Structural Dynamics

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Lecture - 30
Damping for Multi Degree of Freedom Structures

Hello there, you know we have been looking at a dynamic response of quantitative freedom systems. And in the last lecture, we solved one particular problem and highlighted certain you know, useful facts about the mode superposition method for example, we actually solved on damped problem. Because, I still have not looked at damping in it is entirety, although some way along the way, I have always said that you can incorporate damping this way. And one of the thing that we saw of the mode superposition method was that you did not need to consider all modes to get responses.

And secondly, for displacement responses, you needed fewer modes to be considered than if they were for force responses. So, this is in a sense you are a basic now, how many modes should be considered, now this is a very, very difficult problem to answer, I will answer it later a when we take the specific case of earth quake response analysis. The reason behind that is that, you know in earth quake response analysis, we can actually identify a parameter, which you can get for every mode and then when you add up the parameter, you could pretty much decide, how many modes you should consider to get a very good estimate of the force responses.

But, how many modes should you consider, well you know you really cannot tell, you know, I mean well you the best way to do is try out 5 modes, try out 6 modes, you know, I mean and see how I works that is all solution. In general loading, it is very difficult to identify how many modes to consider for force responses and how many modes to consider for displacement responses. Today, what I am going to do is, I am going to go back to damping.

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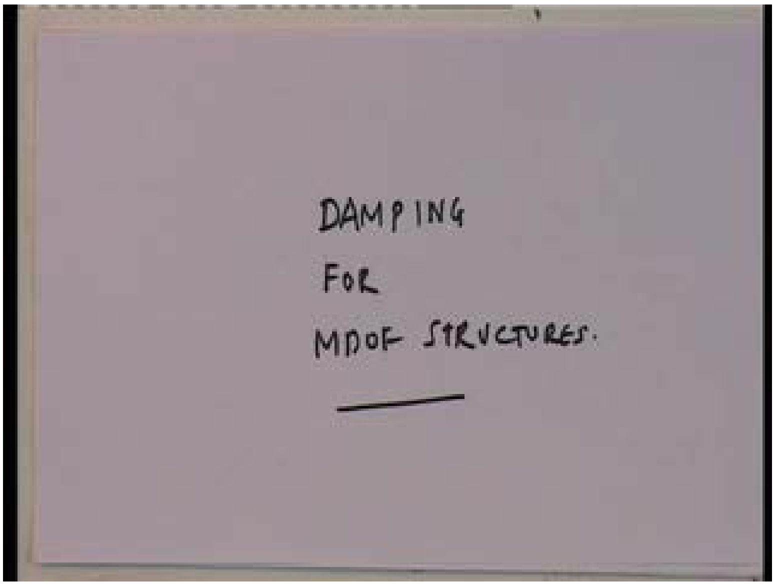

And I am going to look at damping for multi degree of freedom structures, you see a we talked about the fact, that if I include damping, now you have this situation, you have a situation where, you have the equation, which looks like this.

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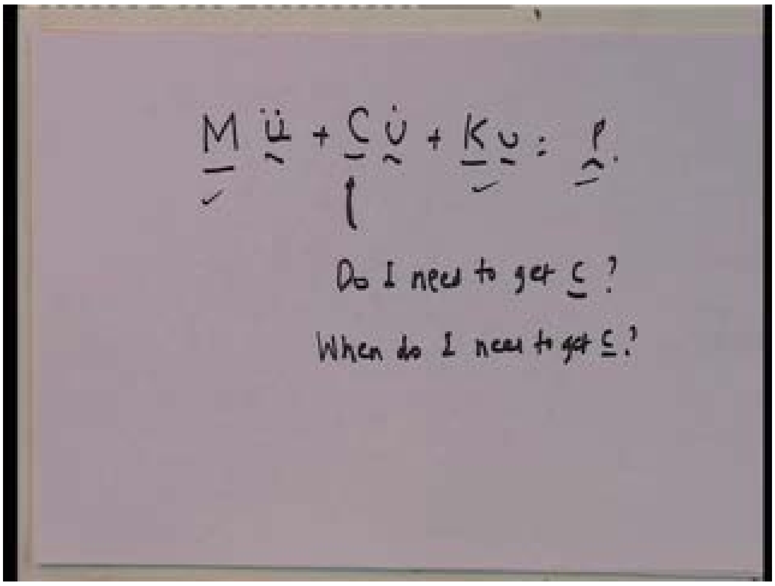

By now, this and this is something that we know how to get today in this lecture, I am going to concentrate on this, first question is do I need to get c, when do I need to get c. These are some of the questions that I am going to be asking and answering more importantly, answering asking questions is always such a good idea, but getting answers to them is a far superior a concept.

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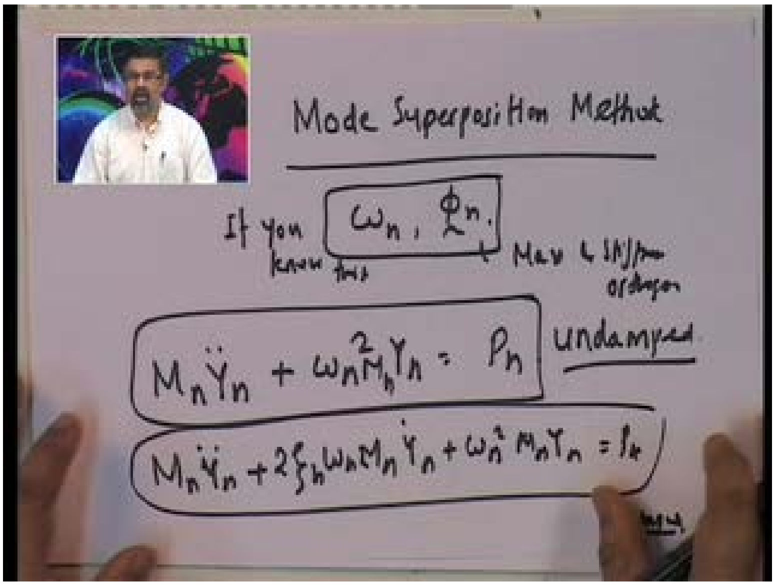

So, let us see what did we do, we talked about mode superposition and the point that, I make is the following, that as long as you know omega n and phi n. If you know this, the mode superposition method is the only way to go, even if you do not use this it is a far, far superior computationally, to go through this rather than solve this equation directly. You know, I have shown you how to solve and when we go back to looking at it how to solve this equation, but mode superposition, how does mode superposition look it looks like this.

