Structural Dynamics

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Lecture - 31
Earthquake Response for Multi Degree of Freedom Structures

Hello there, we been looking at response analysis of Multi Degree of Freedom Systems. And you know, I have basically looked at the overall concept and we also looked at one example, to highlight some of the issues related to analysis. I am not doing any more you know dynamic analysis, because as I said since it you use a mode superposition method. And since we are only talking about linear systems, if you use the mode superposition method.

Essentially, it boils down to a single degree of freedom system problem. And if you have a single system single degree of freedom system problem, we have solved enough about how to you know analyze it for different kinds of loads. However, today I am going to look again at a specific kind of load basically, because the form of the equations of motion and other associated issues, become highlighted. And that is we are going to look at earth quake response of multi degree of freedom structures.

So, that in a sense is what we are going to be looking at today. So, now if we before I start looking at multi degree of freedom system problems, since I looked at only earth quake response of single degree of freedom problems, what I will do is, I will start off with looking at the generalized a single degree of freedom. And define how the earth quake response is obtained for a generalized single degree of freedom, because again it is single degree of freedom a problem.

So, let us look at again you know, just to derive the basic let us take the cantilever, and here so this is the state of the cantilever. So, this is what it is and then I displace it, I subject it to a ground acceleration, so what happens is that any instant of time, this is the state of the structure. So, if we look at it essentially what happens is that, if you look at the purely at the, you know essentially this problem becomes, this let us look at let us go from first principles right.

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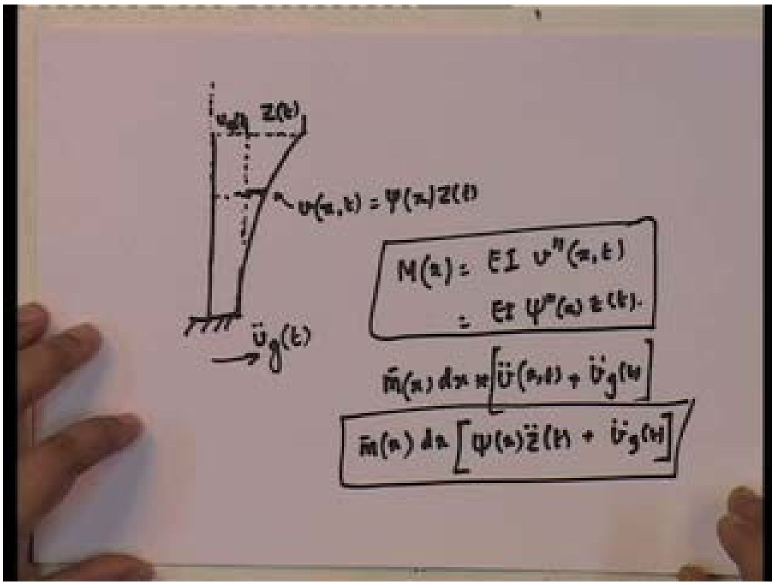

So, we look at the forces that are in the system and so if we look at the forces in the system, one is the a you know due to this deformation and this deformation as we know is given by the following, where from here to here is given by z of t, the deformation is given by z of t. However, this part is v g of t, so if you look at it, so this is z of t and this v x of t which is essentially this displacement. It is always this displacement v x of t equal to psi x into z t, we have already talked about this right. So, therefore, if you look at it the m x due to the deformation is equal to E I v double prime which is what d square v by d x square v we t call it as v x double prime.

So, this is equal to E I into the second derivative of psi x into z of t, so this is my m x due to the deformation and let us look at, what is the a acceleration that this point undergoes acceleration that this point undergoes is the following it is equal to m bar x d x that is again the mass, times the acceleration and the acceleration is equal to v double dot x of t that is this part plus because that is the acceleration the mass was here and it is here. So, it is been subjected to that acceleration. So, if you look at this essentially becomes the following, I will put this as m x d x psi x into zee double dot plus v g. So, this is the due to the deformation and this is the inertial force down now there are no other lodes on this system.

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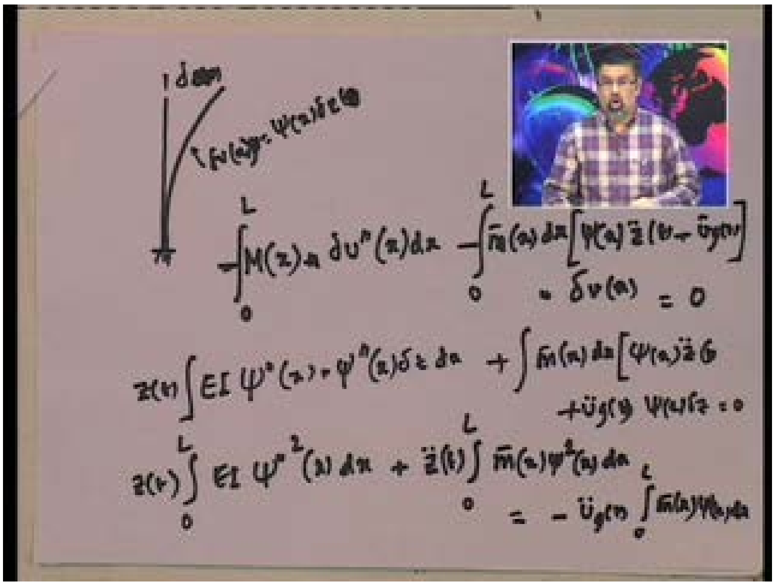

So, if I take if I look at it from that perspective that we have, now you know these are the forces. And so now, what I am going to do is I am going to subject this to a virtual displacement and note, that the virtual displacement is this right. So, this is equal to virtual displacement and del v x of t is equal to psi x into del zee of t sorry note del zee of t, this is just del zee of t this is the virtual displacement psi x into del zee. So, this is no not this is only with x because this is the virtual displacement, it is a virtual displacement.

And what we do is, we now given is virtual displacement we find out the work done and we have already, done this that the work done by these m x is going to be equal to m x into now, you know it is going to be opposite. So, it is going to be minus m into x into what is it going to be it is going to be equal to del v, so it is going to be del v prime x d x that is the relative motion of one end respect to the other because this is the internal force. And then I have minus m bar x and this is integrated over the whole length because that is again for a particular infinitesimal element. And then the s m bar d x into psi x z double prime plus v g of t this is the force, this into del v x that is and that integrated from 0 to L. So, this in a sense is the work done by the forces and that is equal to 0, so now, if I put this in what do I get, well I can take minus, minus out because I can put it on the other side.

And, so this becomes essentially E I psi prime I am going to put z of t outside. So, E I double prime the z of t goes here that is n x then del v is equal to psi x double prime del zee into d x plus I am going to put the term I here. So, this is going to be equal to m bar x d x into psi x z double dot plus v g into psi x into del z is equal to 0. This is dell z exists in both of them, essentially I can take z outside and then my equation essentially becomes this that it becomes z t o to L E I squared x d x plus and I am going to put the z x term outside. So, this is going to be z x 0 to L m bar x psi squared x d x because this is i x d x this one and this I am going to take this term, which is v g term on the other side. So, this is going to be equal to minus v g time 0 to L m bar x psi x d x.

