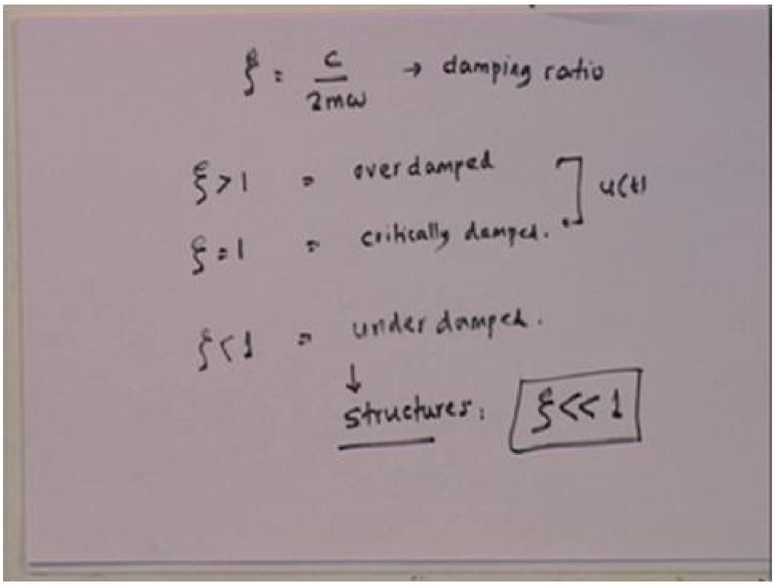
Structural Dynamics
Prof. P. Banerji
Department of Civil Engineering
Indian Institute of Technology, Bombay

Lecture - 3
Dynamics of SDOF Structure

In the last lecture we discussed the response of an un-damped system and we found out that there was an issue related to un-damped system, because it was a conservative, and also it could not model the energy loss, and also we introduced the concept of viscous damping in the structure, and from that we obtained this equation. We saw that the parameter C, which is the dashpot constant C upon 2 pi omega is a critical damping ratio and if we take the ratio of the actual damping to the critical.

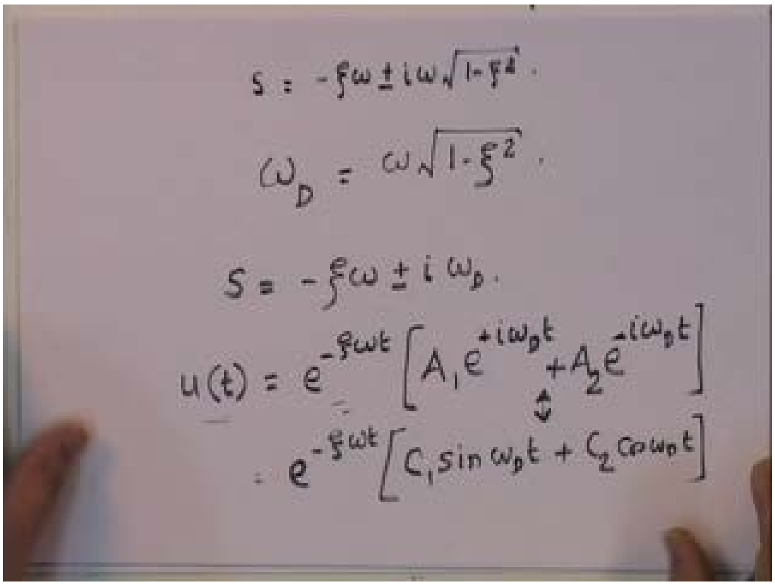
(Refer Slide Time: 01:40)



Damping, which is C upon 2 pi omega we just define the parameter called x I, and we call that although that should strictly that be called ratio of the viscous dashpot constant to the critical dashpot constant. We specify it as viscous damping ratio or damping ratio, because if we modulate as viscous damping the energy loss viscous damping then we just call it as damping ratio.

So, the parameter zeta, which is equal to C upon 2 m omega is the damping ratio, and e saw that x i was greater than 1. It was over damped over critically, which is over damped x i equal to 1. It was critically damped, and we got a qualitative assessment of the response, and we saw that in both the over damped system, and critically damped systems you essentially got one directional response. So, there was no cyclic response once. So, there was no vibration and the we say that look for x i less than 1, which is under critically damped or as we say under damped all structures exhibit under dumped. So, there for since, we looking at only structural dynamics we are strictly interested in x i less than 1 kind of a situation and. In fact, in structures it seen that x i is actually significantly less than 1, we will see what that entails let us now try to solve the free vibration equation for under damped system and this I will go through in detail. So, let us see, what does it becomes.

(Refer Slide Time: 03:42)

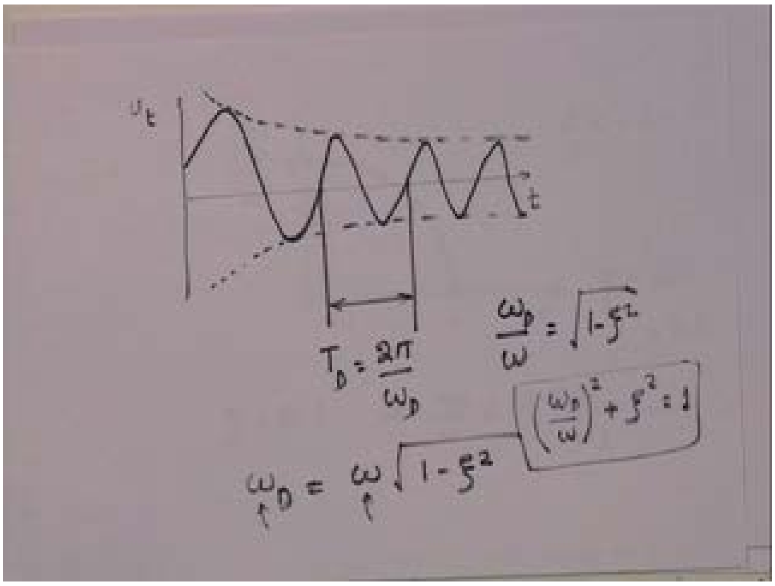


If you think back became minus x i z plus or minus i omega into 1 minus x i square. So, if this is the situation I am going to define a parameter omega D, which I will call by omega 1 minus x i square. Then s becomes equal to minus zeta omega plus minus i omega D. So, the u of t essentially becomes e to the power of minus x i omega t into A 1 e to the power plus i omega D t plus A 2 e to the power of minus omega D t, let us assume that we ignored this term, see if you look at this is the solution right get this plug.

