Structural Dynamics

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Lecture - 32
Earthquake Response of Multi Degree of Freedom Structures

Hello there, in the last lecture we started talking about Earthquake Response of Multi Degree of Freedom systems, and I introduced you to two different ways of solving the problem.

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And let me just rephrase, that we are again looking at earthquake response of multi degree of freedom structures, please note that I it is used be to add over here that there is a lot more to earthquake response analysis. Then what I am doing, I am just taking an overview of just 2 lectures to give you an overview of earthquake response of multi degree of freedom systems. Please note that, there is actually a completely separate course available in this particular format, which is known as introduction to earthquake engineering.

And that is the course, in which a you will be given a much more detailed background behind earth quakes, why earth quakes, what are the characteristics of earth quakes, how do we take those, how do we analyze the response to those are things to be done in earthquake engineering. I am just using earthquake response as really an example and it is a and it is a very important example.

Because, the entire idea of structural dynamics of multi degree of freedom systems, actually came out in structural engineering, actually came out of the need to analyze the earthquake response and wind response of multi degree of freedom systems, buildings that is where entire thing came about. So, I am just visualizing that as an example or to introduce a several concepts; and let us now go back and look at something that I discussed last time. But, I am going to a you know put it down in a more formal way, the formal way is this that you are given that.

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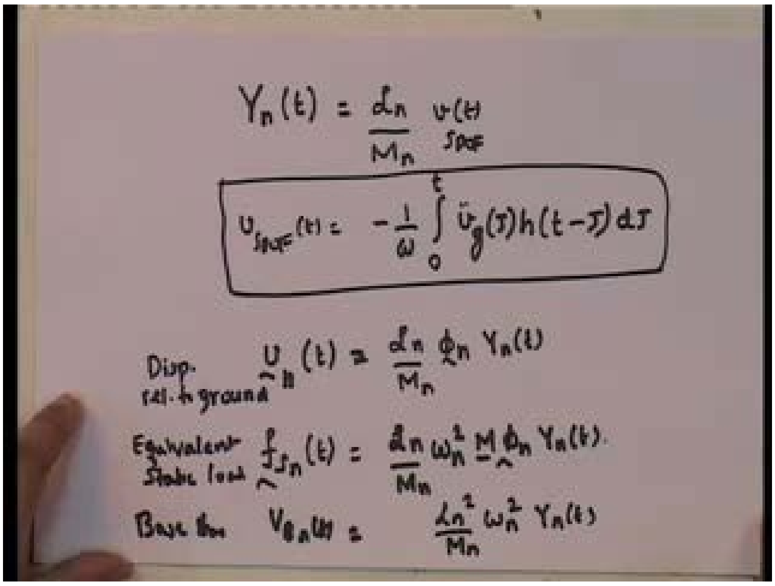

If I say, I know that my Y n of t is equal to L n upon M n into I will call that now, as the single degree of freedom. So, this is the v single degree of freedom t is nothing, but one upon omega 0 to t v g tau h t minus tau d tau, so this is we know that this is the response of a single degree of freedom, so no I can call that you know in a way, so that is my y n of t and my v n of t is equal to L n upon m n into phi n into y n of t we also found out that the f s n of t that is the, so this is the displacements relative to ground, please understand that these are the displacements, these are the equivalent static loads.

These are the equivalent static forces that give the displacement relative to the ground that we have over here and this we saw was equal to L n upon m n omega squared M m n into phi n y n of t and the specific form of base shear in the n th mode is equal to L n squared m n squared into omega n squared into y n of t. So, these are the things that we have if fault and therefore, if I look at what I call as the response history method.

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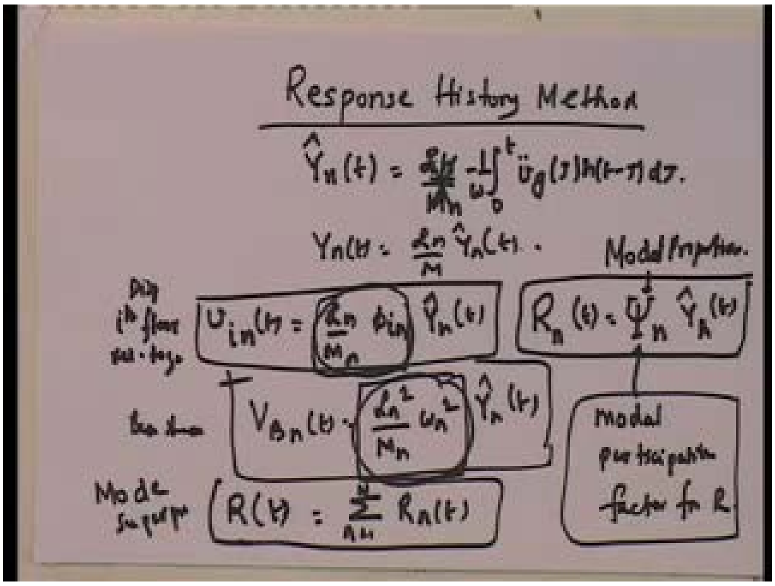

So, this is called as the response history method the response history method is based on the following, y n of t is equal to L n upon M n let me define particular term let me define a y hat n a y hat n is the that single degree of freedom problem that I have which is minus one upon omega. So, that I am just kind of a, so what I have is Y n of t is equal to L n into y hat of t.