Omega n square M n for un damped and this comes from the fact that this right, the phi n r mass and stiffness orthogonal, that is why, I get this equation and I say that, the corresponding damped system with the equivalence from the single degree of freedom system would look like this. This is what a damp would look like and we also say that look.

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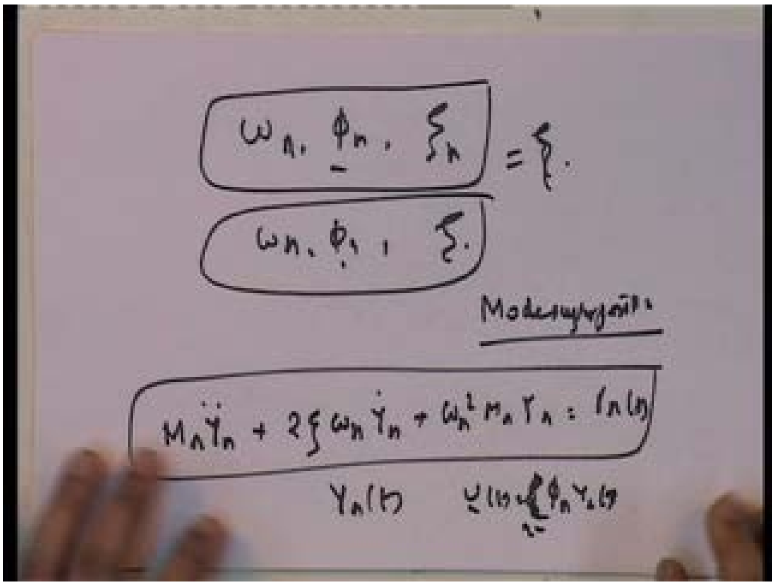

So, all we need to do is along with omega n phi n, for mode superposition method all we need to do is define this parameter. So, if we define the psi n and we also talked about the fact that look, since I do not know, I will take that every a mode has the same damping, because you do remember, you know a we discussed this structures made of the same material, then you know damping is obtained numerically and it is the same. Irrespective of whether the structure is stiff structure or a b, you know flexible structure a damping pretty much remains the same. So, we can just extrapolate that and say that look, I am going to do omega n phi n and I am going to define psi.

If I am using mode superposition, this is all we are using the mode superposition method that is all you need to do, because y then you have and this is your standard single degree of freedom problem and you can always get Y n of t. And once you get Y n of t v of t is equal to phi n Y n of t summation or which we saw was equal to phi into y all of those things and I am going from 1 to n. And we have done it, we have considered damping trivial absolutely, trivial to consider damping in this particular manner. Now the question becomes that well, that is it mode superposition, but suppose, you have a situation and this is equally.

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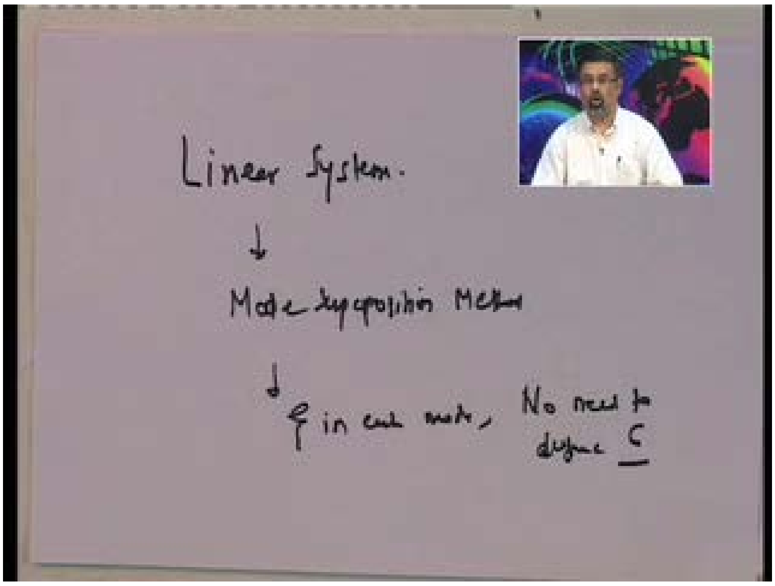

So, linear system, mode sorry, superposition method psi in each mode no need to define c matrix at all. So, it's a linear system mode superposition, you define psi in every mode, you do not need to define c at all where do you need to define c, if your system is going non-linear and you see a system does not go non-linear straight away when you have let us say, when you have a loading.

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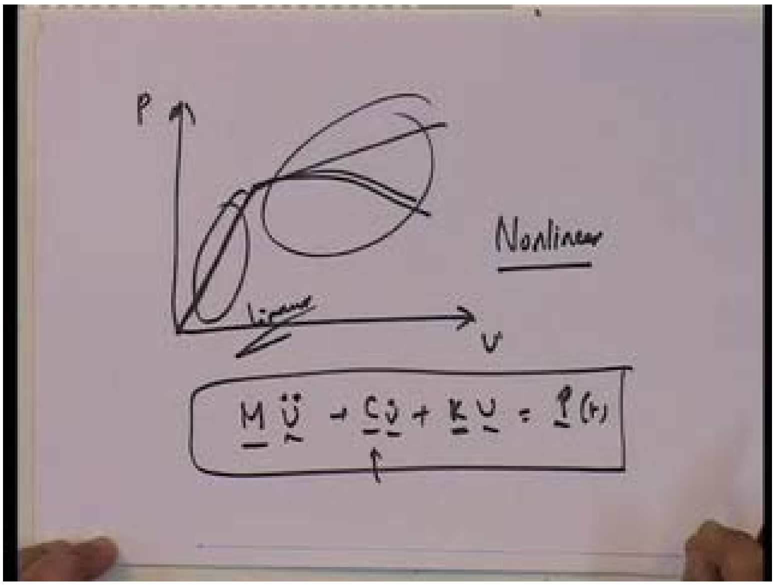

A how does how you know, you have loading, you are loading the structure, if you will look at K V versus lets say p, how does it go, it actually stays straight and then it goes non-linear or something that sort you know I mean non-linear, this way or this way, whatever this is the non-linear part, this is the linear part. Now, the question becomes that look, even in the linear part, you might reduce what is superposition only when you go non-linear you need to use direct methods. Where direct methods are based on M V C V dot plus K V is equal to phi.

