Structural Dynamics
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Lecture - 29
Dynamic Response of Multi Degree of Freedom Structures

Hello, again we were talking about the response of response analysis of multi degree of freedom systems to a dynamic loading.

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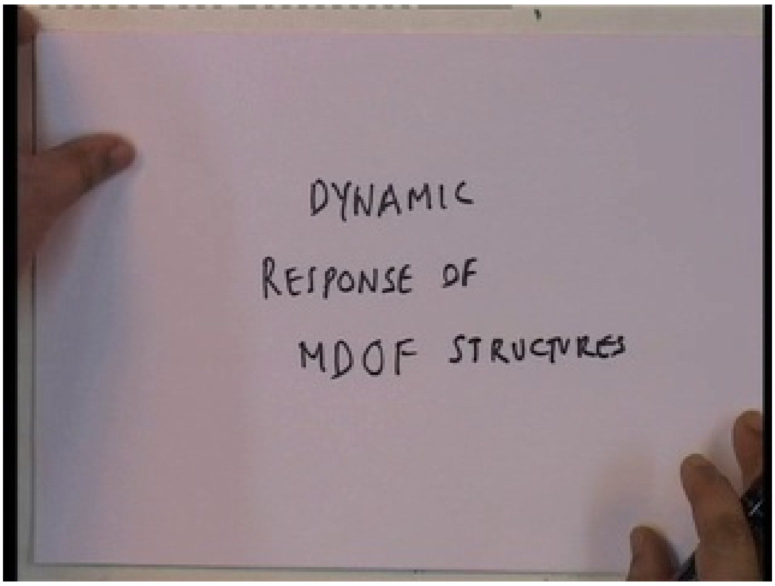

So, we are looking at dynamic response of, and here I have been talking about the fact that the kind of structures that we are referring to in this particular course, is essentially that of your buildings building kind of building frames buildings that kind of a thing. So, just to review the mode super position method, the mode super position method is based on the fact that, where these are displacements corresponding to degrees of the freedom. These are modal amplitudes and this is the mode shape matrix. This is N by1 because there are N degrees of freedom this is N by1 and this is N by N. Now, we can also say remember last time we said that essentially the fact becomes the problem becomes that in reality you know you do not require all the modes to be considered and therefore, if you were look at it.

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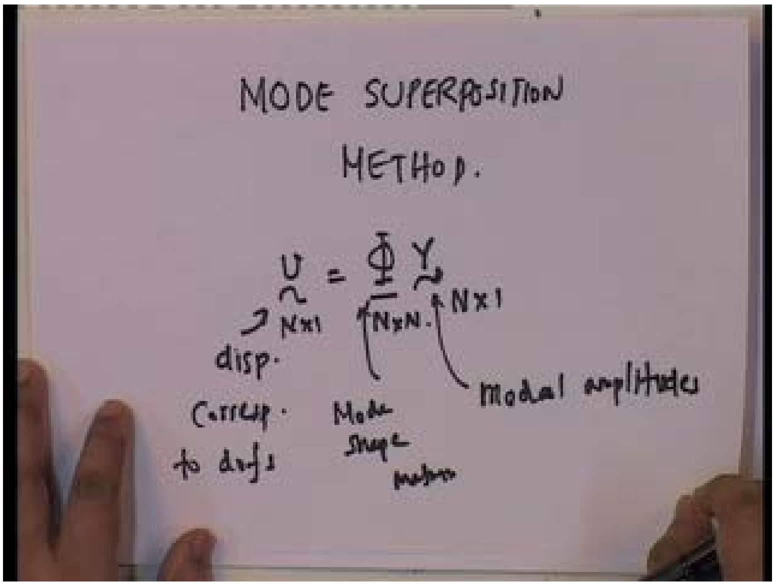