So, then I can say that look just like I have this situation where if I take let us say v i n I want to find out the displacement of the i th flour displacement relative to ground what would that be equal to v n of i t it will be equal to L n upon m n into phi i n into y n hat into t because, it is just nothing, but phi i into y n of t, so this is what I get. Suppose I want to find out these shear that is equal to V b n of t is equal to L n squared upon m n omega n into Y n hat of t.

The point here is that we see that there is a quantity given any response quantity given any response quantity what we get is r n of t is equal to I will call that the participation factor, into that single degree of freedom response. Where this I will call it as the modal participation factor for r note that the modal participation factor, for R note that participation factor depends on which response you are looking at for example, if you are looking at the i th floor displacement my this is my psi n, if I were looking at the base shear this is my psi n for any response quantity.

You can actually find out a scalar which will have you know L n m n well all kinds of things you know depending on it would definitely have L n upon m n, but it would have a other tones also associated a with it would have definitely L n upon m n because, y n is your L n upon m n into y hat. So, therefore, L n upon m n is always there, but there is other terms for example, if it is i th floor displacement the other term is phi i n if it is base shear the other term is L n into omega n squared.

So, that is all I mean you we can always find out a participation factor and if you find out the a this thing participation factor, then the total response time history is actually by classical mode superposition this is nothing, but mode superposition. So, this in a sense is your response history analysis what do you do well given a acceleration time history base acceleration time history you first find out the Y n hat which is given in this form you somehow find out y n hat which is essentially the single degree of freedom displacement.

Then you want to find out any other quantity well a it is always a participation for any response quantity find out the participation factor and this participation factor is independent of the ground motion, it is just a modal participation factor which depends on a you know L n L n is nothing, but a phi n transpose m into one m n which is phi n transpose m phi n it is or phi i n or it is omega n. In other words it is parameters that are given once you know a particular mode and they are different for different modes.

So, you psi n that you have over here which is the modal participation factor, will be different for different modes and it will be different for different response quantities, but any way it is a quantity that depends only on modal properties, this depends only on modal properties. So, this you can always find out given and this thing and you multiply that with the single degree of freedom and what you have is the modal response and, then you use mode superposition to find out the response time history a of that response this in a sense is the background of the response history method.

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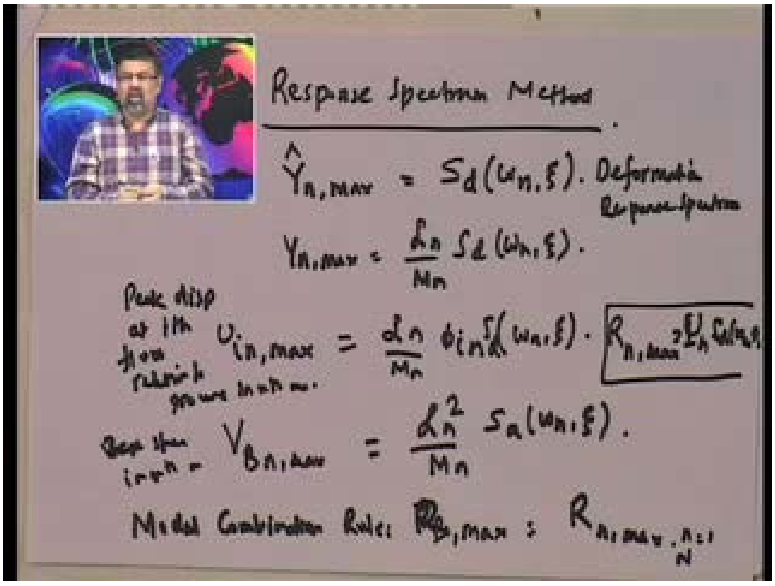

Now, let us look at the other method that we have looked at, but I want to you know put down on paper and that is the response spectrum method, in the response spectrum method what you have is actually these parameters. Let us see suppose you want to find out Y n hat max that is the single degree of freedom Y n hat max is nothing, but s d omega n n psi, so these depend on the modal parameters and, so and the deformation the response spectrum.

So, Y n max is this, so what is Y n max equal to L n upon m n into Y n hat max, which is s d omega n psi if I were to look at v i n max that is the peak displacement at i th floor relative to ground please remember that, but this is always relative to ground. So, this is going to be equal to L n upon m n into phi i n which is the i th term of the n th mode into s d omega n psi n and you know if I look at the base shear in n th mode, so this is base shear in n th mode is equal to L n squared upon m n into s a omega n psi and here now, you cannot use mode superposition.