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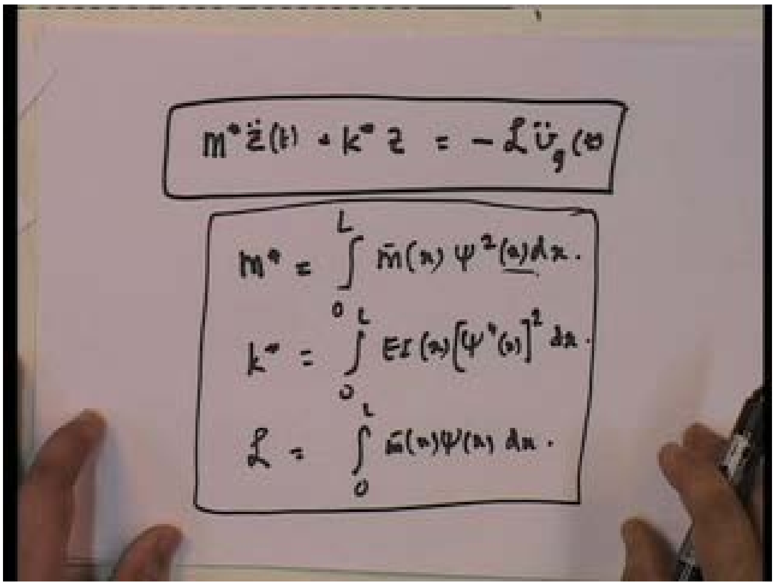

So, this becomes the equation and so if I rewrite it in this form I am going to rewrite it in the form following form, I am going to rewrite it in the form that m star z double dot plus k star z is equal to minus L into v g t this is my equation. Where; obviously, m star is equal to 0 to L m star x psi squared x d x k star is equal to 0 to L E I x psi double prime x whole squared d x and L is equal to 0 to L m bar x psi x d x.

Now, if I look at this, this looks exactly like the single degree of freedom accepting that in the single degree of freedom it was m z double dot plus k z a sorry m v double dot plus k v double dot is equal to minus m v. So, the only difference between the generalized is that you don't get m star over here, m star note is m bar psi squared x d x what you get is an L which is m bar into psi x d x. So, that is what you get and then if I look at the solution to this a problem, it becomes a fairly trivial kind of a solution.

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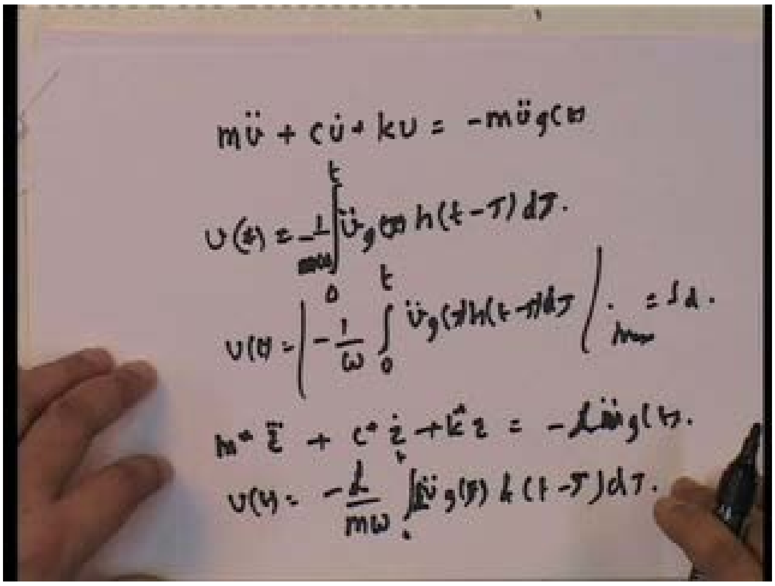

Because, see we have already looked at it, let us look at what we have solved. We have seen that this plus, well you know I mean I can put now my c star you know I mean this one is minus m v g double dot. And what we got was that look v of t was equal to what is was in this particular case equal to here, you have minus v g double dot this is m upon k. So, what do you get you get i m, so you get this minus sorry 0 to t v g tau h t minus tau d tau.

And the only thing that was there, was that remember this that what we had, was that this was the unit impulse response function and it was 1 upon m omega remember that, this was 1 upon m omega into m. So, this thing disappears, so what you get is essentially v of t minus 1 omega 0 to t v j tau a t minus tau t tau and we said that the max of this was equal to s d. So, now let us look at this particular problem, this problem essentially is m star z double dot plus well does not matter c star, k star z minus L v g double dot upon n.

So, if you really look at it what happens over here, is that this what will be the response of this, this one will be minus 1 upon m omega 0 to t v g tau h t minus tau d tau and the max of this, if you look at it there will be an L here, L v g double dot 1 upon m omega. And so this then becomes the solution L comes out, so L actually comes out here. So, that this one if you look at it, this part is identical to s d.

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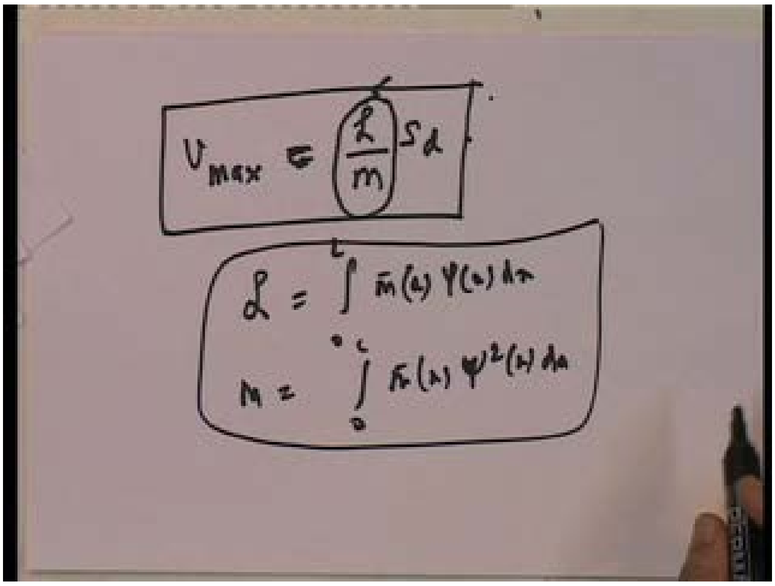

So, if you look at it the peak displacement is equal to L upon m into s d. So, that is becomes my solution and then of course, you know we can find out what do you want to find out, we want to find out the shear force. The shear force etcetera, we can always find out, but in a sense the point that I am trying to make is that there is a term, which is been brought in where L is equal to 0 to L m bar x psi x d x and m is 0 to L m bar x psi square x d x.

So, they are different quantities because psi x is different and although both have m bar. So, in a way it is still is an inertial load, it is just that inertial load instead of being it a actually L upon m into m v g double dot, so that is the whole point that there is an L upon m which is non zero. In fact, L upon m if you look at it you will find it that it will actually land up being, depends on the psi x that you choose that you get different values.

So, this in a sense is a generalized single degree of freedom and I just introduce that concept of the generalized single degree of freedom to understand for you the fact that, in such situations what you have is that you do not have, you know it is still inertial loading you know. In other words it is still L, L is still an inertia a term because it is m bar x d x where into psi x with psi x is dimensionless.