It in you get put it in you get e to the power of minus x i omega plus i omega the whole terms becomes D. So, that the whole term becomes minus t. So, let us forget this term, if you look at this is identical to what we had got for the undamped system accepting that there we had omega, and here we have omega D, and without going to the entire process all over again since the r you know complex functions, and this is the real function this real.

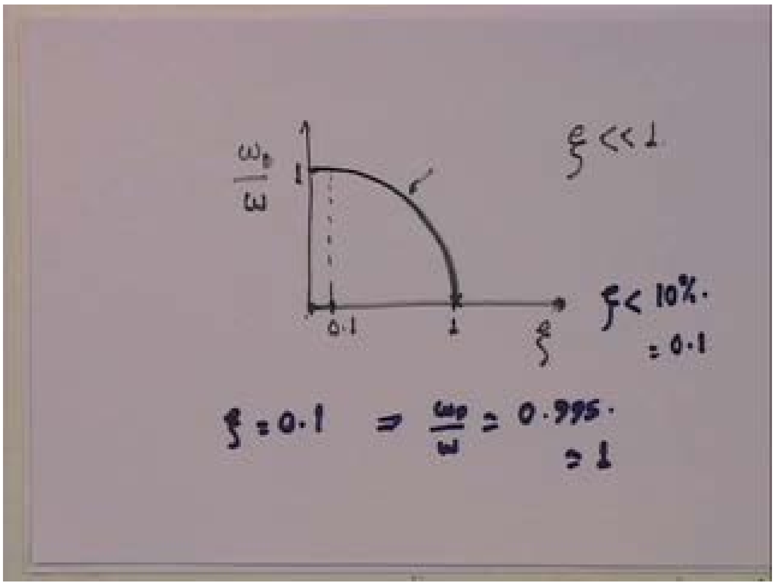
So, only this is the complex, we know that this will actually be of the form you know. I already define this in the last lecture the last lecture before last and this will essentially becomes, because these have to add up the real function the same thing. So, this basically becomes this. So, from here to here the procedure is exactly the same. As we got m u double dot plus k u equal to 0 term. So, these this going from this exponential to this I am not repeating the same thing that I have already talked about last time. So, essentially what does it look like qualitatively look qualitatively.

(Refer Slide Time: 07:05)



This looks like there is a time u of t. if you look at this if you look at this term, then is again identically it is an harmonic function. So, it is an harmonic function accepting that what you had is a harmonic function modulated by exponentially decaying function. So, if I plot this the harmonic function is modulated by an exponentially decaying function this is what we have and; obviously, T D, which is 2 pi by omega D. So, in a sense what we are saying is that this is the time period, and the only difference here from the previous undamped situation is that omega D is omega into one minus x i square this was the natural frequency for an un-damped system. This is the natural frequency for a term, and relationship between these two if you look at it is what let me plot this omega D, upon omega is equal to square root of 1 minus x i square is basically means omega D, upon omega square plus x i square is equal to 1.

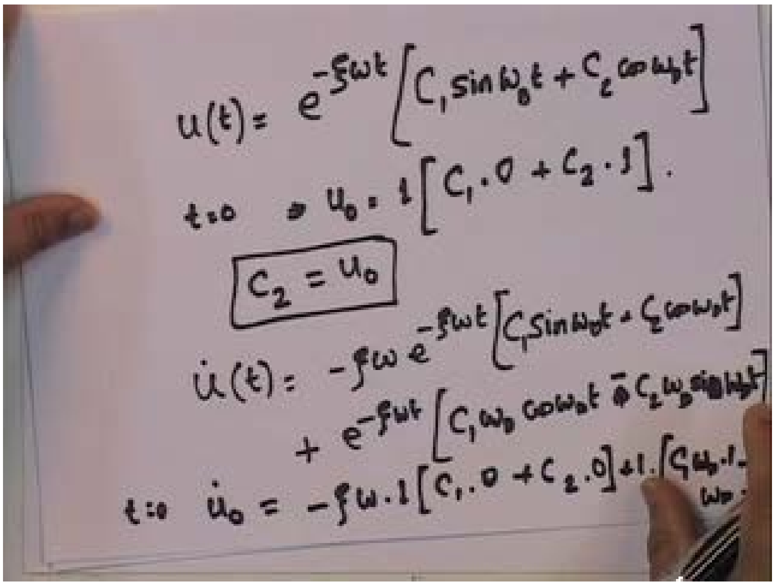
(Refer Slide Time: 09:31)



What is this? The equation for this is the equation for a circleonly problem is that x i is positive by definition. So, is only from here to here omega D and omega are both positive by definition. So, what we actually have is if we look at it this is one, this is one, and this. Defines remembered the words when x i equal to 0 omega D by omega is equal to 1 well; obviously, x i is equal 0 you have an un-damped system and omega D and omega are the same the damped system and the un-damped system frequencies are identical when x i is equal to 1 which is critically damped system what happens to omega D 0 why well we saw right critically damped system you do not have any vibration we do not have any vibration, what is omega what is omega D the damped system actual frequency if it do not vibrate damped frequency of natural frequency of the damped system is 0. It does not vibrate.

So, that is what we have here. So, there for essentially what we have is that, I say it I say it something that most structural damping x i is very much less than one in other words x i is in the zone. The probable x i in the zone if I look at it let me take this as 0.1 if I take 0.1 substitute into this what do I get I get for zeta for zeta equal to 0.1 for all practical purposes one. So, as long as x i less than 10 percent 10 percent implies 0.1. the damped frequency and the un-damped frequency are practically the same. So, there is almost known for in structurally damped system, because most structurally damped systems have less than 10 percent damping most structural systems have less than 10 percent damping. in which case this frequency or the time period is identical the damped system and the un-damped system are practically the same for all practical purposes. So, in other words all that happened is due to damping is this exponentially decaying, and this in essences represents the energy loss in a system. So, what we are doing is we are actually defining the energy loss in the system. So, now, now the question becomes remember I said I could not find out C for a structure can I somehow find out x i can I find out x i before I do that let me actually solve it because it is in its an interesting to actually solve for the system.