So, you have to use what are known as modal combination to go from to get v b peak value of R, so therefore, even here I can write that R n max is equal to actually psi n into s d omega n I could do that, so that psi n parameter still remains. So, R b max will be given in terms of R n max all R n max n going from one to n using modal combination rule, so this in a sense is the response spectrum method where you are only given the response spectra displacement spectra a you know acceleration spectra. Typically, you are only given acceleration spectrum well you can always find out displacement spectrum from the pseudo acceleration spectrum and the modal combination rules.

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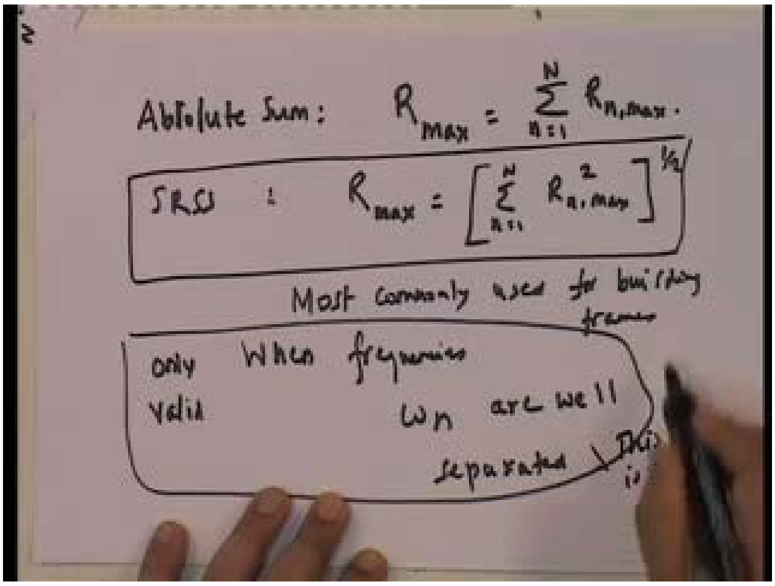

Just again to a restate that particular thing modal combination rules there are the absolute sum which is R max is equal to summation n going from one to n R n max and you have the square root of the sum of the squares. Where R max is equal to, these were rules that were developed and which seem to give very good results and, so they were used SRSS is the most commonly used most commonly used.

For building frames why because, well you know what they did was that they did the response history analysis obtained a you know, sorry response history analysis got R of t and found out the peak of R of t and, then saw well let us look at R n of t look at peak R n of t and they saw that well, if I use the SRSS rule the R max that I got from the this came closest to the true R max that is how they did it. Till this was used in the sixties and seventies extensively, till in the you know mid early to mid eighties when they were looking at random vibrations etcetera.

I do not want to discuss that they did some very interesting things and found out that this is only valid when frequencies omega n are well separated now, typically why you know see for frames the first few modes which is what is considered you do not co consider all the modes, you actually consider the first few modes the first few modes in frame structures the frequencies are well separated. So therefore, this was done in a eristic sense and when this checked that s r s has worked fine it was that they used it you know this is valid for building frames. So therefore, it was a full filling self-fulfilling prophesy recently they have looked at another modal combination rule, i will just layout the parameters.

The parameters it is called the CQC rule this is complete quadratic combination, and this rule says that look R max is equal to double sum m going from one to n, n going from one to m rho m n R n max into R n max. Where rho m n is a correlation factor, that is a function of omega n omega m and psi, I am not going to give you those values because, it is not relevant it is just that it suffices to say that it depends on omega n omega m and psi. Now, let us look at this suppose in this particular one I take the situation that rho m n is equal to one.

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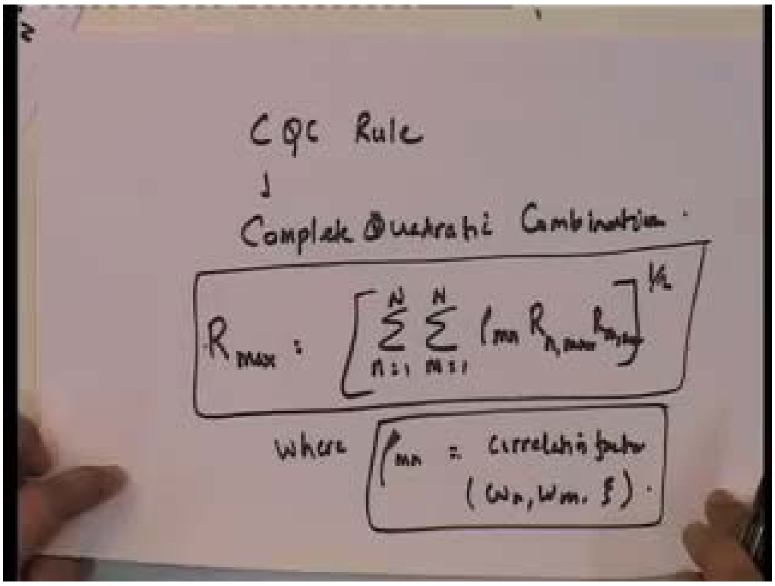