So, that is when you know, that is the only time, I cannot use mode superposition purely because it is m you cannot use mode superposition mode superposition only valid for linear system. So, that is when I need to solve this and the problem is how do I get C when I have non-linear, I need to have C, which is consistent with my definition that, I have used in the linear part mode superposition. Mode superposition what did I say, I said well take psi in all the modes that, I am going to consider and that, then my c matrix has to be consistent with that, because let us put it this way. Since you have linear non-linear, that is in the linear part, this solution has to be consistent with the mode superposition method, you understand my point.

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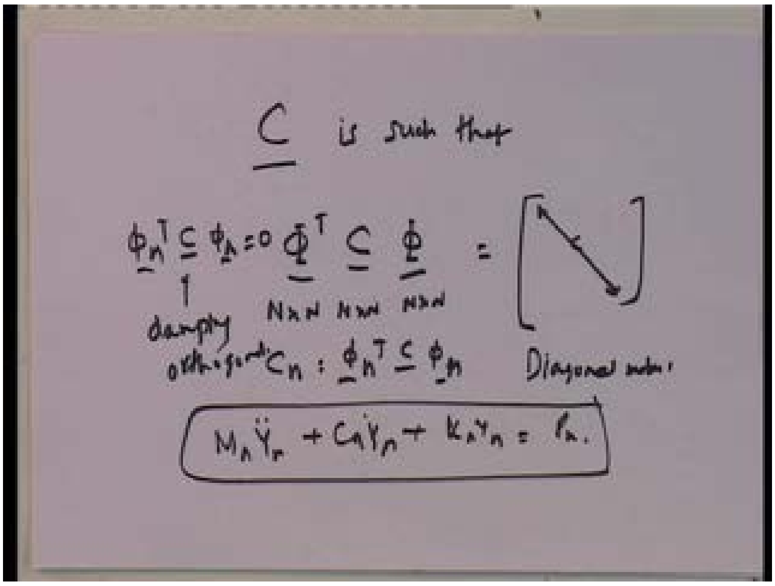

So, therefore, that is where you have a situation where, you need to define C, such that C is such that phi transpose C phi, this is N by N N by N N by N is diagonal matrix. So, that we get M n plus C n plus K n is equal to p n now that K n is equal to omega n square. So, it is the same thing and this C n is this one, so in other words C n is equal to phi n transpose C phi n and phi n transpose C phi M is equal to 0 in other words, it is also damping orthogonal.

So that means, the it is orthogonal with respect to the damping remember, we already showed that, it is orthogonal with respect to mass matrix is orthogonal with respect to stiffness matrix. Now, we are saying that for us to ensure this, which is the uncoupled modal equation, modal amplitude equation, we have to have the damping matrix also that, the mode shape vector also orthogonal with respect to the damping matrix. Now this is the requirement and this is called orthogonal damping or as I said remember, when is the now, you know, these do not the phi and omega are for un damped systems. So, it is only m and k that come into picture, so therefore, now here thus phi n have to satisfy C, it is just that for us to make it, you know for us to ensure that, we get uncoupled modal equations, we need it to be orthogonal.

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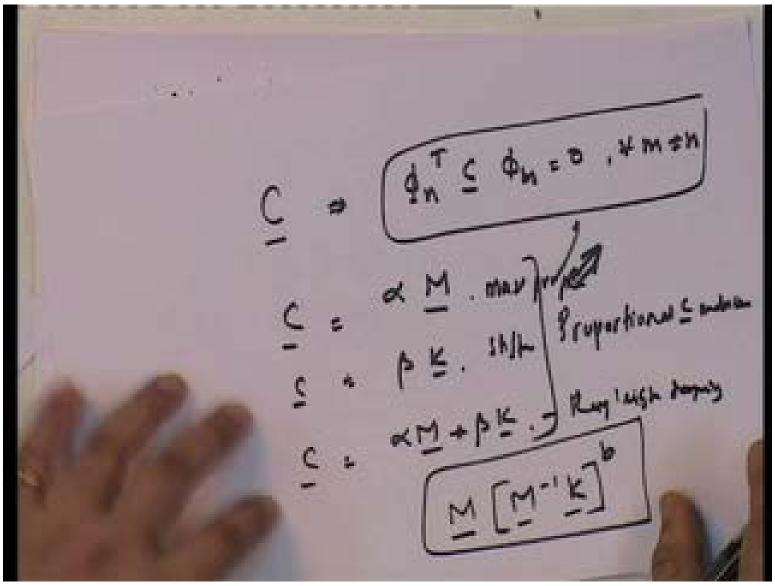

So, therefore, C matrix has to be of a form such that, so once we have that, we see that look the remember, we said that C can be of the form, this form C can be of the form, its mass orthogonal, yes its stiffness orthogonal, we know that, if this is this then we have the situation where, it is a linear combination of mass and stiffness a proportional. So, these are what are known as proportional C matrices, if these are there then this automatically is satisfied automatically, because we know that, you know phi is mass orthogonal stiffness orthogonal, so if they are of this form, we know that, this is a true.

There is another remember, we said that, there is a general orthogonality and I remember, I wrote down the form of the general orthogonality that, it was the mass matrices where, orthogonal with respect to this also. So, I can actually write down another, this is known as mass proportional, stiffness proportional, this is known as rally damping. And we have another C matrix, which would ensure, another C matrix ensure, this and that C matrix is of this form.

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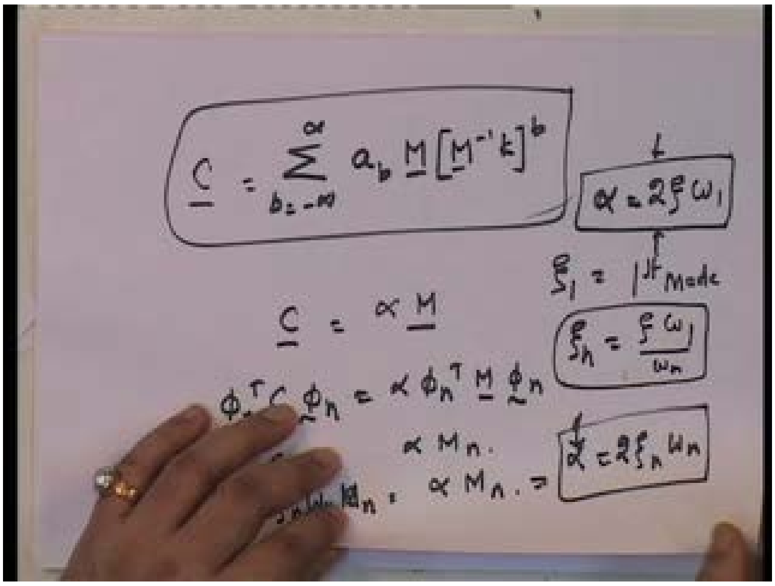

If C is of this form, it is also proportional, because this is we have shown that, this is proportional, I mean in other words that, this is the a orthogonality, you know the this orthogonality condition of this, we have shown. So, if it is a linear combination of all of these automatically, C is also orthogonal, I mean the mode shaped vectors are orthogonal with respect to C matrix. So, let us look at what happens note that the only thing is we can always find out the C matrix, if we can find out alpha beta alpha and beta here and in a more general sense the a b.