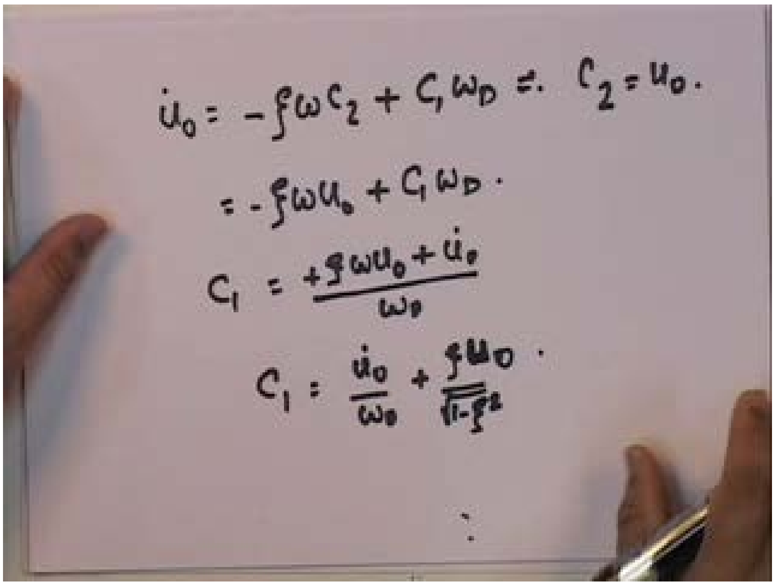
(Refer Slide Time: 13:53)



So, let me go back to my U of t is equal to e to the power of minus zeta omega t C 1 sine omega D t plus C 2 cosine omega D t, and now what I am going to do is substitute T equal to 0. So, what we have is u 0 is equal to 1 into C 1 0 plus C 2 into1, what we have is C 2 equal to u 0. We get this we get this solution next how do I find out C 1 well differentiate now I going to differentiate two terms right. So, first I will differentiate this term. So, this will become minus zeta omega e to the power of minus zeta omega into C 1 sine omega D t plus C 2 cosine omega D t plus, now I will differentiate this one. So, this becomes e to the power of minus x i omega C 1 omega D cosine omega D t plus C 2 sorry minus C 2 omega D sine omega D t. So, this the two term I differentiated both terms.

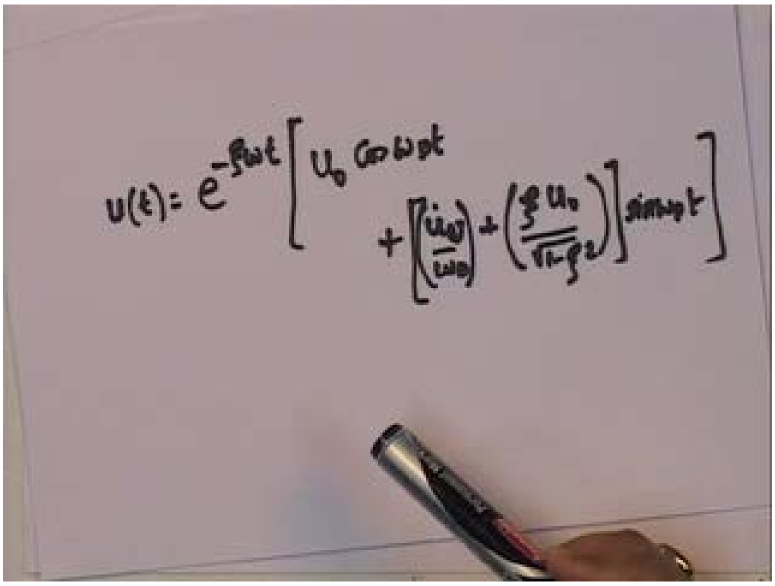
And now I am going to put t equal to 0. So, what I am going to get u dot 0 is equal to first. look at this here we get minus x i omega this is 1 into 1 plus C 1 into 0 plus C 2 into 0 plus 1 into C 1 omega D 1 minus C 2 omega D 0. So, let me just rewrite this I will rewrite this entire thing in a in a proper way write in the proper way ,what do I get I get.

(Refer Slide Time: 16:52)



U dot 0 is equal to I am. So, sorry here you do not have x i 2 it is its one. So, you get minus x i omega C 2 plus C 1 omega D is equal to u 1. So, we know that C 2 is equal to 0. So, let us substitute that in here. So, this becomes minus x i omega u 0 plus C 1 omega D. So, what does C 1 become C 1 becomes x i omega u 0 plus u dot entire thing divided by omega D. So, if you look at this is becomes omega we have already we are seeing that omega D upon omega is equal to square root of 1 minus x i square. So, this basically becomes then C 1 becomes u dot upon omega D plus x i omega sorry x i u 0 upon 1 minus x i square.

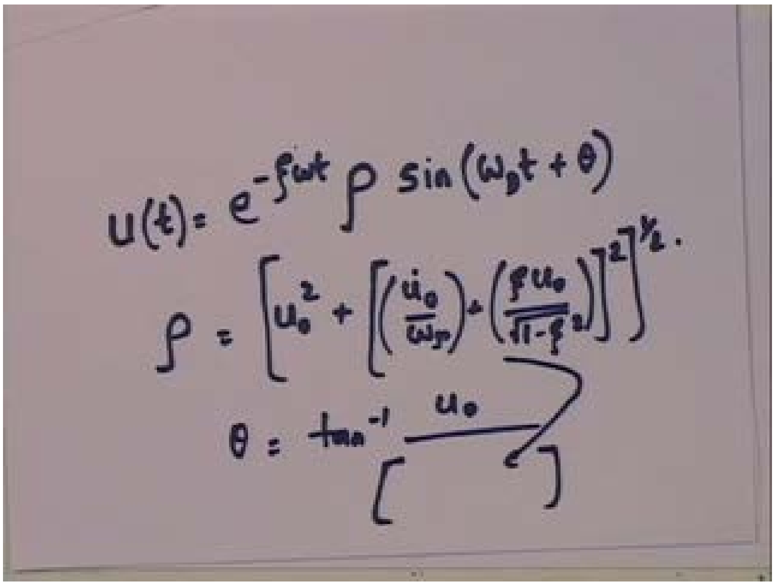
(Refer Slide Time: 18:43)



So, having plug those in what I get alternately as my solution is u of t is equal to minus x i omega t plus u 0 cosine omega D plus u dot upon omega D plus zeta u 0 upon square root of minus zeta D sine omega D a fairly complex aha equation isn’t it. So, what understand that t that last time if you looked at the solution what did we get we got for the un-damped system u 0 cosine omega D plus u dot 0 upon omega into sine omega D t right, because of the since its x i ,which represents the damping is non zero here.