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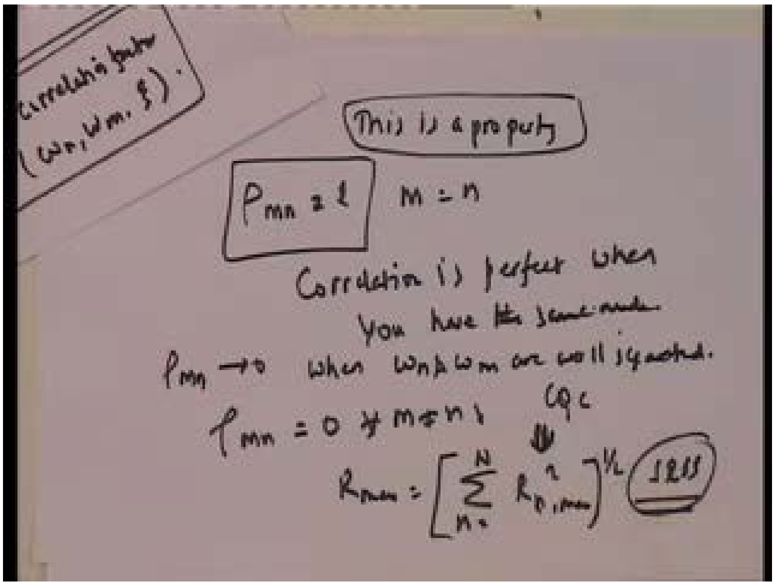

By the way correlation rho m n is equal to one is it satisfies rho m n is equal to one when m equal to n this is a property because, correlation is perfect when you have the same mode, so this is the property, suppose I say now, that rho m n is equal to zero for m not equal to for all m not equal to n. Let us see what happens let us plug it in I know that rho m n is equal to one it is known when n is equal to m, so that means, it becomes R n in m that is R n x squared and note that all m n for not m not equal to n is zero.

So, if you look at this double summation what does this become this double summation, then CQC implies that R max is equal to note that the double summation only the term m equal to n lasts. So, this then becomes a single summation n equal to one R n m x squared SRSS now, this correlation factor rho m n tends to zero when omega n and omega m are well separated this is part of the function, so as it tends to zero when they are well separated that means, it becomes SRSS.

So, SRSS is actually included the square root of the s sum of the squares modal combination rule is actually embedded into the complete quadratic combination rule CQC rule. Which is why today in the world we tend to use the CQC rule why because, well if they are not well separated CQC rules gives the better estimate and if they are well separated it kind of automatically becomes an SRSS, so we do not have to completely separate it out and use it.

So, this in a sense is modal combination and modal combination rules are essential in any method that computes only modal peaks and not modal time history and the response method is based on only computing modal peaks and, so if you have modal peaks the only way that you can tackle this problem is by applying a modal combination rule. So, this you know, so therefore, coming back to it we have two methods for earthquake response analysis among multi degree of freedom system problems, we have the response history method, which I have enumerated and we have the response spectrum method, where in the response history method you can use the standard mode superposition and get away with it the response spectrum method, own to be only get modal peaks and if you get modal peaks, then the only way that you can combine is by using modal combination rule. Please note something that I have used earthquake response as an example let us say that you want to find out you have a shock a load and you have given the shock spectrum which is essentially what t n upon t d t upon t d t.

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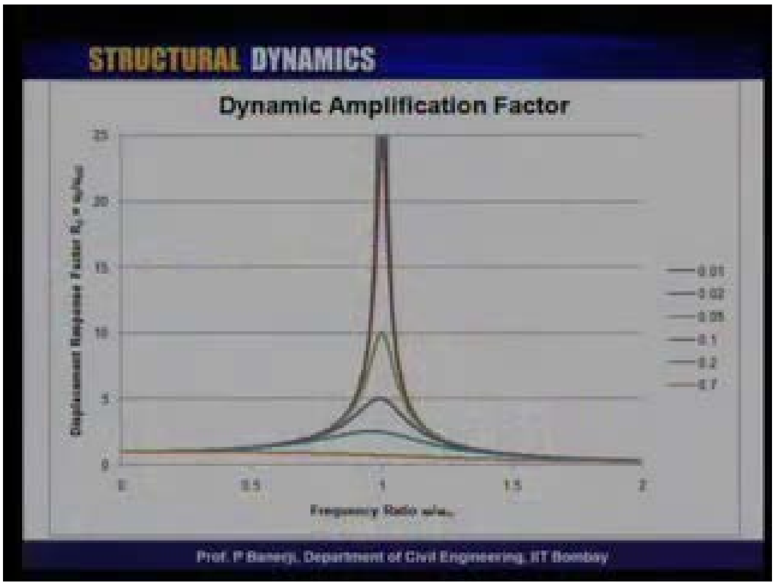

So, it is really t n upon t d now, there also what are you computing, you are computing the modal peak single degree of freedom you get peak response see we showed when we looked at it. We tried to develop time history and later on you know if you look at harmonic, simple harmonic what did we do we said we draw the mode modal amplification factor. And, then we are able to look at the variety of things that we have and therefore, the point then becomes that the entire thing if you look at my single degree of freedom problem, the entire thing just look back at it if you have a harmonic response you have the displacement response factor.