So, we can find that all, we have got C matrix, which is consistent with our whatever, we have assumed. So, let us see what happens, when C is equal to alpha M. So, let us look at this, we have a situation where, I am post multiplying and pre multiplying both sides by phi n transpose a post multiplying with phi n and pre multiplying with phi n transpose, so I have done both. So, this is equal to C n is equal to alpha M n and note that C n is equal to 2 psi n omega n C n, which is equal to sorry, M n, which is equal to alpha M n.

So, this implies that, alpha is equal to psi n omega n, note something very interesting and that is the there is only one alpha. So, for it to be K, there is only one for, which it will be valid and so therefore, if I were to look at it, I could say this, that psi 1 is equal to a sorry, alpha is equal to 2 psi omega 1, if I say this note that, this defines alpha. Because, I am saying that look at the first mode, this is psi, if I put that in this become alpha becomes the unique number.

Now, if I want to look at some other, so this is equal to this. So, if I look at it what would be my psi n in another mode, well let us see, alpha is equal to 2 psi. So, it would be it would become then alpha I will put 2 psi omega n 2 and 2 cancel and psi n would become psi, omega 1 upon omega n, note something very interesting, what form does this take, if I note let's look at this is the very interesting point, the interesting point is the following.

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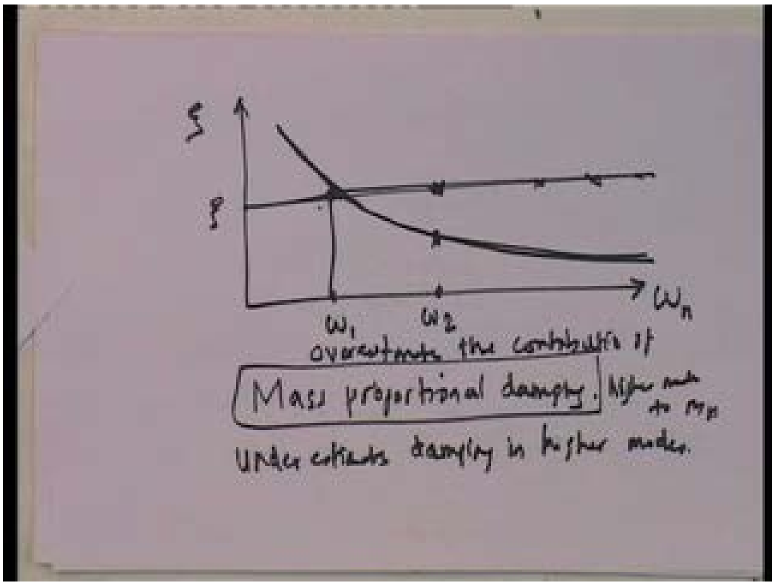

I have omega n and I have psi, suppose I have omega 1 and I will fix psi for that value, so I have it here. Now, if you look at this, now once you fixed alpha is given, so therefore, psi n is equal to psi, which is this value omega 1 upon omega M. So, if I want to look at 2, what would that be, that would be look at this, this would be less right. So, and how would that go, if I would look at omega n, this would become like, so omega n over here.

So, let us say that, omega 2 is to twice, so if I do omega S twice, then psi n is half, if you look at this, this is a constant. So, this is like psi n is inversely proportional to so if I look at it, it would be going like this find, this it is a hyperbola. So, mass proportional damping, what does it do, we know that these have to be here, but if I take mass proportional you know, this is where, I know all modes have to have 5 percent, but once I make it mass proportional damping, it underestimates damping in higher modes.

And once it under estimates damping in high modes, what happens? If I take mass proportional damping, it over estimates the contribution of higher modes to response. Understand that, if you under estimate response, what are you doing low damping, what does it do, it always increases the a accepting for remember in transmissibility actually, you want to be damping, but let us not go there, that is it design, let us take structure.

Structure is being subjected to a arbitrary loading, I assume damping to be mass proportional, I can only fix it in 1 mode and that, typically is the fundamental mode, because that has the largest value. So, I fix it there, then what happens to higher modes, I under estimate, the damping in the higher modes, because you know, you it is supposed to be 5 percent actually, take it 1 percent 2 percent what happens, I overestimate the contribution of those modes to a response right. Because, you know I have to take psi n has to be lower and lower. So, actually, I may actually adding more of the high modes than necessary, that is the problem with mass proportional damping, lets see what happens when, I have stiffness proportional damping, If I have stiffness proportional damping.

Let us see C is equal to beta K. So, if I look at it lets go through, it this is pre multiply and post multiply by so this becomes phi n transpose k phi n and so this by definition is 2 psi n omega n M n is equal to beta and what is this is equal to omega n square m n. So, M n cancelled out and what do I get beta is equal to 2 psi n upon omega n what do that mean, I again beta is 1 value. So, I can fix it in 1, so I can say that, look I am going to say that, beta is equal to 2 psi upon omega 1, once I have that.