So, it is still a inertial force, in other words the base motion can be transferred into an inertial force. But, the only the only thing is that the inertial force is not a directly related to the mass, but a to a factor L which we call as mass participation, we call that as the mass participation of or rather inertial participation of the load. So, this is just to illustrate the point generalizing degree of freedom, was just to illustrate the point to you that a is not the same as that you had for single degree of freedom system.

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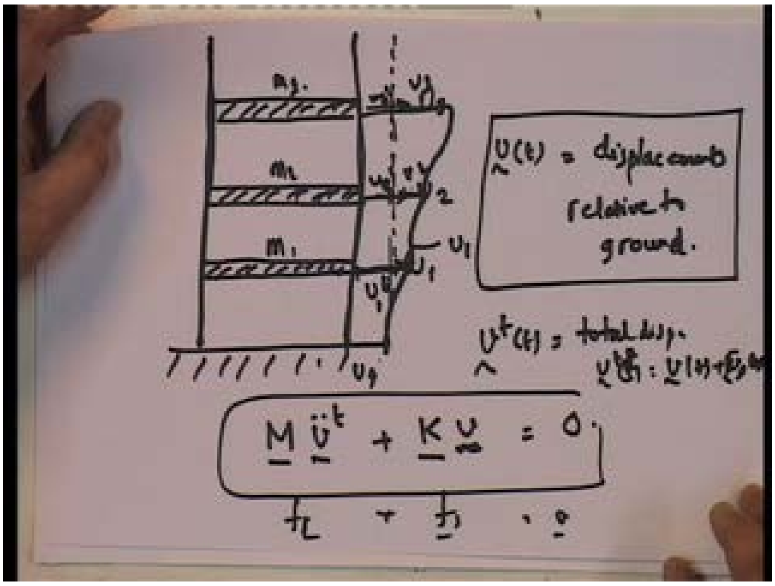

So, the now let us look at multi storied structure and I am going to as I said you know, I am now specifically talking about plane frames. So, what I am going to define right now is plane frames and then in the next class, we will see how we can tackle the problem of when you have building, so this is not a building it is a plain frame. So, here this is let us say v 3, v 2, v 1 actually you know you could I could call this v n, you know I mean because you can make n storied building.

And this if we look at any inertial point, this is the inertial system. Now, it is subjected to this is also the inertial point that we have, now due to base acceleration what happens this moves here. So, I am going to look at this as the shifted inertial frame and due to the base, this actually starts going, so you have what we call as v 1, v 2, v 3. Now, v 1, v 2, v 3 the way they are defined. So, this v of t is actually displacements relative to ground. So, this is how we define the displacements relative to the ground.

If we find the displacement relative to the ground, in other words the ground has moved here. So, the displacements are all considered relative to the ground, then what do we have well let us look at it this is m 1, m 2, m 3, so if you look at the inertial force, what we have is if you look at this particular thing, you also have v total of v total is the total displacement at the point.

So, this is the total displacement from here to here, that is called as v total 1, this is v total 2, this is v total 3. And if you look at it the masses has been subjected to these acceleration, so they has been subjected to the relative a the accelerations. So, then what you have is M into v total that is the inertial force plus, now k that is the deformation into v is equal to 0 because there is no external load. However, if you look at this what is v total t equal to it is equal to the relative plus everywhere is the same. So, it is equal to v g t, but 1 right because v g t is a scalar, so it has to be one. So, if you look at the you know now, this is the equation of motion because this is the inertial force and this is the spring force and this is equal to 0, we are assuming that there is currently no a damping.

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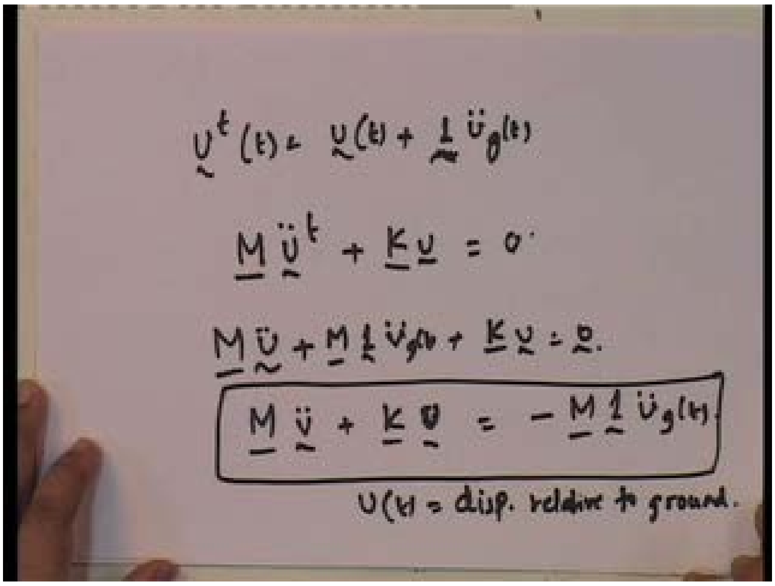

So, therefore, if you look at it, we get this kind of a situation that since, v total is equal to v of t plus now, each one is being subjected to this is a vector of one’s into the ground, every one of them is one, one, one and so this is the vector of ones. So, if I look at m v double dot t plus k v is equal to 0 this becomes then m v double dot plus m into 1 dot v g t plus k into v is equal to 0 this is the n by 1. So, I will take this one on the other side.

So, my equation in terms of the displacements, so what we have here, is minus m into 1 into v g, note this then becomes the equation of motion at the where note that v of t is displacement relative to ground, please understand that, that this is the displacement relative to the ground. So, if this is the equation, so this becomes our basic equation and then what we do is what do we do, we say that look this is like s loading right.

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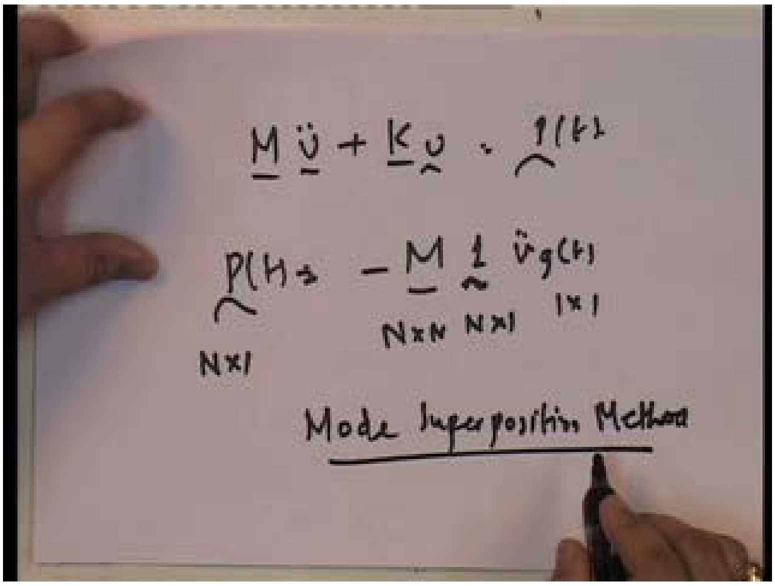

So, we have a situation where, we have m v double dot plus k v is equal to p of t where p of t actually if you look at it, this is the vector the p of t is nothing but minus m into a vector of one's into v g of t. So, this is in N by 1, this is in N by N, this is in N by 1, vector of one's all of them are one and this is the scalar. So, just note that this is what you get and p of t is this, if I look at the mode superposition method. So, I am going to look at now, solution using mode superposition method because I am looking at linear systems only. So, if I look at mode superposition method what do you get, well you see the basic point then becomes is that I have found out p of t, I know what the a modal amplitude equation looks like.