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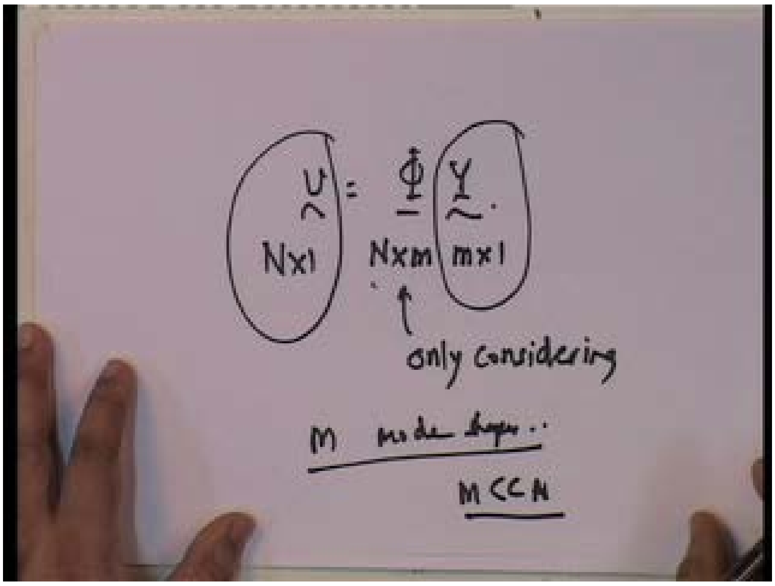

We are saying that look v is equal to phi into y, where this is N into1 this remains n into 1 purely because it is it is an N degree of freedom structure. So, these are displacements corresponding to each degree of freedom this is no longer N by N it is M by N by M. So, what happens is only considering M modes. So, where M is significantly less than N. So, that is the problem that happens and note you only have N is this is a mode shape and there are only m mode shapes and therefore, there are only M modal amplitudes. So, this is the beauty that we taken this problem we have made into a smaller problem and further more. Because of the orthogonality that you have of a you know mode shapes essentially, what you come down to is the following problem.

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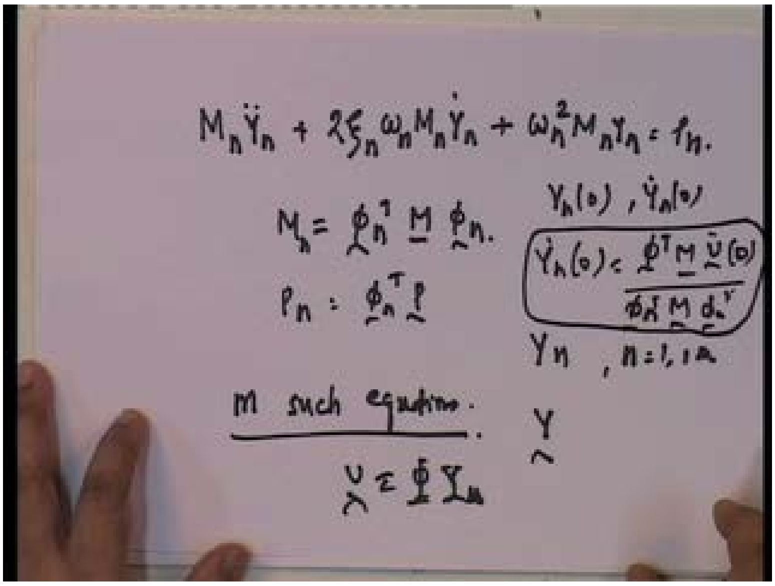

Then we talked about the fact that we could that is how we include damping and we also saw where M n is equal to phi n transpose M phi n and P n is equal to phi n transpose P. So, this we have M such equations and you need to solve M such single degree of freedom problems and you solved the equation. So, this in and then once you have got Y n you have got different Y n n going from one to m then that becomes the phi and then V you get by phi transpose Y n. And of course, this is subjected to Y n 0 and Y n dot 0, where Y n 0 and Y n dot 0 are given by phi n transpose M V 0 or V dot 0 both are given exactly the same way.

The only thing is that this is not dot then this is not dot it this dot then this is dot into phi M transpose. So, this in essence is how you compute how you go from this is how you go from the original space to the modal space and this is how you go from the modal space to the original space. So, these are the two opposite kind of relationships and all of these are based on nothing but the fundamental fact that all modes are orthogonal with respect to each other.