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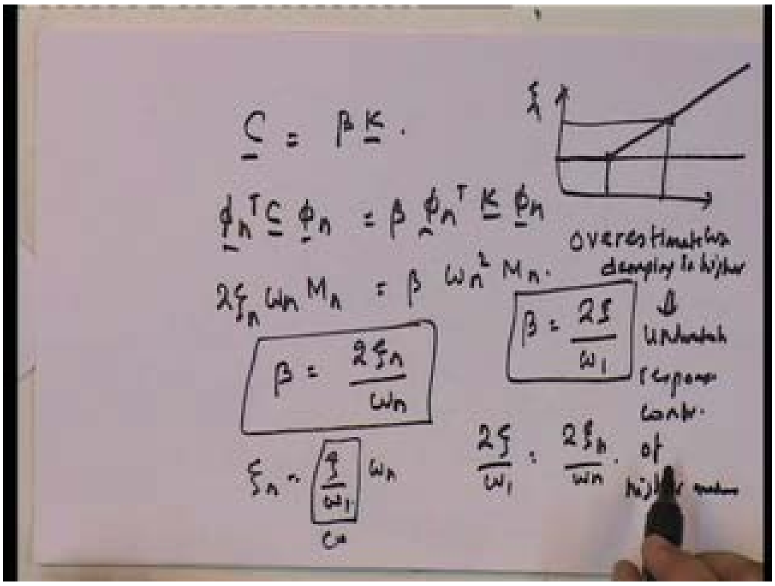

So, that means, this is equal to 2 psi upon omega 1 is equal to 2 psi n upon omega n and therefore, psi n in all the higher modes is equal to psi upon omega 1, which is a constant into omega n. So, if you look at it, if you are going to plot psi n with omega n and I fix it to this, what happens it goes up in a linear fashion, so that means, the second mode, if this is what supposed to be its actually this, over estimate damping and therefore, under estimate in higher modes and therefore, under estimate response contribution of higher modes. You see, so therefore, it becomes very, very interesting, if you have mass proportional, you are over estimating the response of higher modes over estimating the contribution of the higher modes why, because you are under estimating the damping. If you have stiffness proportional, you are over estimating the damping in the higher modes and therefore, you are under estimating the contribution of the higher modes.

So, now let us look at valid damping rally damping should solve that problem right, because it is a linear combination of the 2. So, let us see what happens, phi n transpose C phi n is equal to alpha into phi n transpose, M phi n plus beta into phi n transpose K phi n. So, now, if I will do this, this is equal to 2 psi n omega n, M n is equal to alpha M n plus beta omega n square M n.

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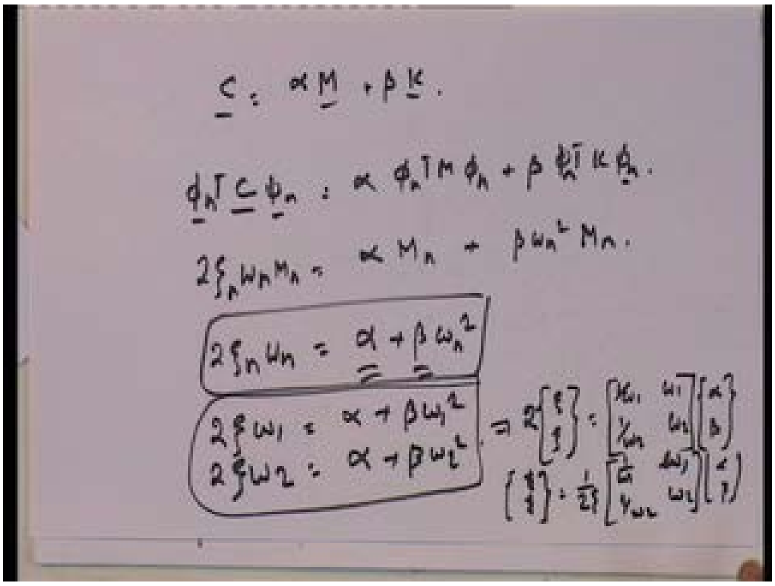

So, let us get a rid of it, what we get is 2 psi n omega n is equal to alpha plus beta omega n square, there are 2 parameters to get 2 parameters, what can I do, I can fix, the damping in 2 modes. So, therefore, what I will do is lets see, what I am going to do, let me assume that, I fixed that thing in 2 modes, so I say 2 psi omega 1 is equal to alpha plus beta omega 1 square and I am going to say 2 psi omega 2 is equal to alpha plus beta omega square, these are 2 modes.

So, let us see what happens, I am going to take this thing out and I am going to take you know the 12. So, what do I get the following, this is implies that, I have 2 psi psi 1 upon omega 1 omega 1, 1 upon omega 2 omega 2 alpha, note that the first equation is 2 psi is equal to alpha upon omega 1 and put alpha here plus beta into omega 1, that is this equation bottom equation is 2 times psi is equal to 1 upon omega 2 alpha plus beta omega. So, this becomes then that I can find out psi psi, so I can actually take psi psi. So, I can put it 1 and 1, I can put 1 upon 2 psi here 1 upon omega 1 sorry, omega 1, 1 upon omega 2 omega 2 alpha beta, I can solve for alpha and beta from here. So, I can solve for it.

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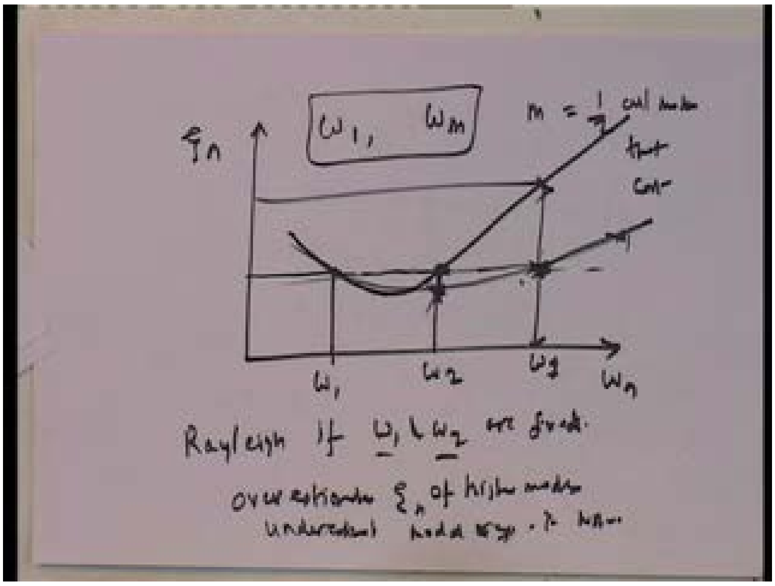

So what have I done lets read look at it lets look at it, when I put this and I draw omega n versus psi n, what have I done, I have fixed it here, I have fixed it omega 1 and I fixed omega 2. So, I fixed these 2, so how does it actually go, if I am going to draw this initially, the mass proportional damping starts thing. And then you see because higher modes, the mass proportional cont contribution goes down and only the stiffness proportional damping remains.