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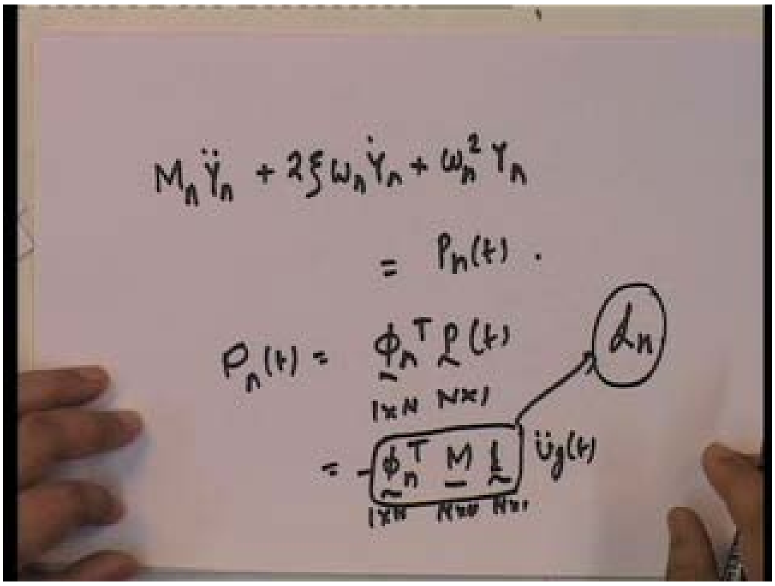

So, the modal amplitude equation looks like, M n y and double dot and here I can incorporate my damping, y n double dot plus omega n square y n is equal to p n of t well what is p n. P n if you look at it is nothing but p n of t is equal to phi n transpose p of t this is 1 by N, this is N by 1, this is 1 by 1 scalar. Now, here in this particular case p n of t a p of t is equal to, so I am going to say phi n transpose I am going to put minus m into 1 into v g of t.

So, this becomes my p n of t and if I look at this, this is 1 by N, this is the N by N, this is the N by 1 should we do get a scalar. So, this is the scalar and this term, we call as L n remember yesterday when I solved that specific problem, remember I had I had defined that the base shear was given by L n. So, the same phi n transpose N into 1, now this is L n this is the same L n that we defined yesterday.

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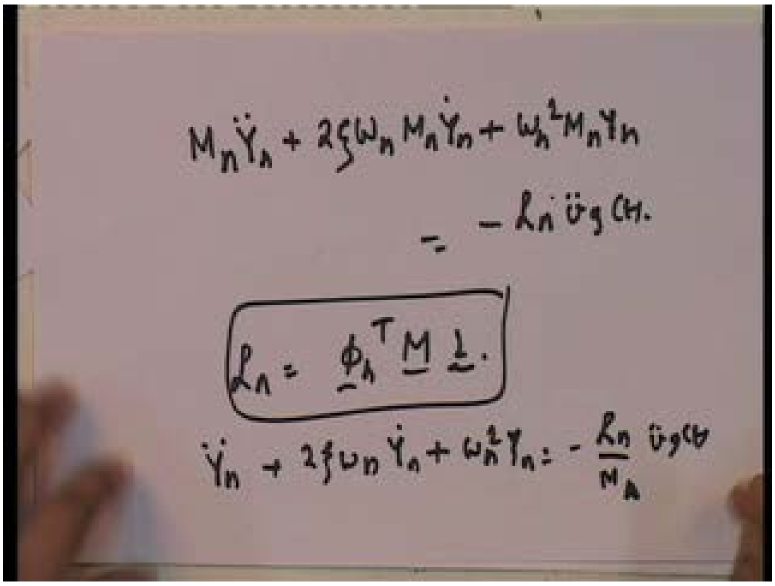

So, if you look at it then the equation at the n'th level becomes what 2 psi n omega n m n y n dot plus omega n squared y n is equal to minus L n into v g dot, this is what we get. And so therefore, where L n is equal to phi n transpose m into 1, so if you look at this, this is identical to the single degree of freedom, accepting that there is a term. So, if I rewrite this becomes y n plus 2 psi omega n y n dot plus omega n squared y n is equal to minus L n upon m n v g double dot. And what is this L n upon m n, this is very interesting see how why I did the generalized single degree of freedom.

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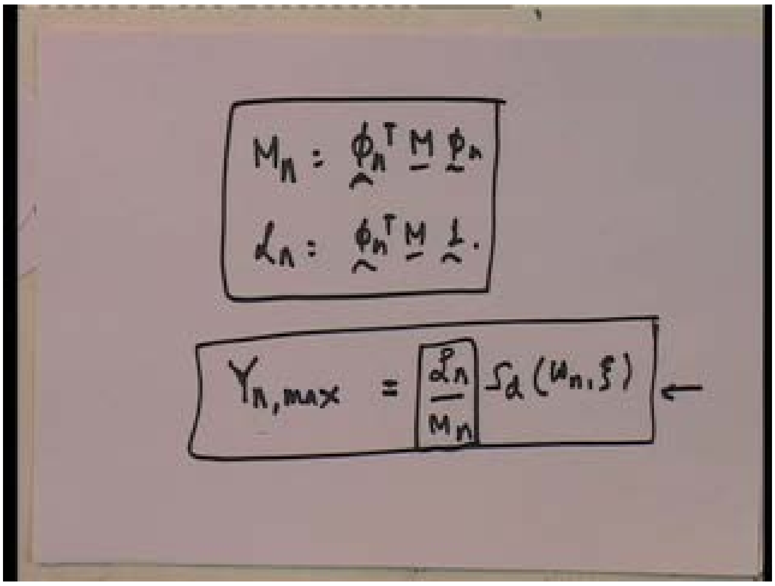

If you look at M n it is equal to phi n transpose M phi n, in a way you can look at this is like a the two together and L n is equal to phi n transpose M into 1. Do you see the correspondence between the generalized single degree of freedom, there what did we have m bar x psi x d x. So, psi x into 1 d x m the m star was m bar x psi square x is exactly the same kind of concept, and all that we get is if I want to look at y n max.

What is my y n max, if you look at it y n max is nothing but L n upon M n into s d which is a function of omega n and psi. In terms of the displacement spectrum, your y n is equal to what L n upon M n into s d because in the previous case we had said that v max was equal to s d. So, since this is been multiplied by L n upon M n it is just that factor continues. So, this in a sense in turn, so I can get the maximum modal amplitude in terms of the deformation spectrum, only thing is the deformation spectrum is now, multiplied by a particular additional term. Now, what does this mean in terms of v and max, what is v and max v and max is nothing but the contribution of the n' th mode to the displacement.