Now, you plug in plug into this x i equal to x i equal to x i equal to 0. if you plug in for x i equal to 0 this becomes 1 omega D b comes omega. So, becomes u 0 cosine omega t plus u dot 0 upon omega this term disappears sine omega t that exactly, what we got in the un-damped system. So, you see the damped system few vibration response actually includes the un-damped system response, but just substituting x i equal to 0 we can get the solution and if you look at it you know I have just got you the expression, but you know the expressions do not tell you about anything other than the fact that this essentially is a simple harmonic motion modulated by an exponentially decaying function.

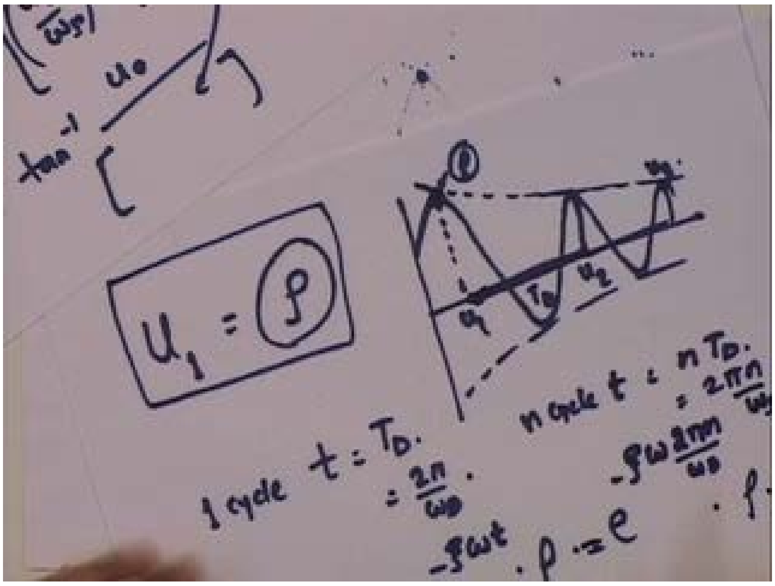
(Refer Slide Time: 20:51)



.

I can rewrite this as the question becomes that what is rho is a amplitude. Rho is equal to that is the amplitude rho and what is that theta, this term if you look at this particular thing this is u of t and so if you look at the maximum response the maximum response is nothing but Rho.

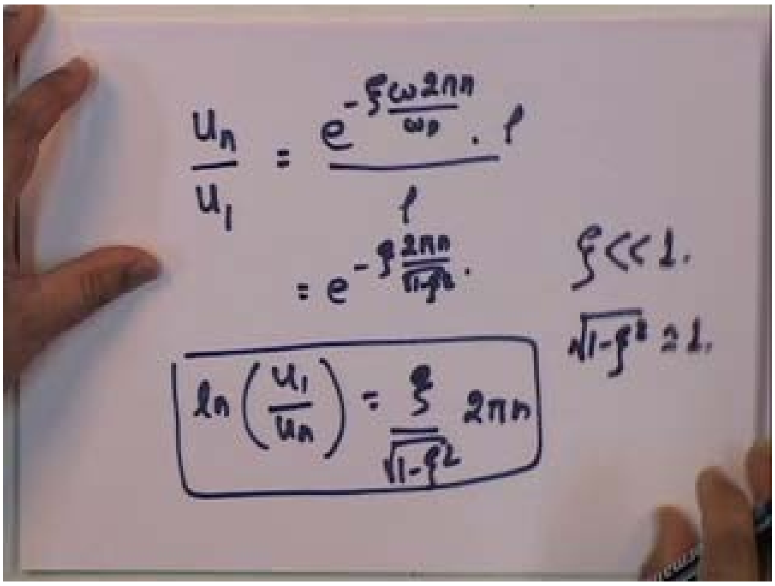
(Refer Slide Time: 22:34)



Let us take a situation, where this is my u 1, in other words I am tying that loop that if you look at it this is what I am saying that this is the in the sense something like this something like this it is starts go like this then starts. So, this if you look at it at this point this is rho, and I am calling that as my u one. So, I take this peak and this is my u 1 after one cycle the one cycle is what up to this point. it is one cycle right that is you see is T D this is u 2 then after threes cycle this is u 3 and. So, on after every cycle how much is one cycle one cycle, which is equal to 2 pi upon omega D. So, let me look at u n. So, if you look at this is the solution what I am saying is that this is this is one you I mean at that instant time by rho, and this is one that is what this is point.

So, there for this will always be 1 after every T D, so this will always be 1. So, if I look at u n. it will only have e to the power of minus zeta omega t I left define t is. So, n cycle t is equal to n T D right. So, if I take that instant of time as 1. So, then this one is going to be , and to rho and now t is equal to T D. So, I am going to put in n TD over here. So, this is equal to minus e to the power of minus x i omega and t is n T D, which is 2 pi n upon omega D. see n cycle t is n T D T D is going to be equal to 2 pi n upon omega D. So, this becomes u n becomes this. If this is rho then this is equal to this into rho if I take.