So, rally damping, if omega 1 and omega 2 are fixed, what happens, we still over estimate see we over, we omega 1 and omega 2 fixed no problem psi and psi are there over estimates psi of higher psi n of higher modes. So, this is like omega 3, it over estimates right, this is supposed to be values higher modes, under estimates modal response in higher modes.

So, in other words in this process, what are we doing, if we do this, now the there is another option, the other option is just of omega 2 you do, some other omega n, but then what happens is you know, you fix this all that happens is that, it will become it underestimates the middle ones and omega over estimates the higher ones. So, you may say that look, I am under estimating a few over estimating a few, which is why actually in a way rally damping has been used for so long. The reason rally damping has been used is that instead of omega 1 and omega 2, you actually take omega 1 and omega m.

The 1 omega m is not really the omega m, actually this m is approximately about half of all modes that contribute, because then what happens is, when you know, if you take that mode. Let us say 4 modes contribute and I take omega 3, if I take omega 3, what is happening is omega 1 and omega 3 are at 5 percent omega 2 is un damping is under estimate. So, its contribution is over estimated and then the higher modes, you know omega 4 and all, there is a damping is over estimated and therefore you know under estimate other response.

But, that is you know in some way people are saying that, underestimate a 1 or 2 modes and you over estimate some 4 or 5 modes, in a way the total response, if you put it together, it is how its good. You know in essence it sounds very good in reality, see what are we trying to do, you need to understand again that, I am trying to see that, If I use what is known as the direct integration procedure, you know, which is m plus m v double dot plus C v dot plus k v is equal to p, I can solve that equation.

Using the direct integration procedure that, I establish a for single degree of freedom go back all of that happened is instead a single equation, we will get a set of simultaneous equations, which we will solve for does not matter. Direct integration is m v double dot plus C v dot plus K v is equal to p, you solve it directly, this response for a linear system has to be identical to the response where, we assume psi in every mode.

So, these 2 have to give the same answer, so you have to have a C, which kind of give similar results and that is what, we are saying that, you know, if you do some omega m where m is half of all the modes, you consider. You under estimate a few you over estimate the others and you say that it acts but note that the higher modes responses are typically lower.

So, when you are under estimating some modes, you actually boosting up, you know that middle ones, which you will see that in reality, your response to the direct integration procedure will turn out to be slightly more than, that from the modality, if you use rally damping. If you use, if you use mass proportional stiffness proportional is a disaster, because you can only fix in 1 mode, you have always fix it in a fundamental mode, so either you hugely over estimate hugely underestimate.

Rally damping at least it logically seams to make sense, but I just told you, that underestimating some over estimating others when, you are over estimating the lower modes underestimating the higher modes, it does not help, they do not balance each other out, because each mode contributes differently. So, how do I solve this particular problem.

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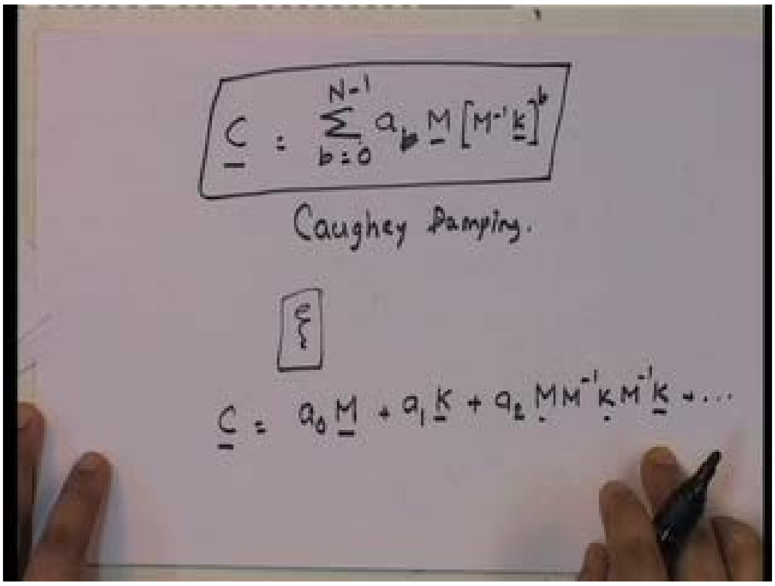

The problem is by considering C to be given as let us say, I am going to say b going from 0 to N, I consider N a series remember, the this is goes from minus infinity to infinity, but I will consider only n of those. And I will say that look a b MM inverse K into b, if I take this note, what happens, this procedure is was developed by Caughey and that is why it is called Caughey damping. Rally damping 2 modes, Caughey damping N modes how many unknowns do I have, so look at it sorry, this is a n minus 1, because 0 is also there, so this N minus 1, so you have N unknown a b.

Now, in principle this looks beautiful, because I can fix N modes, I have only to define N psi and I am done, but how do I get the a b s and that is the question that comes up. So, let us see, I know that, if I were to take this, so this in a sense, if you look at it basically becomes the following, C becomes equal to a 0 m plus a 1 K plus a 2 M M inverse K M inverse K and so on and so forth. So, this becomes my solution space, so let us see what happens in such a situation well.

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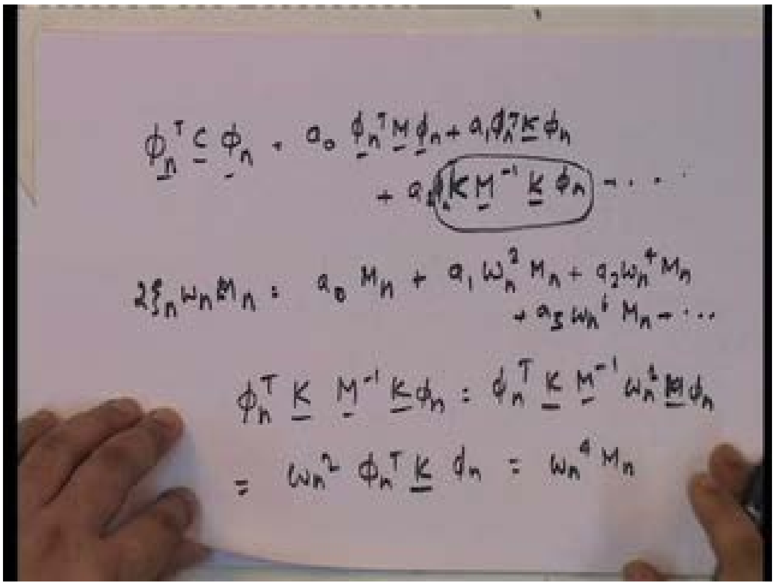

If I were to make get phi n transpose C phi n, I get a 0 phi n transpose n phi n plus a 1 phi n transpose K psi n plus a 2 M into M inverse, which is i. So, this becomes K M inverse K phi n phi n transpose plus, o, what does this become this becomes 2 psi n omega n M n is equal to a 0 M n plus a 1, this I know is omega n square M n, what is this, let us see what this one is equal to lets see what this one is and if I look at this phi n transpose K M inverse K phi n is equal to phi n transpose K M inverse into omega n square M phi n.