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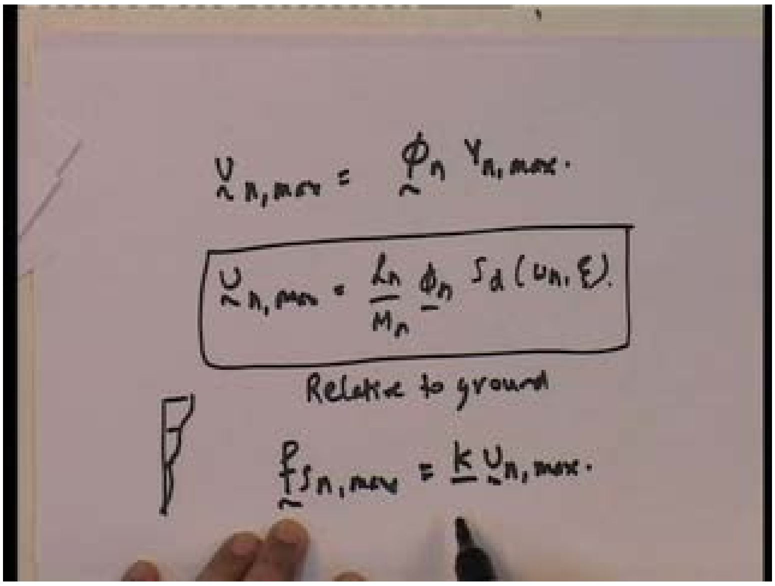

So, if I look at v n max this is nothing but phi n into y n max. So, if you look at this then v n max is nothing but equal to L n upon M n into phi n s d omega n psi, since we typically take the psi same psi value across the modes I am going I am not putting psi n it is psi. So, this in a sense is the peak displacement in the n'th mode, now can I find out, so this is the displacement pattern that we have, the definite and by the way this is relative to ground. Please note that, this is just the essentially the deformation of the structure is what we are looking at…

And, so if this is the displacement pattern, we also again say what displacement pattern gives this what force gives the this is displacement pattern. And so if we look at that force f s n max, so that this is the spring force or rather force that is going to give rise to this peak displacement. So, if we look at it this is going to be equal to what it is going to be equal to k into v n max do you agree to that right. So, now, this is f s n because this is the load that statically gives this displacement.

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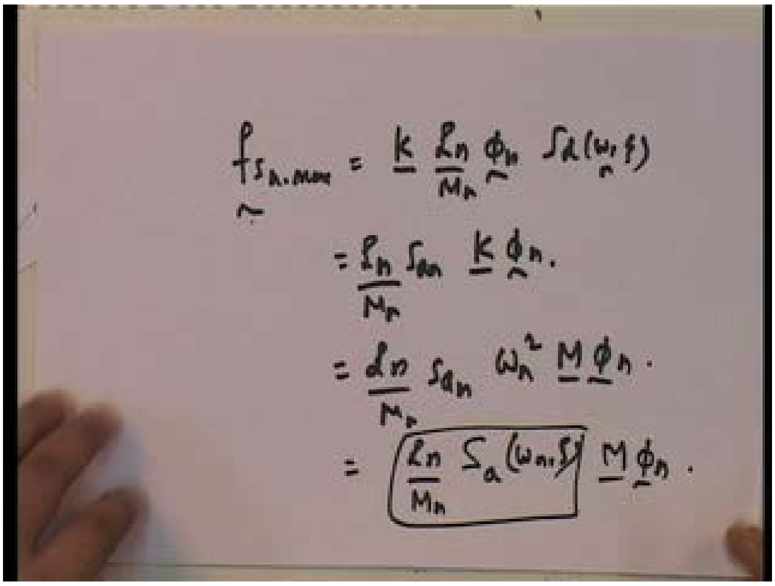

So, this is the load, so if I put k into v n I am going to substitute that term for v n in here and if I substitute that and look at that particular thing what do I get, I get f s n max is equal to k into L n m n which is anyway outside phi n into s d omega n psi, I can take these factors L n upon M n and s d n I will call that s d n because it is for omega n and take those are scalars and keep this k phi n in here.

So, what is k phi n equal to it is equal to s d n, if phi n is equal to omega n squared into m phi n by definition right, this is k phi n is equal to omega squared m phi n is by definition we have already seen that. And so therefore, this thing then becomes scalar and if you look at this what is omega n squared s d n, it is nothing but s a n. So, it is L a n s d n omega n squared remember, it is a pseudo acceleration velocity. So, that is s a omega n psi and this is the parameter and this is being multiplied by m phi n that is the load that you have…

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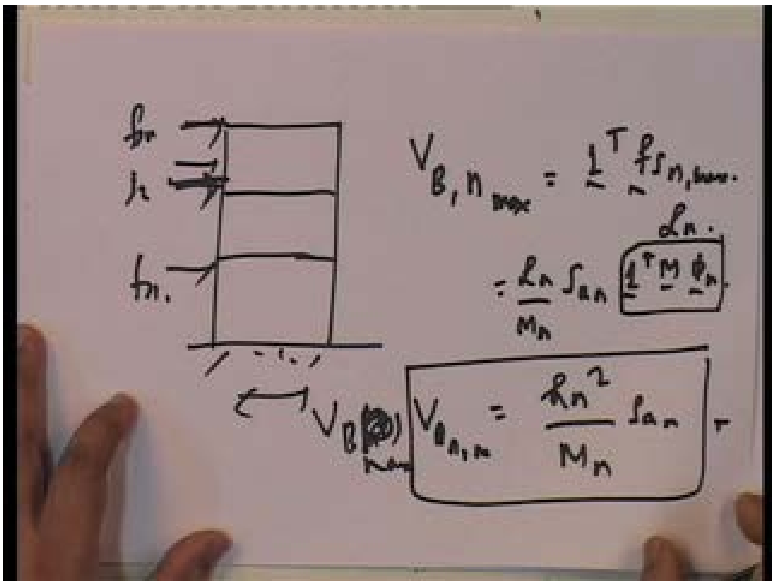

So, if I want to draw this; that means, in the n'th mode I have this structure and you know again I am just saying that I have, at all de corresponding to all degrees of freedom and f s n f s 2 f s 1, so here are the loads that I have. So, if I look at the base shear or rather not thin e max because these are all in the n'th mode. So, this is going to be the base shear in the n'th mode and; obviously, maximum what is that going to be equal to that's going to be equal to 1 transpose into f s n max.

So, what is that equal to plug in f s n max, what we get is L n upon M n into s a n I will call it again, into 1 transpose m phi n we have already done this phi n transpose m a transpose to 1 transpose, transpose m is a symmetric matrix. So, m transpose is itself, so this becomes nothing but this we have already seen is equal to L n. So, if you look at d b n what you get L n squared upon m n into s n, such an elegant formulation for the base shear due to earth quake load.