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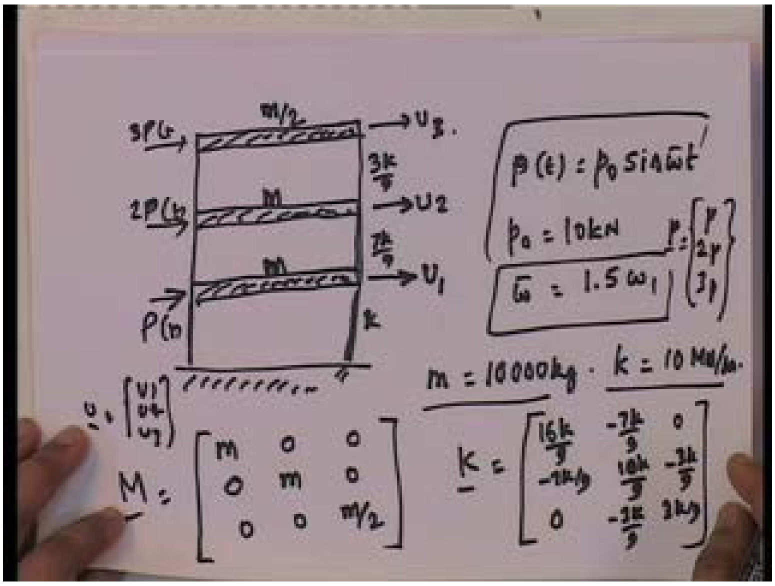

So, let me do one thing let us now look at a specific problem and this is the problem that is fairly well known problem and you know I am going to listed out to you. This is it is a three storeyed building with rigid beams. So, what we have here is this is v 1 this is v 2 and this is v 3 and the loads are given in this form p 2 p 3 p, where p of t is equal to p 0 sin omega bar t, where p 0 is ten kilo Newtons and omega bar is equal to 1.5 times the fundamental frequency, this is the loading and if p this is p. So, this is 2 p this is p of t 2 p of t 3 p of t. So, this in essence is the problem of course, we have to give you know the weight the masses and this thing we will say that this is weight w.

So, you know we will call this mass m mass n and mass m by 2 and the lateral stiffness of this is given by 3 k by 9 and this is 7 k by 9 and this is k where m is equal to 10 tons or 10,000 kgs and k is equal to 10 mega Newton per meter. So, that these are all the parameters that are given to you. So, if we look at it the mass matrix, what is the mass matrix look like the mass matrix you know you go through the procedure. I will not I will give you what the mass matrix looks like m 0 0; these are all mass less all of these columns are mass less all the mass.

At this point, this is the mass matrix and the stiffness matrix is the following 16 k by 9 minus 7 k by 9 0 minus 7 k by 9 0 10 k over 9 this is minus 3 k over 9 0 minus 3 k over 9 and this is 3 k over 9, where k is given by this m is given by this. So and your displacement vector is v1 v2 v3 the lateral displacements at each storey of the frame. So, this in essence is the this thing and what we are what we say is that. Look we are going to now remember that we never find out c over here right we found out m k and the p vector the p vector is the following p 2 p 3 p where p is given with ten kilo Newton’s.

So, what now the thing is of course, to do k minus omega squared m into phi is equal to 0 and solve the Eigen value problem. Here I am giving you the solution to the Eigen value problem. We have already done that and this focus is more on dynamic response than on free vibration response.

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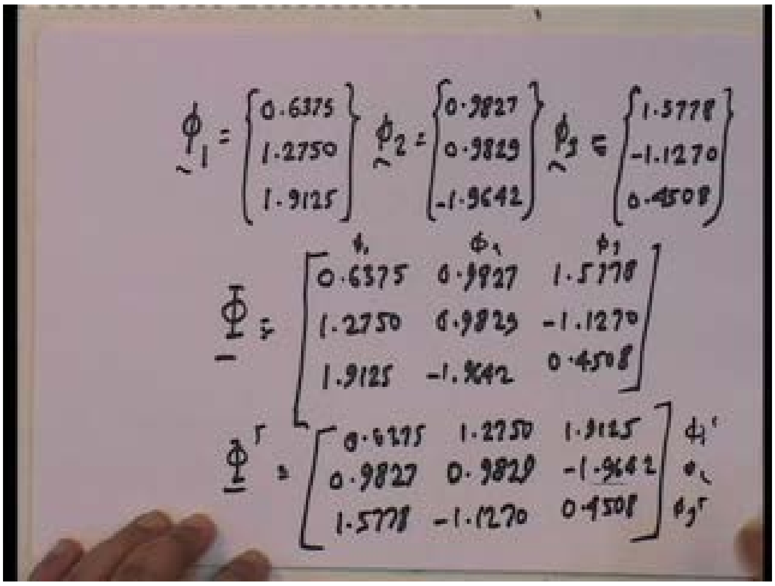