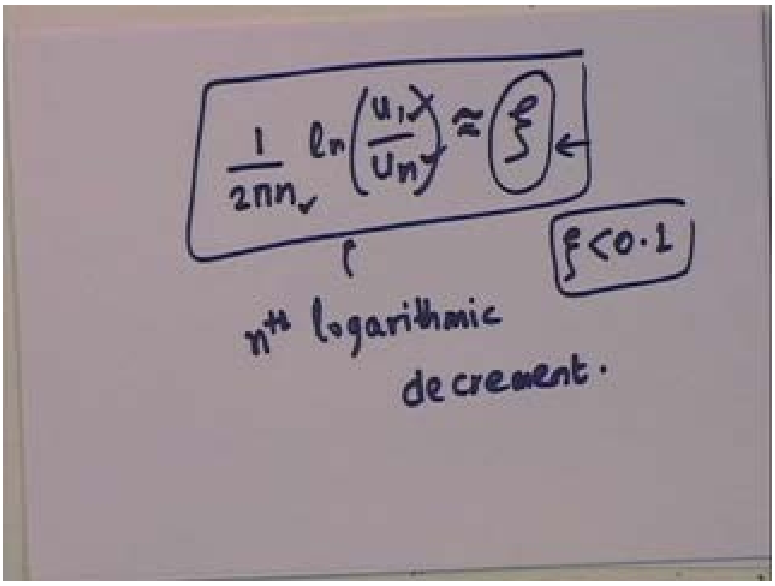
(Refer Slide Time: 25:56)



If I put upon u 1 this is going to be equal to e to the power of minus x i omega 2 pi n upon omega D into rho upon rho. So, this essential become e to the power of minus zeta what is omega D r upon omega this is 2 pi n square root 1 minus x i square rho cancels out.

And. So, I can say that look l n of u 1 upon u n is equal to x i upon into 2 pi now typically x i is very much less than 1 square root of 1 minus x i square for all practical purposes n and. So, there for we have is that.

(Refer Slide Time: 27:14)

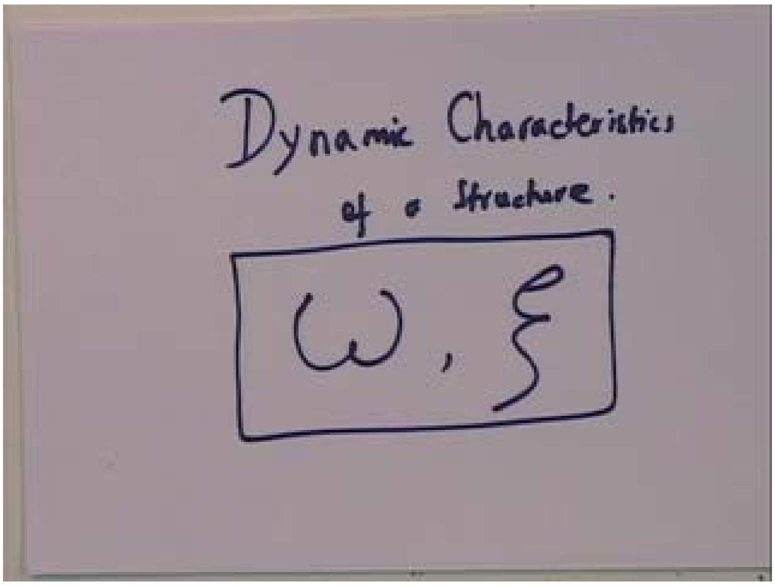


One upon 2 pi n into l n of u 1 upon u n is equal to x i is approximately, because its approximation is I have taken to 1 approximately. So, as long as x i is less than 0.1. This equation is good enough now this for n equal to 1 this is known as a logarithm occurrence. So, this in a way is an nth logarithmic decrement. So, you see now can I obtain x i.

Experimentally well. I can let me set the system into vibration and then you now this is , where my rho is. So, I just take my first point of time as this point, and then I go through an tickles and come back here and measure this from here to here. So, the original was this one and after end cycle is at this point. I can measure those, I can measure I will I will show I said again later on in this during this course you going to be going to be lab and you will actually be looking at this and you can get this and you can get this and this is something that you have taken.

How many cycles of vibration have you taken and based on that vibration of cycles you can directly get x i. So, experimentally we can measure x i for any structure. So, here is the first one experimentally, we can get the mass of the structure experimentally, we can get the key of the structure and experimentally, we can get x i of the structure, and if you look at it the dynamic characteristics.

(Refer Slide Time: 29:46)



now I want to come back to of a structure, and I am going to talk about this dynamic characteristics of a structure over and over again in this course these are the natural frequency of equation note that you know you ask me why omega why not omega D well you see omega D is equal to omega into square root of 1 minus x i square. So, if I if I define my natural and you know we have already seen that for structural damping, where damping is typically less than x i less than 0.1, and you know another thing we do not really talk in terms of 0.1 0.2 0.3 we talk in terms of 10 percent of damping 5 percent of damping in 5 percent damping means 0.05 I 2 percent damping x i equal to 0.02, so for structures.

Damping is almost always less than 10 percent and if it is less than 10 percent omega D and omega are identical. So, there for we do not really talk ever again of damp frequency, because for structures damping frequency are not cannot be measurably separated. it can be mathematically separated you know omega D is not omega D is omega into square root of 1 minus x i square, but u remember put in 0.1. I told you put in 0.1 what happens x i square is 0.01. So, 1 minus x i square is 0.9 square root of 0.9 is approximately about 0.9 0 and 5. So, omega D upon omega is 0.9 9 and 5 you cannot measurably separate it 99 percent the natural frequency is 99 percent 0.5 percent of omega measurable there is no measurable difference, so measurably omega D and omega the same. So, there for we say we measure whenever we actually measuring omega D , because all that is omega because as long as I 10 percent.

So, therefore, omega all omega D, whichever you choose is, but in I prefer natural frequency of vibration of the structure and x i, which is the damping ratio these are the dynamic characteristics of the structure, and these determine how can I determine these experimental sure. I can I can find out how much time one cycle takes that is my time period omega is 2 pi upon the time period. I can measure that oaky x i. I can measure it logarithmic decrement I am measuring. So, both of these are measurable quantities for any structure. In fact, any building you can actually you know, it is very difficult you can do a numerical model of a building and then get the k matrix and m matrix etcetera, but you can actually measure it you know hit the building you need to hit the building till any normal building you really need to hit it with really heavy system, but let us assume that I have a system by which I can hit the building once.