So, if I put omega n outside, what do I get omega n square phi n transpose K M into M inverse is this. So, this is this, so this S equal to omega n 4th M n so if you look at it. So, this becomes a square omega n 4th M n and in this way actually, you can show that, this is equal to omega n 6th M n and so on and so forth. So, now, the number of and this goes up to a n minus 1, so that what does it become it becomes a n minus 1 omega n twice n minus 1 into M n this is what we get. So, now, if I have to plot this, I have how many, I have n number of unknowns.

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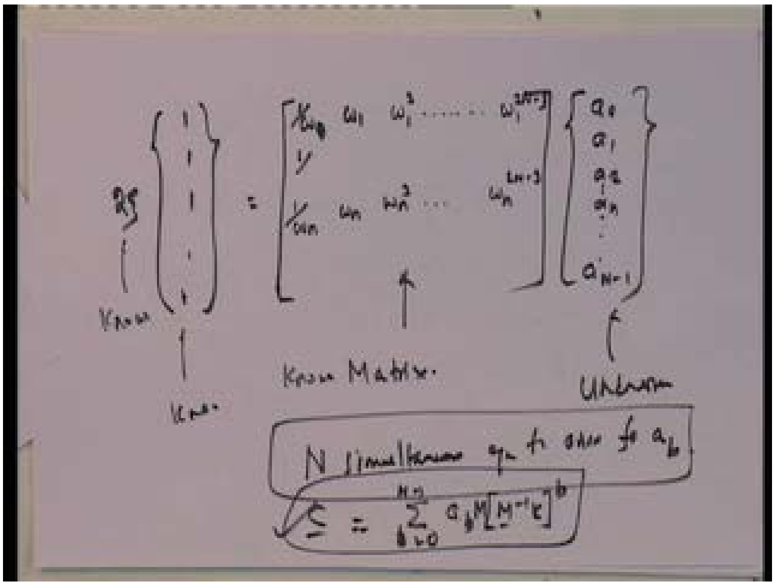

So, I have to specify it n of those, so if I specify in n of those, what do I get the following and I am going to take the omega n this way. So, that I get equal to a the following 2 psi 1 1 up to 1 n modes is equal to let us see a 0 a 1 a 2 all the way up to a n minus 1 and what are the values, this is 1 upon omega ya 1, the first mode omega 1 omega 1 cubed so on and so forth up to omega 1 a to n minus 3. So, similarly I can put and u know same way and if I look at the omega nth time, it will be like this omega n squared and that is my a n is going to be equal to omega n to n minus 1.

Now note that this matrix, I know because I know all omegas up to omega n, I know this, this is the known matrix, known this is known given this is known this is unknown. So, what is this, this is the actually n simultaneous equations to solve for a b, once I solve for that, once I solve for that, I have got my equation, because C is equal to summation b going from 0 to n minus 1 a b M into M inverse K to the power of b, I found out my a b, I can find out my C matrix. The advantage of the Caughey theories is the following, that once you have the Caughey series.

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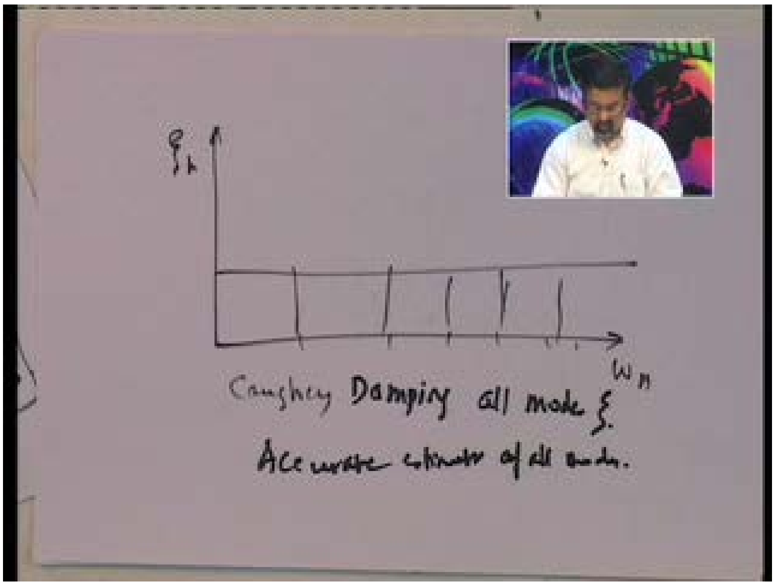

If I were to plot my since, I have forced all of them all of them will be, so Caughey damping all modes psi accurate estimates of all modes and therefore, the ultimate thing is that, if you use Caughey damping. So, finally, let us draw the final conclusion from the damping is that and that is its very u n, you know otherwise becomes very complicated.

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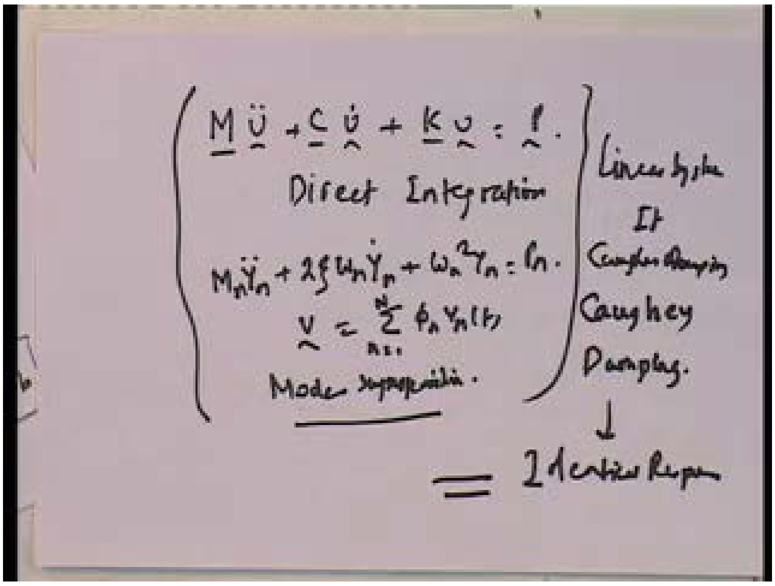

I have 2 approaches M V double dot C V dot pus K V is equal to p and this is direct integration I mean, so I have a linear system. This is the direct integration procedure a just please understand that, the direct integration procedure is exactly the same as what I have, I am going to repeat its exactly the same as what we have developed, remember that time step well instead of a single value, we always say vector, so thats all.