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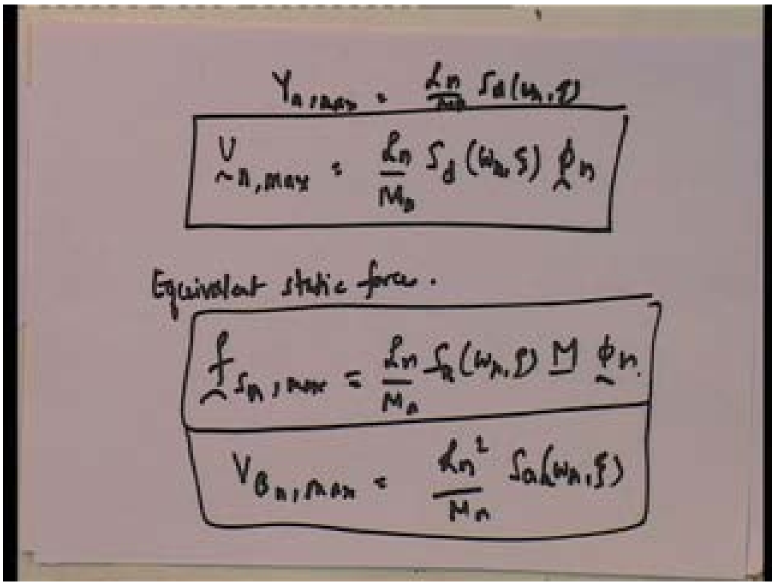

So, therefore, what we have finally, done is in the n’th mode, this is equal to L n upon M n s omega n psi into phi n. So, this is my displacement, the equivalent static forces in the n’th mode is equal to f s n max which is equal to L n upon M n into s a omega n psi n is m phi n, these are the equivalent static force. And you can actually, apply these loads on the structure and get every other force response or displacement response or rotation what have you, you can get all of those.

Because, note that in the n’th mode nothing is bearing with time. So, this is a specific thing and then finally, v B the base shear n’th mode is given by L n squared M n into s a omega n psi. So, everything is and I will write down on top y n max is equal to L n upon M n into s d omega n into psi of course, every time will be different for different modes. The problem that we have here, is that and this is the point that I wanted to make for this particular thing.

Either that we can get every displacement in each mode, the peak value in terms of the response spectrum, the only thing is that here for each mode since it is omega n psi. So, for each mode the value of s d and s a are different, you need not to take that particular omega n and then take it. So, if we look back at the something that we have looked at earlier and if you look at this, we will just go back to the original a thing that I had done.

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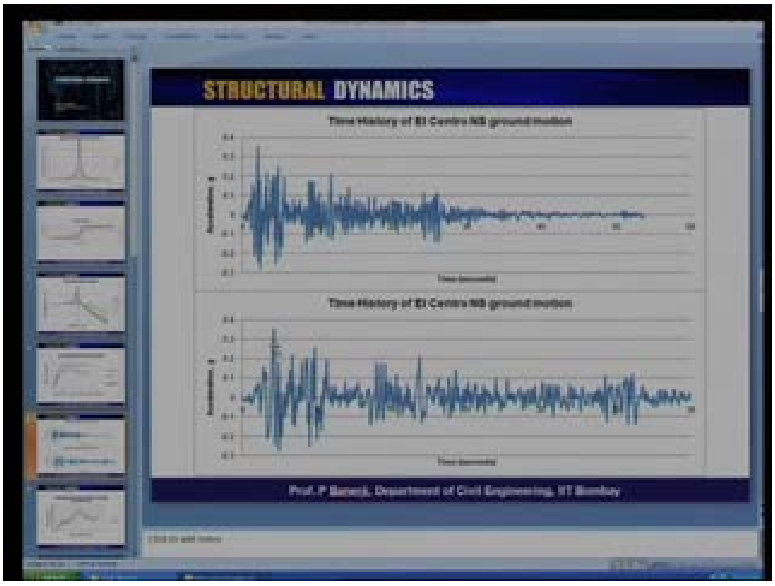

This is the time history.

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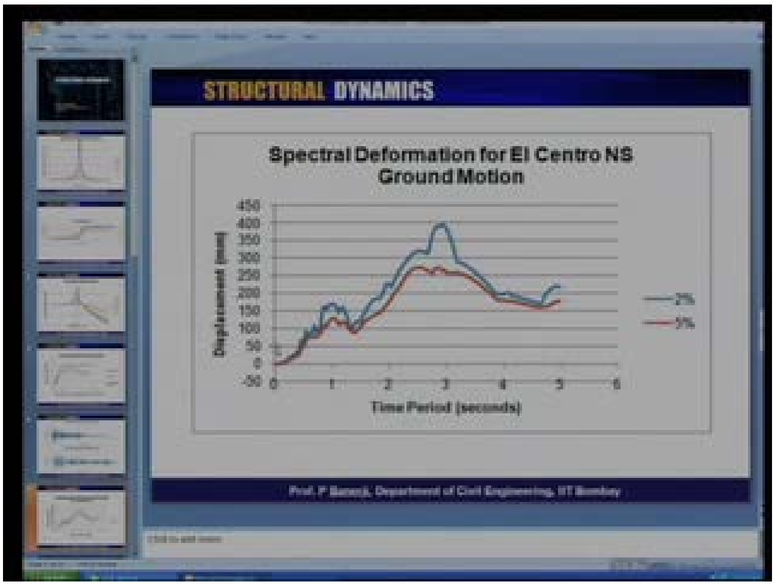

This was the deformation.

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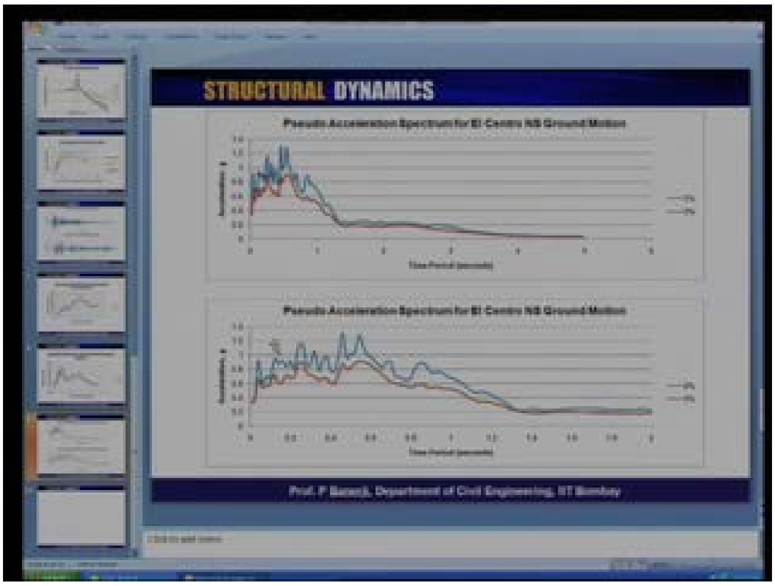

This is a velocity, this is acceleration. So, if I want velocity I mean sorry let us say displacement what do I do.

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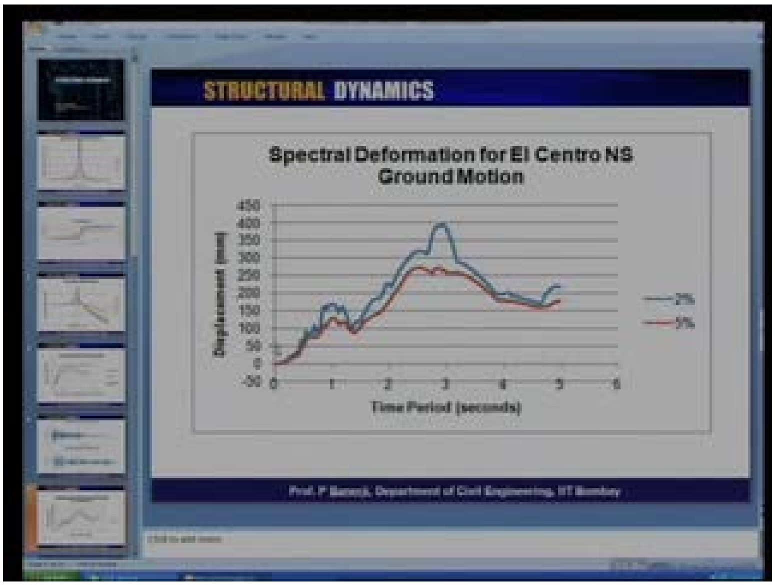

Well I actually take this right. So, now, you now here you have t n (Refer Time: 39:57) if it is omega n this t n t n is equal to 2 psi upon omega n, so t n. So, let us say you want the first mode you find out its t n and read out the displacement value, if you take another mode if you want to take here. So, let us say that here the first mode, let us first mode is you know first mode frequency is low.