So, I will give you the phi vectors this is the first vector, the second vector 0.9827 0.9829 and minus 1.9642. Finally, the third vector there are three degrees of freedom. So, there are three vectors third vector is 1.5778 minus 1.1270 0.4508. So, these are the three mode shapes, I am not going to give the frequencies right now; because we will we will see that we can actually derive the frequencies.

So, if I re write this in the format of phi. So, phi then becomes equal to 0.6375 1.2750 1.92 1.9827 0.9829 minus 1.919642 1.578 minus 1.12 0.4508 and phi transpose is equal to 0.6375 1.2750 1.925 then 0.9827 0.9829 minus 1.9642 and then finally, 1.5778 minus 1.270 0.4508. So, once I have done this is nothing but look that this becomes phi 1 transpose phi 2 transpose phi 3 transpose phi 1 phi 2 phi 3 and this is phi 1 transpose phi 2 transpose phi 3 transpose; now if I do the mass matrix I have got the stiffness matrix.

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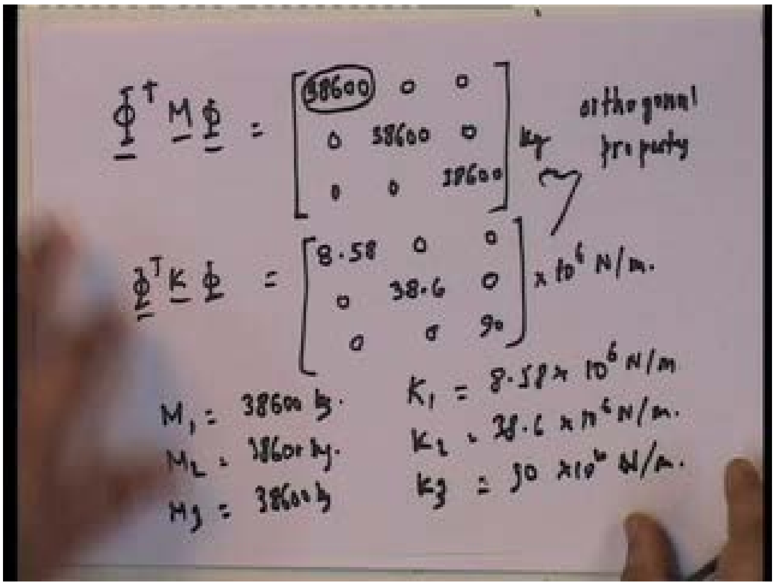

So, if I put phi transpose M phi and go through each of the multiplications; you will get the following. That is phi transpose M and phi transpose k phi is equal to 8.58 this is of course, all kg s this is going to be 8.58000 38.6000 90 into 10 to the power 6 because mega Newton, so 10 to the power 6 Newton per meter. So, now, if I look at it this upon this is note two things diagonal means orthogonal property of both of these is satisfied well; that means, that the you know the phi phi values that we’ve given which is these are indeed orthogonal mode shape dwell mode shapes are orthogonal we’ve already found that out. So, here so; obviously, this becomes what. So, your m one becomes what is k 1 8.58 into the tenth power of 6 Newton per meter m2 k 2, m3 k3, ok.

