 1, because you know 2 becomes pi, pi is not this, so it is 2 n pi, so this is 2 n minus 1. So, this is 3 pi E A square m bar and let us look at the interesting part and that is, so what is psi x equal to, see we are not interested in initial conditions, so we will not have to consider.

So, if you look at it, psi x becomes equal to B 1 sine b x, so this one is nothing but, sine 2 n minus 1 pi over 2 L x, so that one I can say is equal to sine 2, it is psi n is equal to2 n pi over 2 x over L. So therefore, if I have omega 1 and that is equal to pi by 2 E A m squared, the corresponding mode shape and note that, this is mode shape psi 1 x, the corresponding mode shape is equal to sine pi by 2 x by L. So, let us see, how that looks, how does that look, of course note that, I am going to draw u, which should be drawn in this direction.

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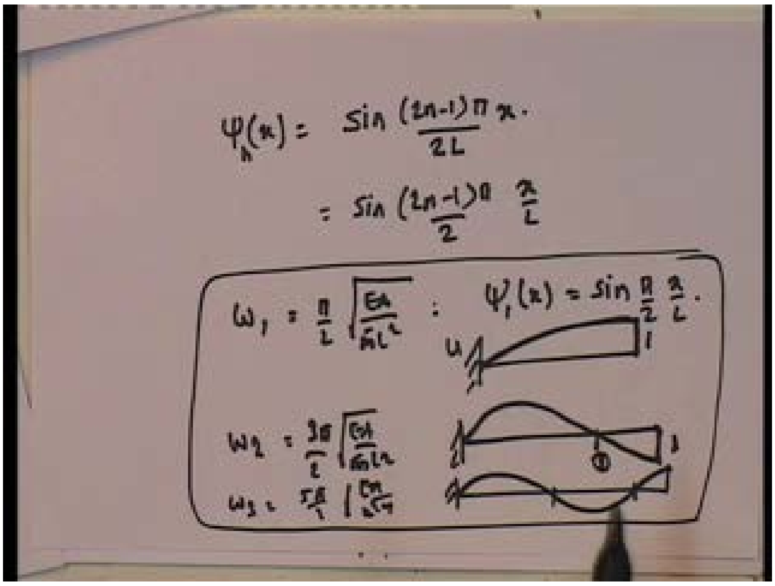

I am going to plot it, sine pi by x at x equal to 0 is 0 and at x equal to L, it is sine pi by 2, so it is 1, so this is the displacement, so it is in this direction, I am just plotting u, so this is equal to 1. So, that in essence, is my mode shape, you see I have got my frequencies and mode shapes. So, omega 2 is equal to 3 pi by 2 E A upon m bar L squared and the corresponding mode shape is pi by 3. So, there is therefore, it going to be in this, so it is going to be this way, 1 because 3 pi by 2, it is minus 1, so that is how, I am going.

The point is, note the initial mode is a simple mode, it is a simple mode and the second mode has one node. If you plot this third one, 5 pi by 2 E A upon m bar L cube, you will see that will be 5 by pi, so it will go here. And so there are two nodes, this is a factor that you have to notice that, if you are looking at a particular situation, the first mode, no node, node is where there are no displacements, second mode 1 node, third mode 2 nodes, fourth mode 3 nodes, this is how it goes.

And this is the free vibration analysis and we have got the mode shape, only thing is that we have got the exact mode shape, because we saw, we have solved the partial differential equations in this particular case. So now, this particular this is for the axial vibration, similarly I will just write down the equation that I have and that is going to be equal to, let us look at the simple flexural vibration.