So, time period is longer, so let us say the first mode is a let us say 3 seconds, so you go up here and let us say you have a 5 percent damping. So, you take over here and you see that the displacement s d because it is not displacement this is s d, s d is like say 260 millimeter. So, what you do is you take this over here, ((Refer Time: 40:51)) as 260 millimeter the first mode and then you find out L 1, M 1 and phi 1 and you have got your v 1 max.

Then you go to let us say this second mode, the second mode let us say is a like 1.5 if you take it as 1.5, we come here and we see 100. So, for the second mode this is first mode this was 260 and for the second mode, this one is a first mode 260 for the second mode this is 100 and you have L 2, M 2 and you can find out the v 2 max. So, this is displacement in the second mode, you can find out the displacement in the first mode in this way similarly if we go here and we look at the pseudo acceleration.

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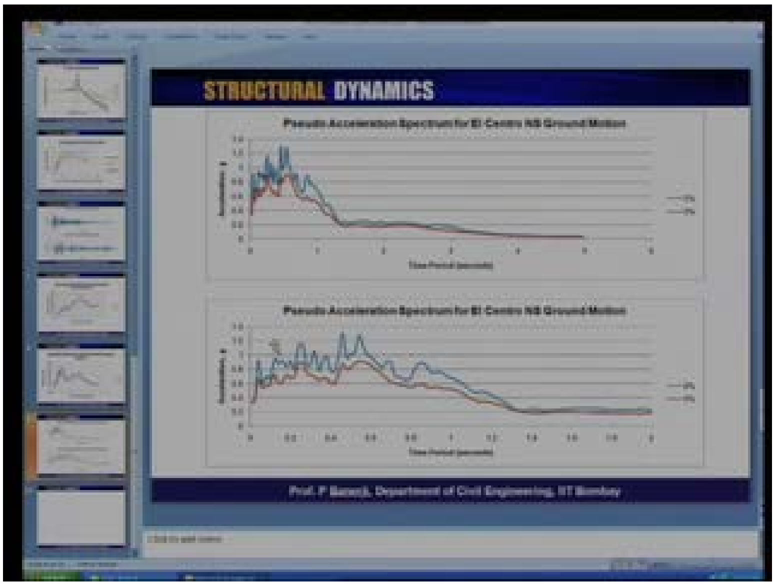

So, we say that look 3 seconds, so we go back over here and take it and we see it is like like 0.18. So, in the first mode this is going to be s a is going to be equal to 0.1 g a which is 1 meter per second squared. So, this becomes one and so these are all the things that you have and or this is 1, meter per second squared and this you get and you get this particular the first mode. For the second mode, which is let us say 1.5 I will come over here look at it and let us see o it is about 0.2, so take this as 0.2.

And similarly, go through with the procedure. So, you understand the basic concept that here, that you have is that this value and these values depend on omega n. So, that is the key point that for different modes, you know you have different values of s a and s d which you read off you know for the particular value of omega n.

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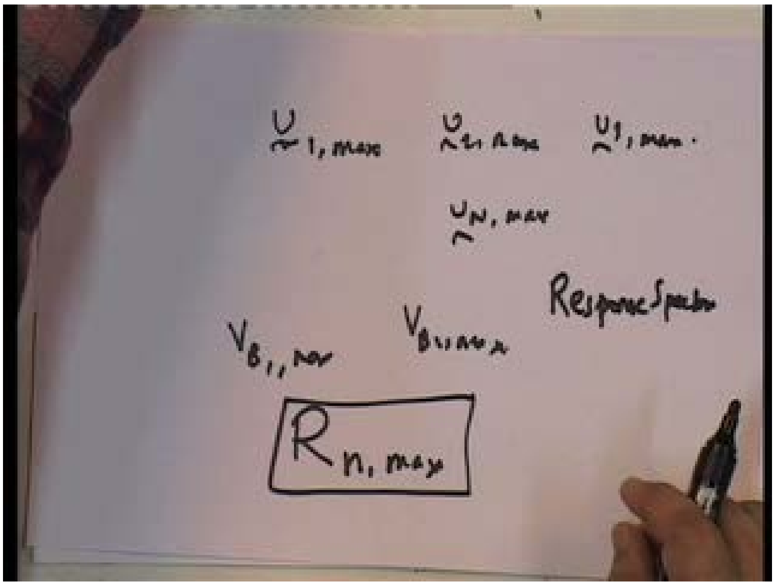

But, ultimately what happens is that look, what we have here is let us say, in a way I am calculating v 1 max, v 2 max, v 3 max right and so on and so forth up to v n max or I say v B 1 max, v B 2 max. So, let me just say, that you know let us say in this particular vector, you actually have particular you know displacement, let us say top story displacement or you know, so you can read it off. So, if I say any response quantity I am getting R n the peak response, because I am using response spectrum, if I was using the time history.

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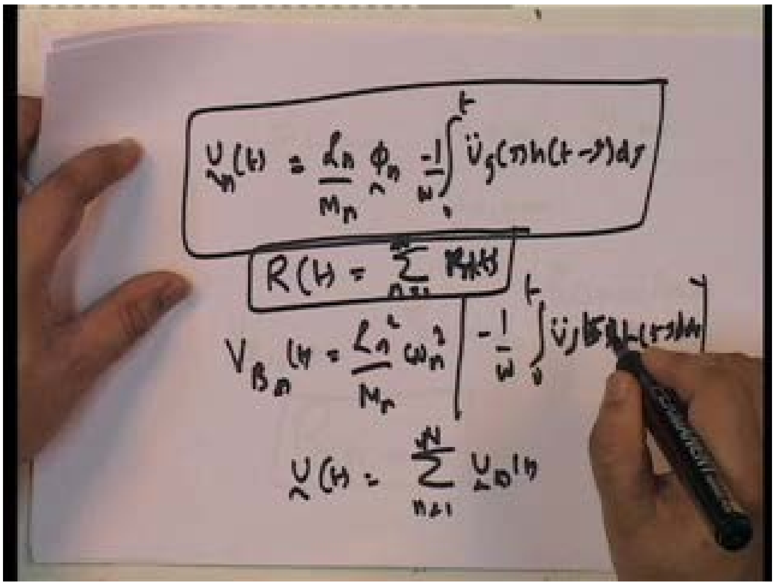

And I found out for example, lets me just go back here and if I was looking at time history v n of t would be equal to, well you know this would be equal to L n upon M n into phi n into an this one would be like, you know minus 1 upon omega 0 to t v g t tau s t minus tau which is the unit in first response function. Suppose, we found out v n in this way, then v B n t would noting be L n squared M n into omega n squared into minus 1 0 to t v g double dot s t upon tau, so all of those kinds of things, you know the you know evaluate it.