So therefore, now you look at it omega bar upon omega n, so for every frequency you find out omega bar upon omega n and find out the displacement peak displacement response factor multiply by the modal factors you got the modal peaks.

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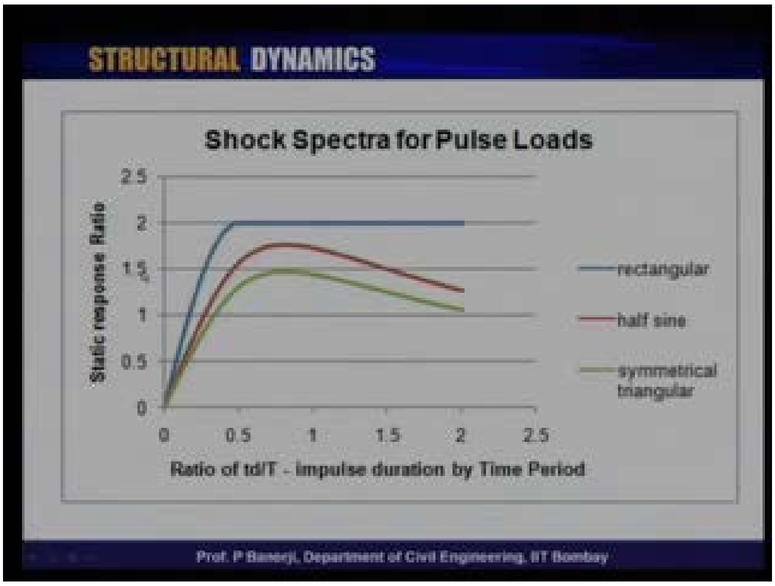

You look at the shock spectrum, what you do is it is t d upon time period of structures it is really t d upon t n, so therefore, you know you look at any specific kind of load you find out t d upon t n and you look at it read it off you have got this static response ratio you multiply that by p n not you know. So, these are all things that are based on finding peaks, whenever you find out the modal peaks the only way that you can find out the total response peak is by using modal combination rules.

So therefore, although I used earthquake response to illustrate modal combination rules, Please understand that this is equally valid when we look at ah other responses also. So, you know these are typically the kind of problems that you will end up in the next classes, next couple of classes I shall actually look at some example problems of shock spectra of harmonic loads and you know earthquake loads earthquake loads.

But to show you that modal combination is fundamental to response analysis of multi degree of freedom systems because, if single degree of freedom systems remember what we said you do not interest in the time history, we were interested really in peak responses. So now, if you are interested in the peak responses we you know and you see the single degree of freedom your multi degree of freedom is really the modal amplitude. So, essentially we get only the peak modal amplitude and once we get the peak modal amplitude every response quantity in a particular mode is connected with the modal amplitude. So, you only get the peak values, so any time you get the modal peaks any response analysis procedure that only gives you modal peaks only way that you can compute the total peak actual response peak because, ultimately mode is just you know ultimately you do mode superposition.

Well the only way that you superpose modes to get the total response the total peak response, is by using modal combination and I have looked at three modal combination without stating anything specific, the absolute sum is something that I normally do not use we always use the complete quadratic combination or the SRSS. And as I said SRSS is incorporated in CQC, so there is no need to go about it, only thing is that you know the first thing in any response analysis is you do free valuation analysis.

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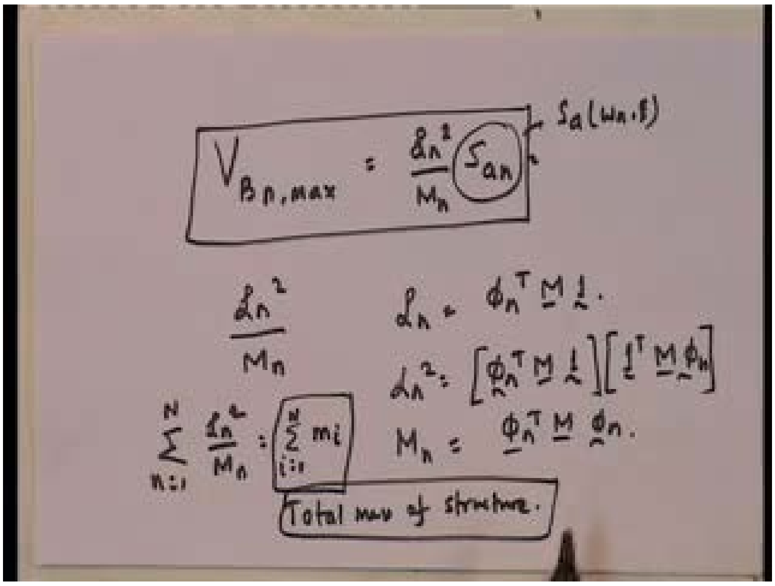

Now, if you see your frequencies are well separated or at least the frequencies that you need to consider are well separated, then you do not need to look at specifically the problem of m using CQC, we just use SRSS and then go about it. Now, I will you know remember I had talked about something last time, where I had said that may be how many modes to consider one of the things of modal superposition is how many modes do I consider and in earthquake you know you can actually find this out very easily.