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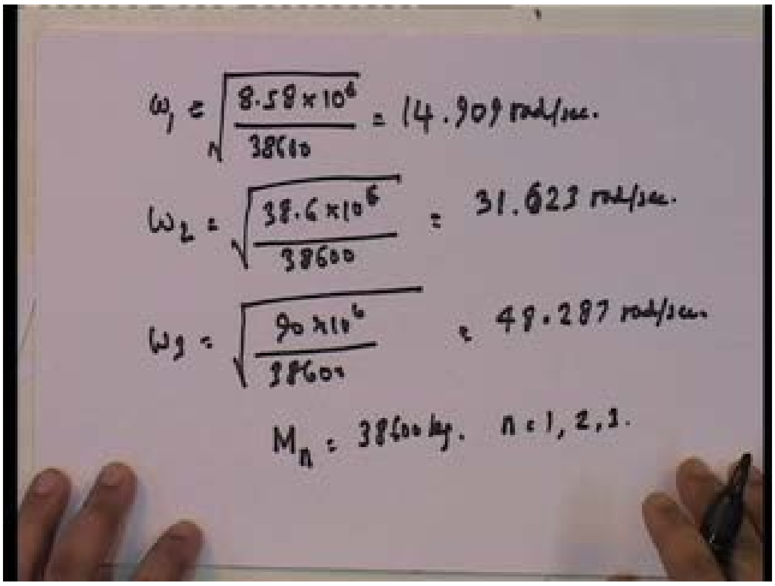

So, having done that I know that k1 upon m1 square root is equal to omega 1 because omega 1 squared is k1 upon m1 because we already seen that k1 is equal to omega 1 squared m1.

So, therefore, omega 1 is equal to k1 k 1 is 8.58 into the 10 power of 6 divided by 38,600 this is 14.909 radians per second omega 2 is equal to three 38.6 into 10 to the power of 6 divided by 38,600 this one turns out to be 31.623 radians and omega 3 is equal to 90 into 10 to the power of 6 upon 38,600 which is equal to 48.287 radians per second. So, essentially what we have here is that we have a situation where your these are the frequencies we have already got the m all the m's m n is equal to 38,600 kg's for n equal to 1 2 and 3 and the only thing that we have left is what the only thing that we have left is actually computation of the loads.

So, p1 p2 p3 let us look at p1 of t which is equal to phi1 transpose into m into p. So, if you do that and substitute the fact that they are this thing this one turns out to be equal to 89.250 kilo Newton sin. Now, since we know 1.5 we know it has to be 14.909 into 1.5. So, that is equal to 22.3635 t that is omega bar that is p1 p2 is equal to phi2 transpose m p and that is equal to minus 2.441 kilo Newton sin 0.365 t and p3 is phi3 transpose m p which is equal to 6.762 kilo Newton sin 22 36 35. So, these are our p1 p2 and p3. So, let us go to the next step.

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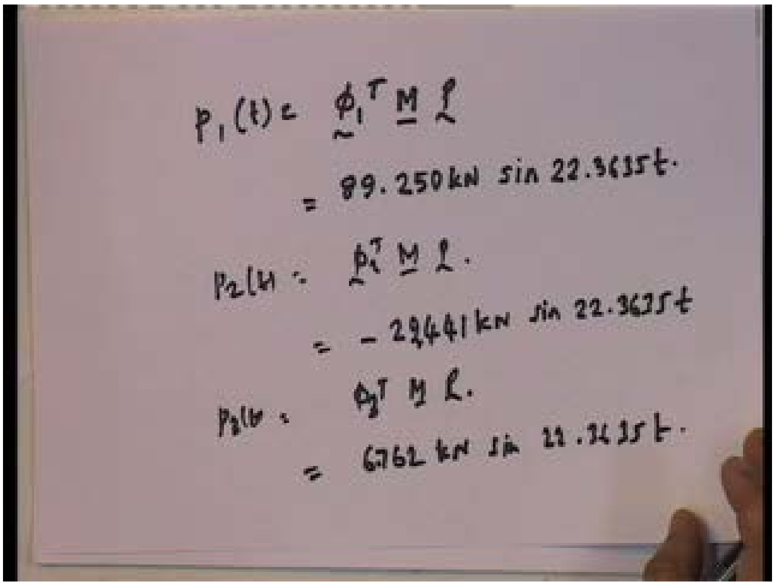