So, what you do is if you get the time histories, then there are no issues because what if you look at it this already phi n y n. So, if you look at it then v of t becomes just summation v n of t n going from 1 to N, similarly any response quantity I could put it this way that if I had the time history, any response quantity would be equal to summation n going from 1 to n r n of t. Note that, this is valid why because at any instant of time I can always add them up. The problem that I have, for this particular problem when I have maximum, when I am using the response spectrum, response spectra I am using the response spectra what happens is I only get max values in every mode.

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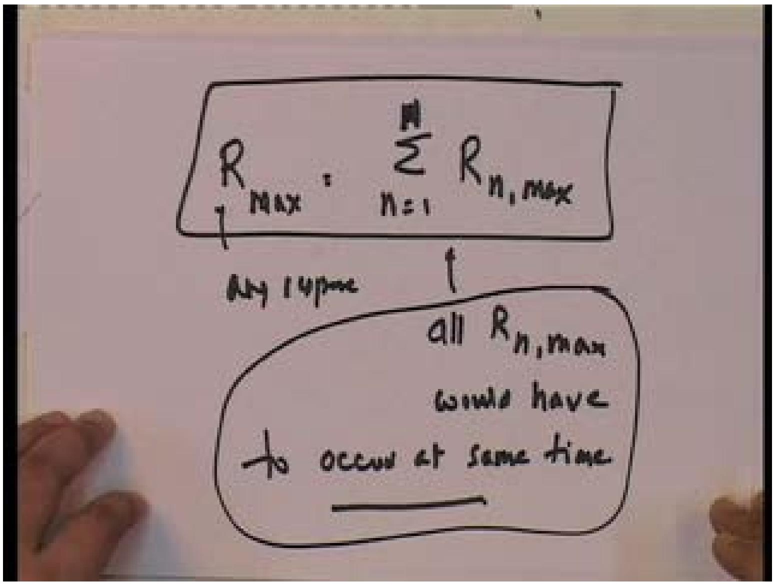

Now, one of the things that you could say is that well. So, what could I find out for example, R max is equal to summation n going from 1 to n R n when, R is any response could I say that, this is the problem with this is that although this is valid for any instant of time, for this to be valid what would happen, all R n max would have to occur at same time right, so this mode superposition all though it is valid when you are doing time history because at any instant of time you can add up.

But, if all of them would have occurred at the same time, then this would be valid. Otherwise, this is not valid and typically what happens is any response quantity, if you look at the time history, look at v b n of t you will typically see that what essentially shows up in each mode is going to be it is own frequency, you will see you know a when you do it a use see that you essentially get, the same frequency.

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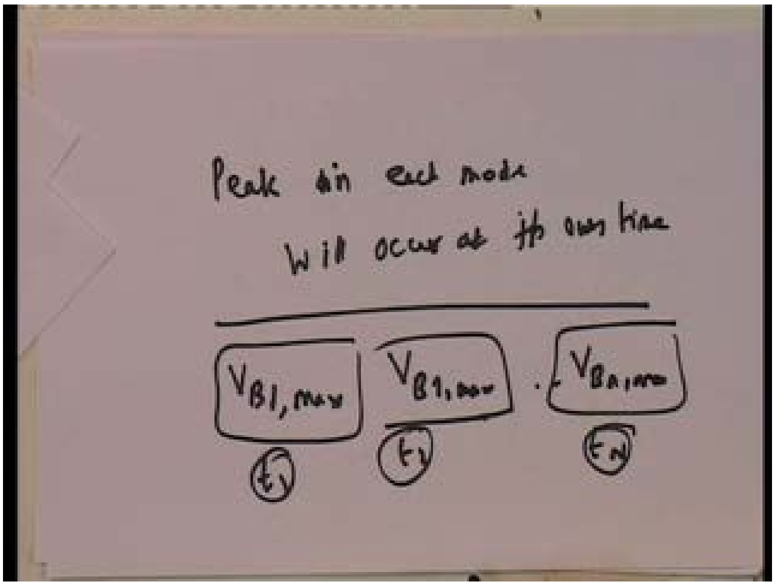

And the peaks you know the fundamental aspect is that the peaks. And this maximum value nothing with the peak in each mode will occur to at it own time they do not occur at the same time. In other words, when I say v B 1 max this is occurring at say time t 1, v B 2 max will occur at some other time t 2 and in that way v B n will occur at some other time t n. Now, what are these we have no clue, these depend really on the acceleration time history of the ground, ground acceleration time history because; obviously, a you know where it where the peak occurs will depend on the time history. And also the you know that frequency of it is we have seen all of these, you know the single degree of freedom I do not want to go into that, we do know that the peaks will not occur at the same time. What happens when peaks do not occur at the same time.

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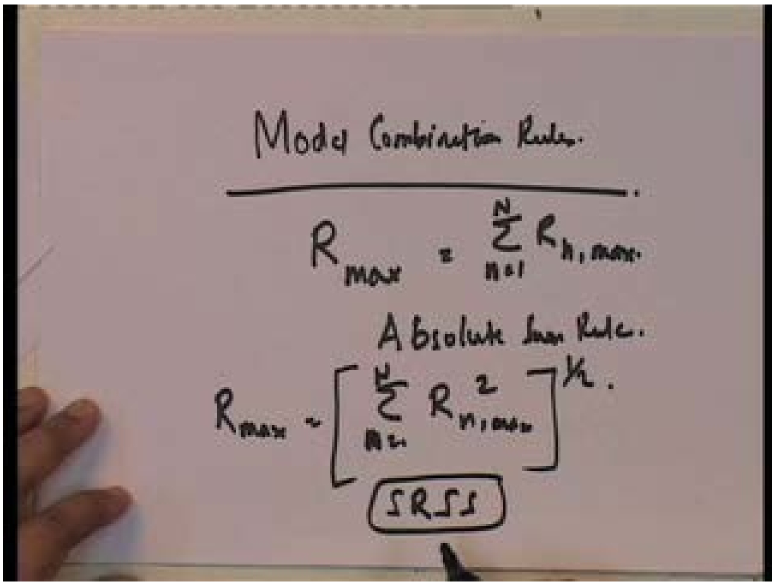

If peaks do not occur at the same time then we have to do what are known as modal combination is to combine the modes, we have to use rules. The one that I show to you which was R max is equal to n 1 n into R m max; this is known as absolute some rule. Then you have R max equal to n going from 1 to infinity R n max squared square root this is known as the square root of the sum of the squares, square root of the sum of the squares SRSS rule, this is the s r s s rule and there are very other rules I will talk about these rules in the next class.

So, just to a sense to tell you that earth quake response analysis, you can do in two ways time history analysis no problem, all that you have is that y n of t is L n upon M n into the single degree of freedom solution. And then we n of t become equal to phi n y n and then you can always sum up, you know any displacement quantity equivalent static force f s n was e q we know we showed it to you that it was equal to L n upon M n into m phi n omega n squared into the displacement y n. So, all of these kinds of things are can be done, if it is time history you also develop what is known as the response spectrum method of analysis. And this response spectrum method of analysis, we show that the you have to use modal combination rule I will keep talking about this in the next lecture.

Thank you very much, bye, bye.