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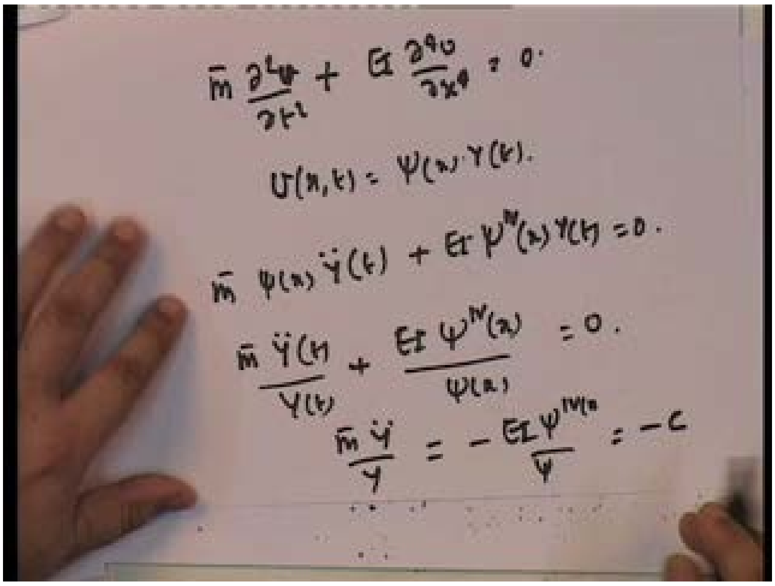

Simple flexural vibration is what, it is given in this format, it is given as m del v by del t squared plus E I del fourth by del x fourth is equal to 0. Now, let us take the situation, where I have the following, I have again my v x of t I take as psi x Y of t. So, if I take it this as then what do I get here, I get m bar del squared becomes just Y of t. So, it becomes psi x Y double dot of t plus E I and here, I have del fourth, so I have the fourth x Y t equals to 0.

Again I am going to divide throughout by this thing and so what I get is, m Y double dot upon Y plus E I fourth upon psi x is equal to 0. So, what we get is the only way, again you see, the only way that a function, which is only a function of t and a function of x can be is if they are and so if I write this in this fashion, minus E I psi fourth upon psi is equal to minus C.

So then what I get is, I get the following equations, I get the equation to be m Y double dot plus C Y is equal to 0, that gives me the fact that, Y of t is of the form A C upon m. So, that is like omega squared, you know omega t plus A 2 cosine omega t, these you can find out from initial conditions. The other one is a more interesting equation, it says E I fourth, so that is actually d fourth by d x squared, so that is the one, this minus this, so this becomes minus C phi is equal to 0.

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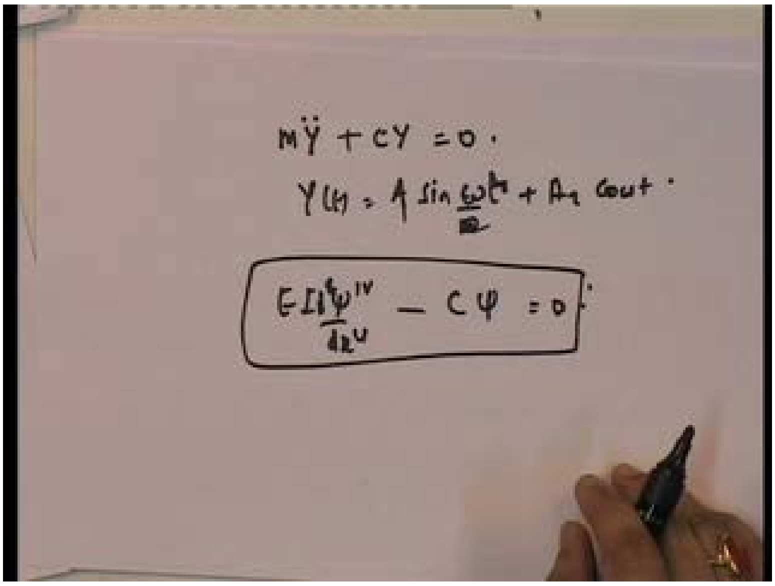

And this one we are going to look at a little bit later, so how to solve this one we will see, this becomes a much more interesting problem, we will see it next time. So, the point then I am trying to make is, today what we have done is, we have looked at overall equation of motion, two equations simple ones, axial deformation and simple flexural. Then what we have looked at is, free vibration and we have seen, that for axial deformations, you required two boundary conditions and for flexural deformation, you required four boundary conditions.

And then when we solved the free vibration equation, the axial I showed you, that you get lovely omega and the corresponding phi and the first frequency has the mode shape, has no nodes, etcetera. And then we went through the process of the flexural deformation and we saw that, you get a very elegant formulation. We will solve this formulation in the next class, so right now I am going to stop, thank you very much, I will talk to you later.

Thank you, bye.