Let us look at for base shear and I am going to drop the max because, any time I put in you know s a it is automatically max V b n max is equal to L n squared upon m n into and I now, call this S a n is nothing, but which is you know this term is equal to, S a of omega n given as I value, so this is your parameter. Now, this L n squared L n squared upon M n if you look at is what is L n L n is phi n transpose m into one, so if I look at it L n squared is actually phi n transpose M one into one transpose M phi n.

Now, this can be shown and what is M n is equal to, phi n transpose M phi n it can actually be shown that for frame structures, summation n going from one to n L n squared M n is equal to summation M I going from one to n th floor mass of structure. Now, total mass of structure is something that we can always evaluate given a structure.

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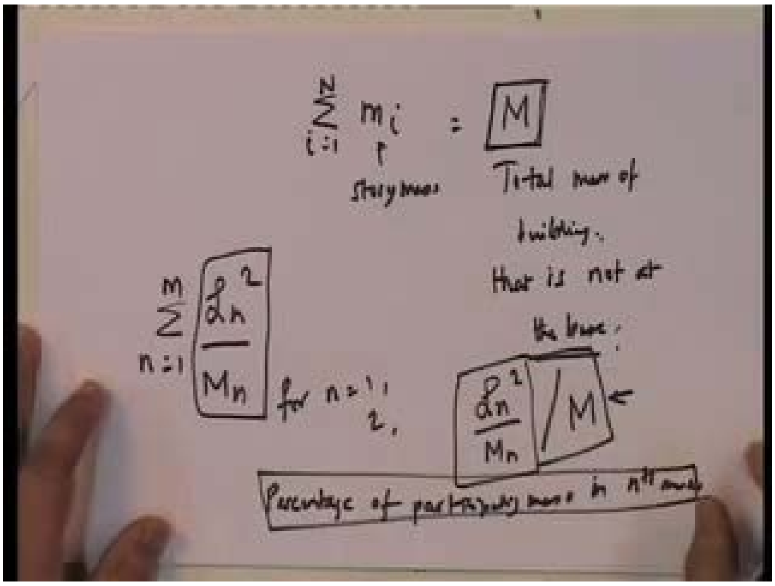

So, if I look at it I will get this kind of a situation that let me first find out I going from one to n this is the n story frame and m i is story mass, so this is equal to total mass building that is not at the base, so it is the total mass it is just any other base that will not to be included in the total mass of the building. So, you can always find out the total mass of the building then, so this is the total mass of the building, then you know this is the term that you know then what you do is you find out L n square upon m n you can find that out and keep doing it adding it adding the first mode second mode the third mode keep adding it.

So, find out this parameter for n equal to one to now, if you do L n square upon m n m n divided by M this gives you noting, but percentage of participating mass in n th mode. So now, this is the percentage because, this is like this will become like first mode it is going to be 83 percent we say that first mode contains 83 percent of this is actually L n square upon n squared upon the total mass that is a factor that into hundred of course, it is percentage.

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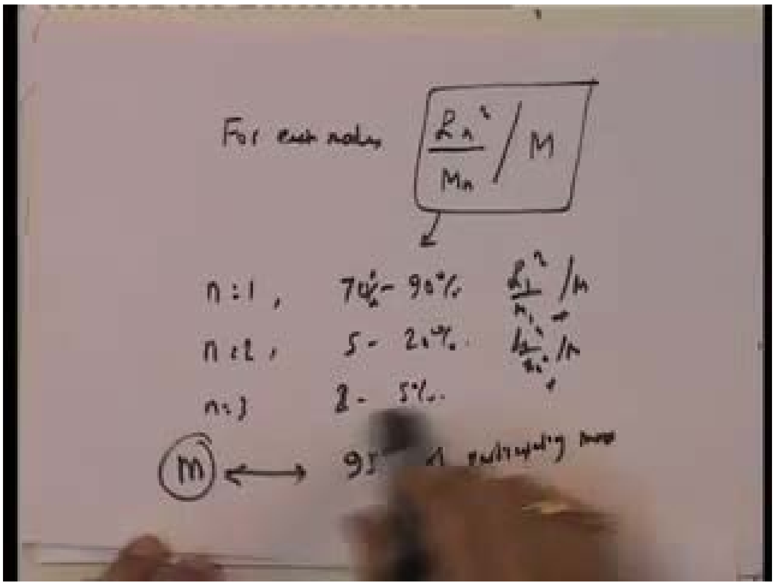

So, this way what happens is what you do is you find out for each mode you find out L n squared upon m n upon m, these are quantities that you can find out for each mode. Now typically the way this goes is that n equal to one this value is anywhere from 70 to 90 percent depending on, how the load is you know how the mass is distributed that show it is n equal to two it can be anywhere from five to twenty percent n is equal to three it keeps going down it will be anywhere from one you know two to about five percent it keeps going as you go further and further.