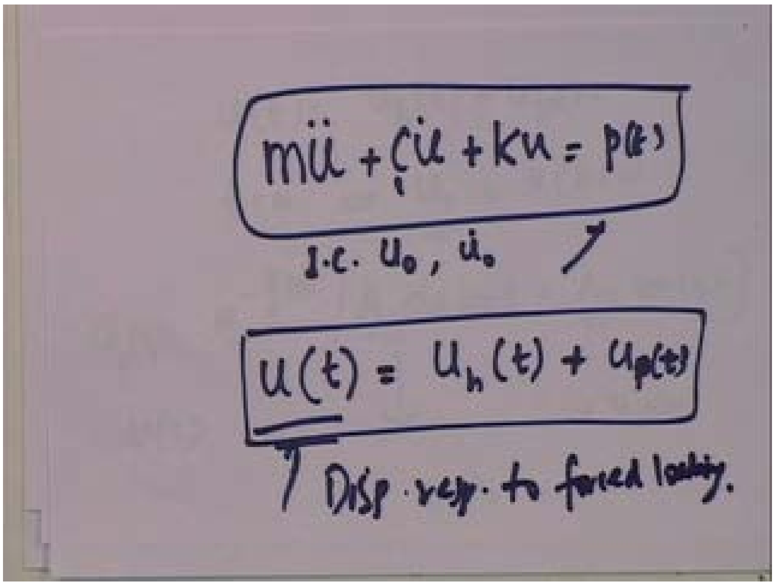
I hit the building and I am making it go, and it is in vibration and I can get the frequency of vibration, and I can get the x i from the logarithmic decrement. So, these are measurable quantities and these are known as the dynamic characteristics of the structures. You see as we go proceed a long that these omega and x i are parameters that determine the dynamic response of a structure to any kind of learning. So, when we say well we will talk about this a little bit later, but this omega and x i; these two terms determine the response and these do not need to be define this can be at least obtain mathematically, because k is a structure I mean k you can measure or you can you know numerical model m. you can measure the numerical model and omega square root of k upon m for single degree of numerical system.

So, you can actually numerically model. This, but this you cannot numerically model this can only be obtained experimentally, and this can also be obtained experimentally we do not need a numerical model. So, this now defines all the characteristics of a realistic structure, which models the energy loss, which models the vibration of free vibration of the structure. So, now, these in essence is the vibration equation for again. I want to go back will show you what structure response looks like when subjected to initial displacement this is what the response looks like u 0 initial displacement initial velocity. If you do not give an initial displacement you only give initial velocity. if you do not give an initial displacement you give only initial velocity then these two terms will disappear and this becomes just e to the power of minus x i u dot 0 upon omega D into sine omega D T.

If you do not give it an initial velocity then the response becomes u 0 cosine omega d plus x i u 0 upon square root of 1 minus x i square t anyways to I mean these are do not matter significantly, but the response can be guard and from the response we get from the dynamic characteristics, which are omega and x i this in essence completes our free vibrations equations of a structure. Now the question is becomes suppose now you know. I just want to go back 1 point and that is here you know I meet this approximation that this is approximately equal to 1 neglect this, and get you now directly from here. I can get this solution right what is zeta x i is not equal to 0.1 what happens well the way we do it we obtain x i from here compute this term and iteratively, and now you know this you can get start getting this.

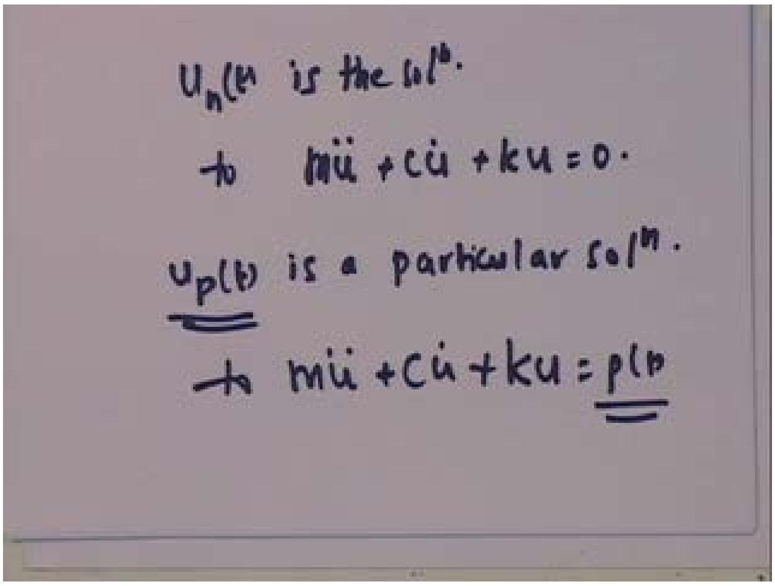
And now you know this will not be satisfied for anymore get a new x i. new x i this term come close you know to solve iteratively x i. So, the initial x i is obtained in this way and then if x i less greater than 0.1, then it need to go back and get the x i iteratively solving for this. So, that is all there is nothing great that much you can always get x i from the free vibration. So, what I will do is I am I am going kind of discussion on free vibration to at this time, and we will we will go back and look at actual you know the problems we will look at the labs I will start up actually with the lab and then look at how to solve problems form the lab. So, now, what I want to do is for the little bit of time that I have left for this particular lecture I am going to start looking at free. Vibration looked at omega zeta now we have to see how do we solve and…

(Refer Slide Time: 38:36)