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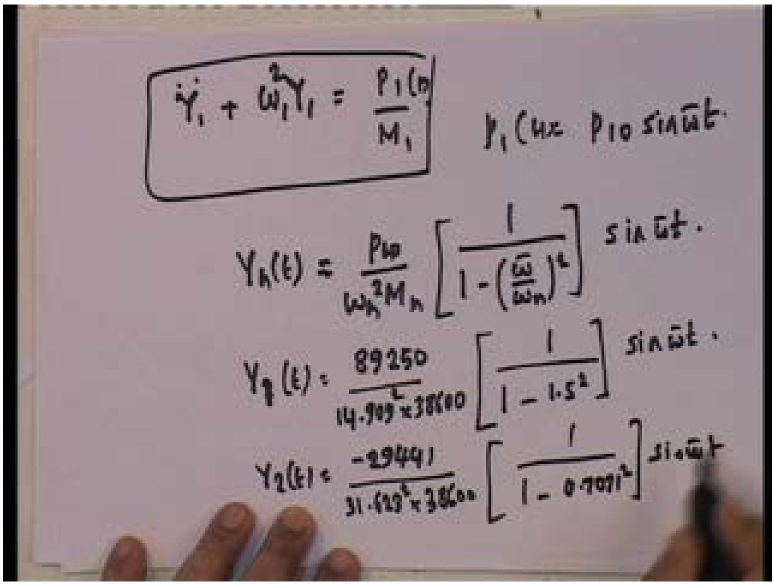

The next step is what is the fact that lets write down the equation get the following equation y1 double dot plus omega 1 squared y1 is equal to p1 upon m1ok.So, that is what we have right and. So, if we go about it omega 1 squared and if you look at this is the this is the single degree of freedom where p1 t of the form let us say p 1 0 sin omega bar t right if this is the format then what is the solution this p1zero is of this format. So, then what is the solution the solution then becomes the following it will be equal to.

So, y1 of t and this is the steady state solution by the way steady state solution only the steady state solution the steady state solution becomes the following p1 0 upon omega1 squared m 1 that is p1 upon k1 into 1 upon1 minus omega bar upon omega1 the whole squared into sin omega bar t ok.

So, if you were to look at it y2 of t I mean let me put it this way it will be y n p n m n into this sin omega bar t that is the solution. So, if that is the solution then y1 is equal to p1 Is 8 now going to put it in Newton’s, because Newton’s and kg’s go together. So, you have into omega n squared. So, that is 14.90 into 38.60 all upon 1 plus 1 minus now if you look at it omega bar upon omega1 is 1.5; obviously, that is how we’ve done it into sin omega bar t.

So, if you look at. So, this is this is how it is right because this is un damped system. So, therefore, it will directly have this now y of t is equal to minus 29441 upon its equal to let me just have a look quick look at it 31.623. So, that is 31.623 squared into 38.001 one minus now the thing is that omega bar is 22 and omega 2 is 30 31.623 if we do that that turns out to be 0.7 o 71 squared sin omega bar t.

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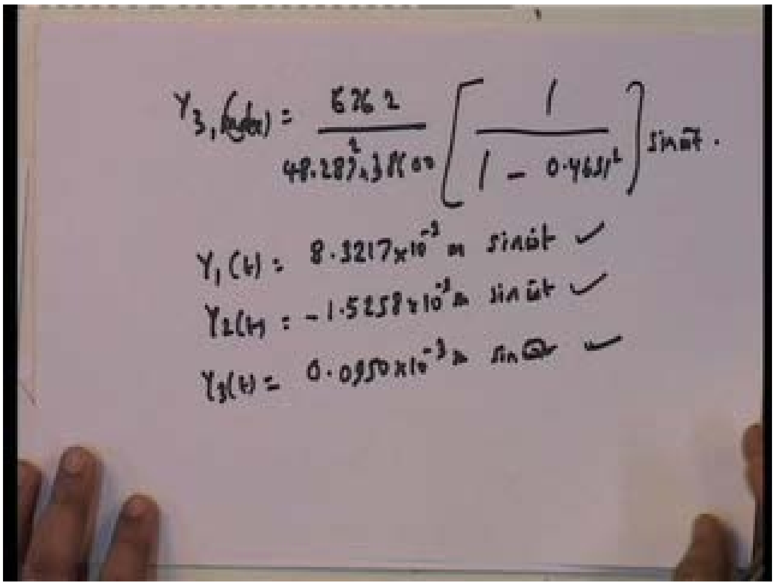