So, what you do is you add you keep adding them and, then what happens is as soon as you hit 95 percent of participating mass stop that many number of modes you consider. So, you see for earthquake response analysis and if it is force response you have a specific way of computing how many modes should I consider and although this is strictly valid for earthquake this is something that is used in general to say that look, but if the you know this is understand that what kind of the loading is it.

It is you know inertial loading it is equal loading for that this one is valid if the loading that you have is fairly uniform over the height of the building, then you can use this irrespective of what kind of load it is fairly uniform. You can use this to get the number of modes, if the loading increases with height this will always under estimate the response of the lower modes.

So, in other words if you use this procedure to calculate the number of modes that you are going to be considering in this a in your analysis in the modal analysis mode superposition mode a response spectrum. Whatever any mode superposition based method a if you compute m based on participating mass as long as it is uniform or increasing in special variation of the load if it increases, then this is a very valid way of computing how many modes should I consider in the analysis.

Now, the problem that happens is if there is any other kind of complicated loading pattern special loading pattern this, then becomes a problem, but understand that in structural engineering the typical kind of loading for which this was considered was essentially wind load and earthquake load environmental loads. So, if you are looking at environmental loads this is a perfectly valid way of computing how many modes should I consider by looking at participation mass, because in terms of earthquake load.

You have uniform load because, m into one remember uniform loads and you have you know in wind load you have increasing load with height and that actually require less modes, but if you consider more modes there is nothing wrong with it and still consider the fact, that since the first mode has about 70 to 80 percent. Even if you have many degrees of freedom pretty much within you know ten or fifteen modes you will have hit 95 percent or if it is a very complex structure.

As, I said remember as I started telling you yesterday either 1500 degree of freedom structure, where I had to consider 50 modes this is how I actually computed 50 modes that I needed to consider, this is almost incorporated in a lot of codes which say that these are the number of modes that you consider. So, this in a sense is gives you an overview of what you have to do for you know modal combination how many modes do you consider how what how do you combine when you have modal peaks how do you combine those.

So, in a sense although I talk I illustrated all of these using earthquake, but I did not go too much in detail into the earthquake accepting to identify that there are two methods response history and response spectrum, but, then you see response spectrum method essentially led me to modal combination rules and the modal combination rules are really valid for all kinds of a loads that you have and you know these are the kinds of things that you have.

So, this in a sense gives you an overview of response analysis for multi degree of freedom problems with use as an example the earthquake response is an example, please note that I have not earthquake response that end here you know there are many other aspects, that are to be included into earthquake analysis and as I said that is best covered this is a structural dynamics course. So, I do not give over emphasis on earthquake.

However, I use earthquake because, it is something that we have a handle on and it also introduces us to the concept of response spectrum and etcetera, but a to for detailed earthquake response. Please do not look at this course, please look at the course on introduction to earthquake engineering, so here earthquake is just an example I do not go in very further with earthquake.

So now, we have looked at multi degree of freedom problems and we actually have looked specifically at framed frames and all my although some of the developments that I did you could do it for any thing you know because, you know as long as you can calculate the mass matrix and the stiffness matrix. And the load vector for any kind of final element model, you should be able to solve it using the response history or the direct integration procedure.

Where you have to now, come back you have to define damping see and we all also discussed on how to develop the damping matrix if you are not using the mode superposition. So, that it is consistent with mode superposition and the fact that psi the damping is really a constant across all modes, so all of these are things that we have looked at, but you know by a large our entire focus has been on framed buildings because, if you remember even in the equations of motion.

I looked at beam column element and how to develop the for the element level stiffness matrix and mass matrix. And how to put it together in overall a sense now, you can actually get for any element as long as you use a fine element you can actually, get an element mass matrix element stiffness matrix element load vector and put them all together a into a structure level formulation So, in other words nothing that i have done is not valid.

However, I have kind of a stayed more and more upon frame kind of structures. So, let me end today is lecture by just introducing you to the concept of a building and let me take the specific example of a one story building to illustrate nothing more than just illustrate the concept let us take.

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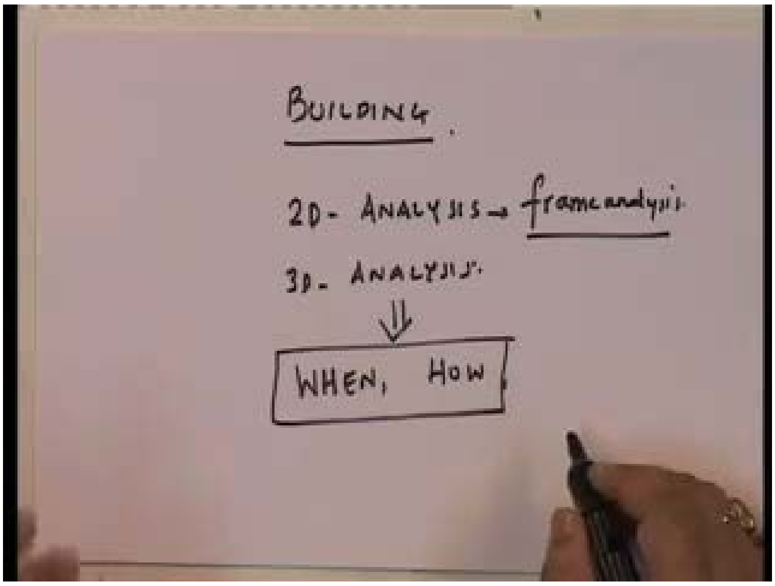

So, I am essentially looking at a building now, and you people must have seen that a building you can do a 2 d analysis or a 3d analysis this is something that all of you must have been aware of, 2d analysis is nothing, but frame analysis. Which we have already looked at extensively now, let us look at since I am you know I am I have always say that you know all that we are going to look at is going to be illustrated with building structures.