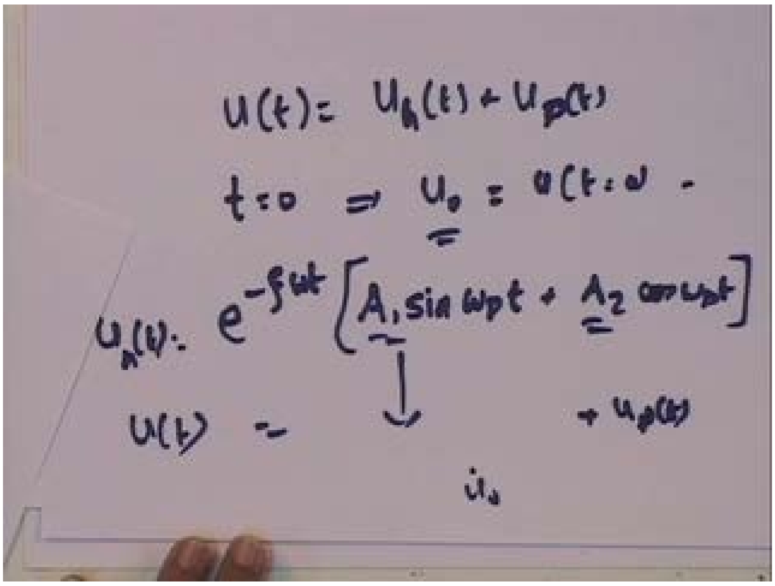
Now, I am going to introduce this in, because this actually you know from if you know x i you can actually find out C, but any way it does not matter oscillator no x i as long as we know x i. So, this is the equation this is the equation that I need to solve for any dynamic loading that, I have how do I go about it just want to spend the next 10 to 15 minutes. Looking at how do I solve this one essentially going to review solution linear second order differential equation. So, how do we solve this and this is of course, with u 0 nu dot 0 the initial condition how do I solve this well mathematics says that u of t is a sum of u h of t plus u p of t, where what is u h of t u h of t the homogeneous path.

(Refer Slide Time: 40:14)



U h of t is the solution to m u double dot plus C u dot plus k u is equal to 0, and u p of t is a particular solution to m u double dot plus C u dot plus k u equal to p. So, p is given by the form of p of t this is the particular solution given p of t. we can actually define a u p of t, which can solve the equation with nothing else solved and u h of t is a solution of this what is this is nothing but the free vibration equation that we already solved. And then what they say is that at once you have got the homogeneous solution, which is the solution to this we have already know it and you have already got it u p is a particular solution, which depends on the form of p of t u p of t is very specific. it is a particular solution of this equation, where u p depends of the form of p, and once you have got that then what we need to do is we need to say that ok.

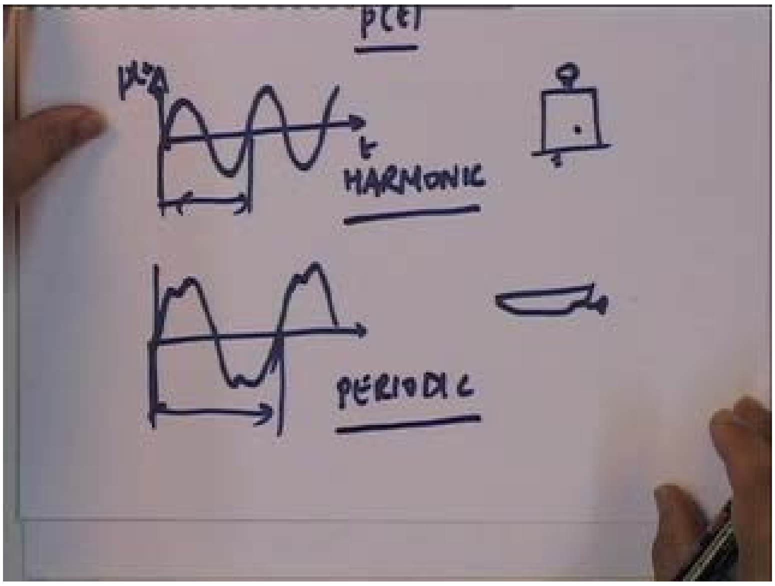
(Refer Slide Time: 41:55)



U of t is equal to u h of t plus u p of t and you know substituting in t equal to 0 you can get you zero is equal to u t of 0 and. So, there for from that the remember u h of t is what u h of t is and. So, u of t is this plus the particular solution, which is completely solved even this has no unknowns, because this has the particular solutions which represents u p of t.

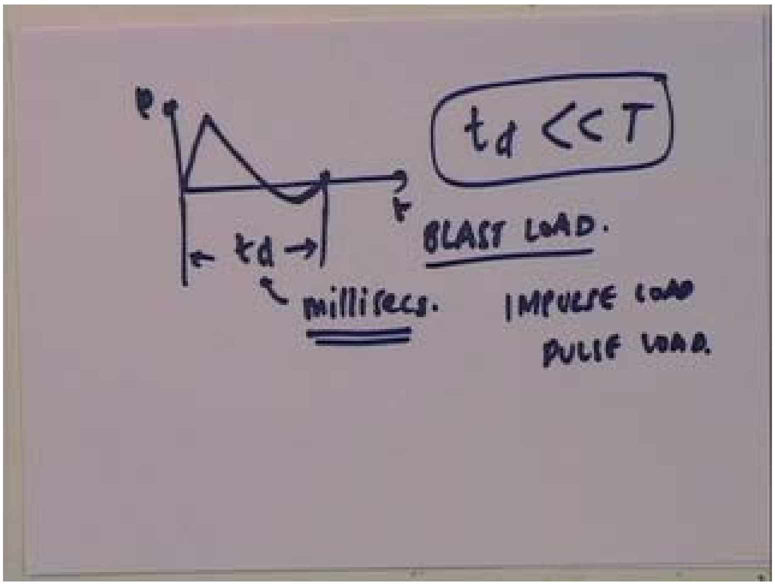
And. So, this plus this and then you substitute t is equal to 0 u 0 and u dot 0 to get an A 1 and A 2, where its u of t, which is this plus this which gives you the solution. So, there for when you are looking at the solution of a first vibration solution what is the first vibration problem that is this problem m u double dot plus C u dot plus k u is equal to p of t this is the first vibration problem. So, here essentially u h of t we already said u h of t is this form we have already solved this, because you see equation of C essentially solves is a solution to this problem, which you have already solved problem u p of t depends on p of t. which is particular solution and once we have that then we can get A 1 and A 2 from the initial conditions and we have the solution in u of t, which is this solution for this loading. So, this is the displacement response to forced loading. So, this is the over view of response of a structure to any kind of load now what kind of loading are we going to consider we just before solving that what kind of loading. Are we look at we going to look at the loading that occurs. in real life what kind of loading happens in real life just look at that.