So, it just becomes a simultaneous equation simultaneous and other is the M n Y n double dot plus 2 psi omega n Y n dot plus omega n squared p n and then saying that, V is equal to summation n going from 1 to n of phi n Y n of t, which is mode superposition. If we use Caughey damping where, we enforce that psi is the same in every, these 2 methods will give, if Caughey damping.

Let me just write it down again, Caughey damping identical response, if you use mass proportional, if you use stiffness proportional, if you use any you know all rally damping you are not going to get the same between the 2, because if you use mass proportional damping, it will overestimate the response from the higher modes. So, therefore, this one will give, since its over a overestimate the response this one will be typically higher than what you had from here, if you use stiffness proportional, you over estimate a the damping and so you under estimate the response.

So, this one is going to be lower than this and if you use rally damping well, it becomes little bit more complicated, but this one and this one at definite not going to give you, the same result. And so only Caughey damping will give you the same result. Please understand that, so therefore the concept of all of these, you know the question of proportional damping.

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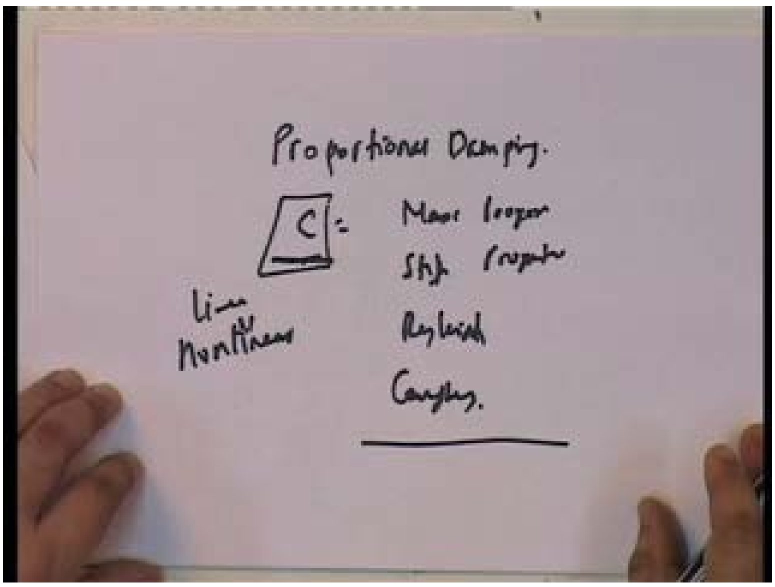

And the fact that, you know your C is mass proportional stiffness proportional rally or Caughey all is relevant only, if you have a system where, you require C, which is when you go non-linear. You go linear and then non-linear only then you require, this if you are dealing only with a linear system understand that, if you are only dealing with a linear system, only linear system all this discussion that.

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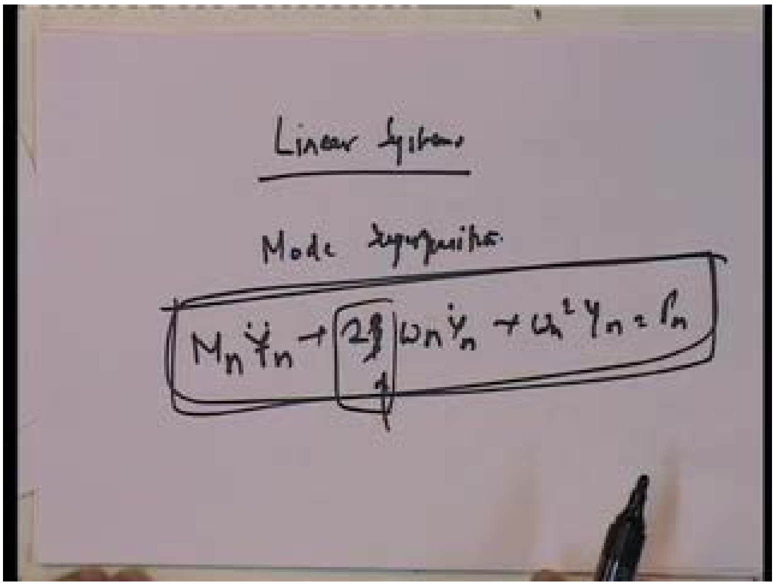

We have having this lecture on mass proportional stiffness proportional all of these are irrelevant why, because you will always use mode superposition. And if you use mode superposition then this is the problem that you solve, this is the problem that, we solved where you take psi as the same in all the modes, you take psi as same in all the modes. If the psi is the same in all the modes then please understand something that, who cares in a way you may say that well by taking psi, I am assuming coughing damping, you can say that, does not matter, but you do not have to you do not have to evaluate a b.

You say that, you considering Caughey damping, but that is also irrelevant who cares, you are solving modal amplitudes and you are assuming psi to be the same in all the modes, this is an assumption, it is an assumption. So, to consider damping, it is not relevant to consider whether it should be mass rally Caughey all of these discussions come into picture only when you have to ultimately look at non-linear response.

And you want to ensure that, linear and you know the linear domain, the 2 methods give you the same result, that is when all of this thing of Caughey rally mass stiffness propose or proportional damping becomes relevant. If you are only dealing with a linear system, there is no need what, so ever of looking at what kind of damping, you just assume it in this particular case do you understand that.

I am going to stop over here for today, because this entire discussion was on how to consider damping in multi degree of freedom system problems, I hope I have been able to get across the point of how to consider. But, again at the end I said that look it is not relevant, because if you are using a linear system and mode superposition, you define psi I to be the same in all the modes and jut solve that is all you said 5 percent damping. So, 5 percent damping, so 5 percent damping in all the modes, you consider solve it and go ahead and go ahead with it. So, I hope I have been able to explain, a little bit of how to consider damping for multi degree of freedom systems from next time onwards, I shall look at the specific case of earth quake loading.

Thank you very much, see you next time, bye.