I have already done the 2d analysis for a building actually because, in 2d analysis you just take frame by frame and you take a participating mass etcetera these are standard techniques and you do a frame analysis something which i have already looked at. Now, let us look at 3d analysis and the question becomes when how, so this is how we are going to look at right.

Now, and I am just going to introduce the concept we will solve problems a later on about how to go about this, I will take some example problems and solve them, but when do we do 3d analysis when we do 3d analysis is typically a situation where you have a building and I am going to just look at the plan of the building.

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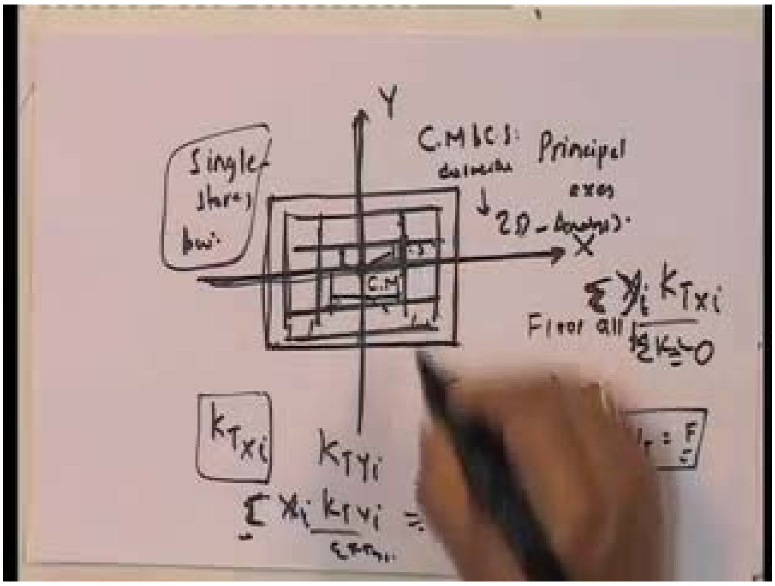

Now, the plan let us just look at I am going to look at a very regular kind of building which has it is x and y these are the principle axis of the and the principle axis are aligned very nicely. Now, typically if you do 2d analysis it is valid when you know you have at a floor, this is the floor a floor you have the center of mass, let us assume that the center of mass it is uniformly distributed this is the center of mass.

Now, you know you have a situation that you have frames aligned along this direction these are the primary frames aligned along this direction and you have frames aligned along this direction. Now, if you have a situation where if you look at frame by frame you look at see a frame always has it is own stiffness remember I said that, if I look at the cells frame I am looking at a single story frame remember I said that if you use lumped mass you can actually get a k t into v t is equal to, in this particular case you can get an equivalent translation.

So, in other words if I were to look at it as a single story building each frame in each direction has it is, so I will have k t x i and k t y i which is the translational stiffness in each direction. So now, if I look at these and I do, so the these are the k t x i s these are the aligned therefore, aligned therefore direction and suppose i find out x i that sorry y i that is the distance from this line the x line y i into k t x i and i sum that over all the frames all frames.

So, in other words this would be positive y i this would be negative y I can find that all frames if that is equal to zero, then I know that this is for the y direction frames this is where the center of stiffness lies because, defines center of stiffness similarly if I do x i k t I and sum it over all the frames in the y direction, then and if this comes out to be zero I know that this is along this. So, if it is along this and it is along this the center of stiffness is so, center of mass and center of stiffness coincide if they coincide two d analysis if they do not coincide three d analysis.

So, if they coincide two d analysis is perfectly valid so, this is very easy, so you can find this out I am talking about single story building now, and you can find this these x i, so, x i s are positive in this direction negative in this direction. So, you find out and if they are up to zero, then the center of stiffness is where it is and actually if they are not zero you can divide them by summation k t x i and will give you the position along the y axis. Where the center of stiffness is and by this also you by dividing by summation k t y i you can find out the position of the center and you actually can find out the center of stiffness, if the center of stiffness does not coincide it is the center of mass. You have to do a 3d analysis why because, I will just describe this now, I will come back to this in the next lecture because, the lateral displacements and the torsional displacement and all those rotation of the slab these are coupled with each other. So, there is lateral torsional coupling if there is lateral torsional coupling you cannot do a 2d analysis will given conservative forces you have to do a three d analysis. So, I will come back to this in the next lecture and we will discuss this in a little bit more detail.

Thank you very much, bye.