(Refer Slide Time: 45:11)



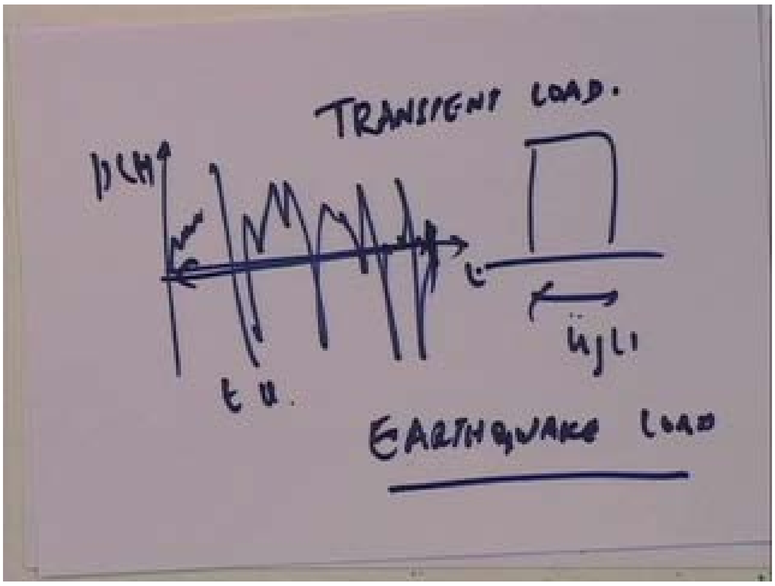
If you have a structure with a rotator machinery on the top then the loading at the structure type is detected to like this ok. If this machinery is rotating at a constant rpm then this is what the kind of loading that we have this is known as harmonic vibration. Then another type of loading see and this is seen for example, in a ship through a the procular keeps is also rotating, but in that kind of a situation what you see is something like this you see in harmonic this gets repeated in periodic this gets repeated periodic loading and. So, the rotating machinery or at a constant velocity at a constant rpm on a building develops a harmonic loading ships procular or any kind of a system, which is unbalanced ships procularis.

(Refer Slide Time: 47:13)



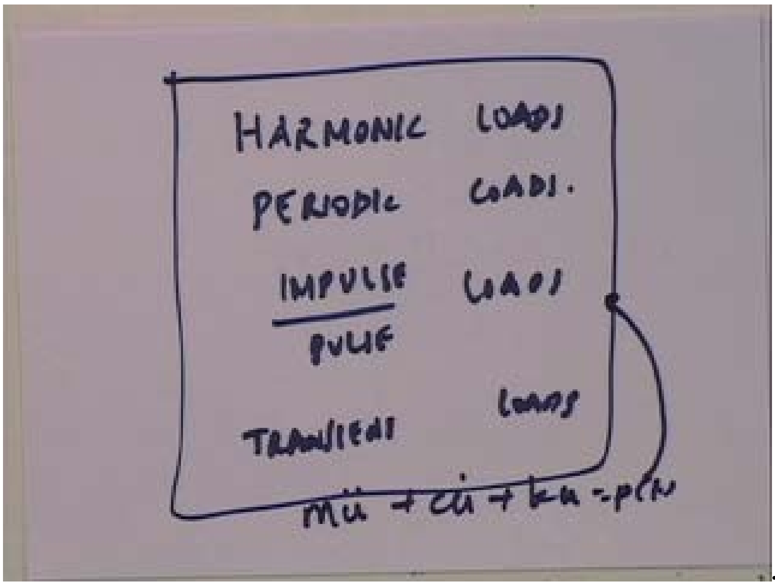
Now balanced subjects the procular to periodic loading what are the kind of loading do we see well many kind of loading that we see is was known as sorry, p t a blast load if there is an exposure away from the structure and that the pressure wave coming and hitting the structure of this form well. Actually, t d is significantly less typically then the kind you know this structure, because this t d is in milli seconds and blast loading typically, can be it looks at as you know we will look at it later as impulse load or for extremely rigid structures as a pulse load.

(Refer Slide Time: 48:33)



So, that is another kind of loading that we see and then finally, another kind of loading with C is a building objected to an earth quake load this is very complex, but it lasts only for a while. So, this is known as transient load. So, this kind of environmental load there could be earth quake load or whether it be wind load is a transient load. So, what are the kind of load that is we are going to looking at the loads that you going to be looking at based on the realistic loads that you see on a structure.

(Refer Slide Time: 49:31)



We are going to first look at harmonic loads, then we are going to look at periodic loads, then we are going to look at impulse and pulse load, and then we are going to look at transient load these last for a while harmonic and periodic loads are defers you have rotationally. You have one forever transient loads these are there forever these are there for extremely short duration these are there for extremely longer duration. So, these are the kinds of loads that we have to analyze the structure for and the this significant next few lecture am going to be looking at the response of single degree of freedom to a molecule, then we going to look at the response of single degree of freedom to a periodic load in. We were in look at the response of single degree of system to harmonics sorry impulse or pulse type loading. And finally, the most complicated kind of loading we going to look at that is transient loading and note that the equation that we are solving is stated over and over again is this problem, where this load is going to be define by these different kinds of loads.

Once we do all these kind of loads and these are the kinds of loads structural subjected to we will be able to then start looking at what are the characteristics again. I want to go back and state the something that I divided in the beginning we are going to solve linear differential equations we are going to find out u of t as a function of homogeneous and particular solution all those we going to do we going to go to that process, but after having gone through that process what will be interesting in doing is not the u of t nor the response time we will be interested in the peak response, and we will see again going back , and if we can find out somehow a p 0 upon k which is static response multiplied by a dynamic factor. If we can find out the dynamic factor for each kind of loading then always we need to do is just prepare a designing chart for kinds of loading, then having looked at the characteristics of the load and the characteristics of the structure we can determine the D.

And once we determine D s multiply D into p naught as the static load multiplied to get the dynamic response of the structure in the entire t to harmonic I mean dynamic loads. So, I will stop here today and I just want to re to write that the reason why we look at loads is not because of there, but because they represent it realistic that a structure ready to see the one is establish the reason why we look at different kinds of loads never forget that we are a genius we are not mathematicians. So, all these mathematics will be looking at ultimately we have to step back why are we looking at the mathematics is important to us, because we need to look at specific responses to specific kinds of loads from the next lecture, we are going to look at forced vibration of a single degree of a system subjected to different kinds of loads.

Thank you very much.