And finally let me do y3 on a on a separate sheet or paper y3 max is y3 t is equal to hmm 6762 upon we had 48.287 squared into multiplied by 38.6001 upon 1 minus omega bar upon omega 322.3635 upon 48.27 that turns out to be 0.4631 squared sin omega bar t. So, if we evaluate these then we get that y1 of t is equal to 8.3217 into ten to the power minus 3 meters sin omega bar t y of t is equal to minus 1.5258 into ten to the power minus 3 sin omega bar t and y3 of t is equal to 0.0950 into 10 to the power minus 3 meters per second now these are all ten to the power of millimeters actually. So, now, if you look at it if I were to put it together well know this is y 1 y 2 y 3 you notice something very interesting that y 1 is large y 2 is less y 3 is even smaller. So, remember I said that the first mode is sometimes good enough ok.

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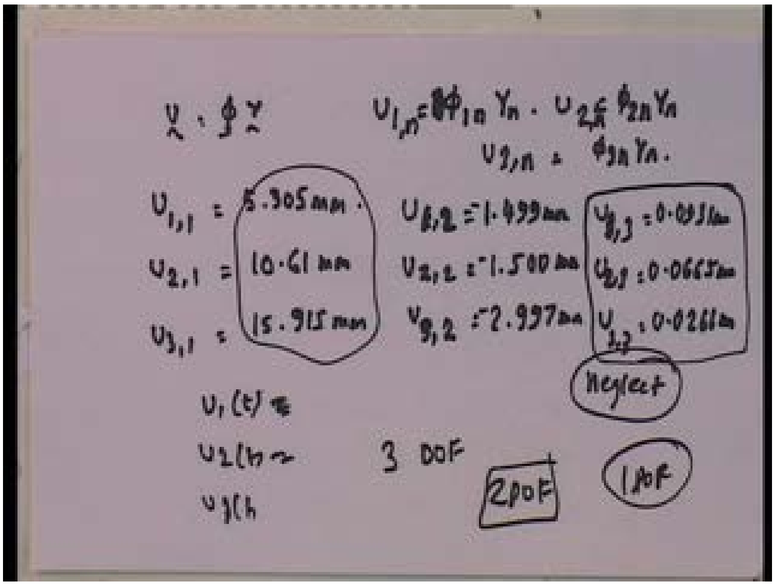

So, let us see what we get if you were to look at it this would become the following it would become that. So, let us now you know once I have got y1 of t a y n of t I can find out v by doing phi transpose y. So, now, I can also say that look v1 of t is equal to phi now here if you look at it is equal to the 1 n y n summation. So, now, if I look at it you will have the situation where that ok. I will say let me look at the first displacement in the first mode that will only be what it will be phi 1 1into y1. So, if I do p 1 1 into y1 I get 5.035 millimeters if I look at the second displacement in the first mode that is essentially equal to phi 2 1 into.

So, the second displacement quantity in the first mode into y 1 will give me this. So, this is equal to 10.61 millimeter and the displacement at the top is going to be 15.95 millimeter. So, in other words v2 is going to be. So, if you are look at v1and it is nothing,, but phi 1 n v2 n is equal to phi 2 n into y n and similarly v3 n is equal to phi3 n into y n. So, therefore, if it is phi3 1 all of this is the first mode the top second and third corresponding. So, if you look at it if I were to look back at this is what you multiply with y 1 this is what you multiply with y1 this is what you multiply with y3 to get your v1 v2 ok.

So, similarly we will get these are the maximum values amplitudes this is the big values. So, v2 1 is equal to 1.499 millimeter v 2 2 this is v 1 2 the second mode for displacement 1.499 second 1.5 v 32 is equal to 2.997 by the way this one is please note that since this is plus this is minus. So, when you are doing it with minus what you get is that all of these are minus and phi 13 is equal to 0.0931 millimeter v 23 is equal to 0.0665 millimeter and v 33 is equal to please note that it is 1.5778 into y 3 and all of those.

So, you get v 33 is equal to what it is going to be smaller and that is 0.0266 millimeter. So, this in essence is your these are your displacements now all of them are with sin omega bar t if I look at v 1 of t it is going to be equal to this minus this plus this. So, that is v 1 it is summation of all 3 of them. So, v 11 v 12 v and all of them have sin omega bar t with them. So, you have a situation where you just add the 2 add the 3 and you get it v two is going to be this plus this plus this. So, this is minus. So, this will be 10.61 minus one point plus ok.

So, have a situation where if you look at it we may as well neglect the third mode now in this particular case please note and similarly we can find out v 3 of t which is this plus this plus this sin omega bar t now the question here becomes is that look since I have only three degrees of freedom I can neglect I do not need to consider the third degree of freedom. So, actually I can make it into just solve the two degrees of freedom and be done with it now this is a very peculiar situation where you know it is a harmonic load and where the harmonic load actually the largest displacement is really in the second mode that is a very interesting point to note because the dynamic largest dynamic amplification that you have in this particular case is actually in the second mode despite the second mode being actually dynamicly more excited because 1.5 omega bar and you know omega 2 upon omega 1 is about two right.

So, it actually comes very close to second mode. So, even though I have taken harmonic which is not tuned to the first mode note that the first mode response is largest of course,, the second mode response is not non negligible because actually you know since it is harmonic load this is exited if you look at it arbitrary kind of load you will always see that this is the extreme situation in which this contributed significantly purely because it is dynamically more excited if you do not have that situation you will have a situation where the total will almost be given by the first mode here I actually call this situation where the second mode contributes more I made it how will you still see the third mode is to be neglected. So, in actual fact in reality, if we just deal with a one degree of freedom system in this particular case you will have solved the problem ok.

So, what is the take away from this particular problem that I solved the problem that I solved are the following one I showed you how to use mode superposition to you know find out the response of a multi degree of freedom system of course,, the multi degree freedom system the first thing that we need to do is the free vibration analysis now here I gave it you because I have done it already and then we went through the steps of the mode superposition analysis of how to go about it and of course,, here I just considered harmonic load as a specific case of the form ok.

So, now one the procedure we lasted the procedure two we saw that actually in a three degree of freedom structure the displacement quantity can given by the first mode in this particular case because I gave the harmonic load you know if I put omega bar very close to omega one you will see that omega you know that the second mode almost does not contribute. So, these are the points that you can actually reduce the number of degrees of freedom to illustrate the issue now this is…

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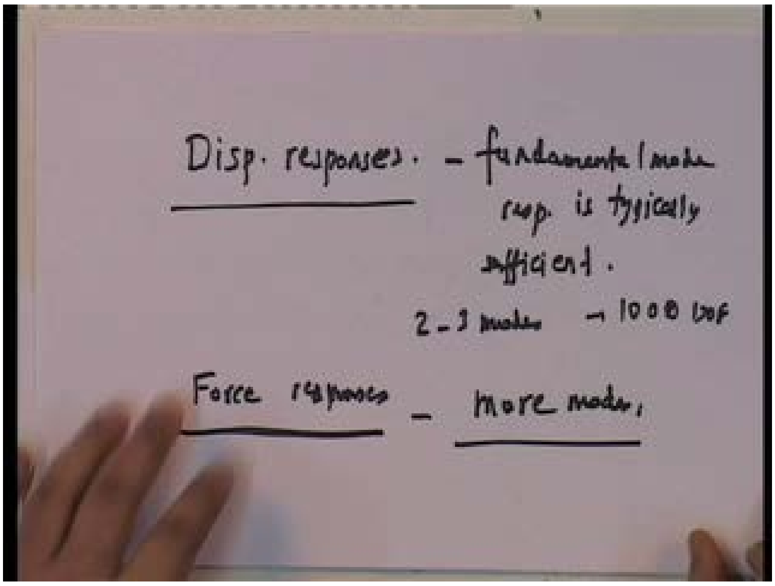

So, much as this now for displacements this is the final point that I want to make in this particular lecture and that is that in these kind of displacements what happens here is the following that for is typically sufficient or at the most 2 to 3 modes even when you have let us say 1,000 degrees of freedom,, but if you are looking at force responses for example, force responses what kind of force response are there for example, the bestial or bending movement at a particular time or storestia, if these are the things that you want to find out then what happens is that you can go ahead and do this force responses you require more modes now why is that let us let us let us look at it let us look at what happens if I were to look at this particular problem ok.

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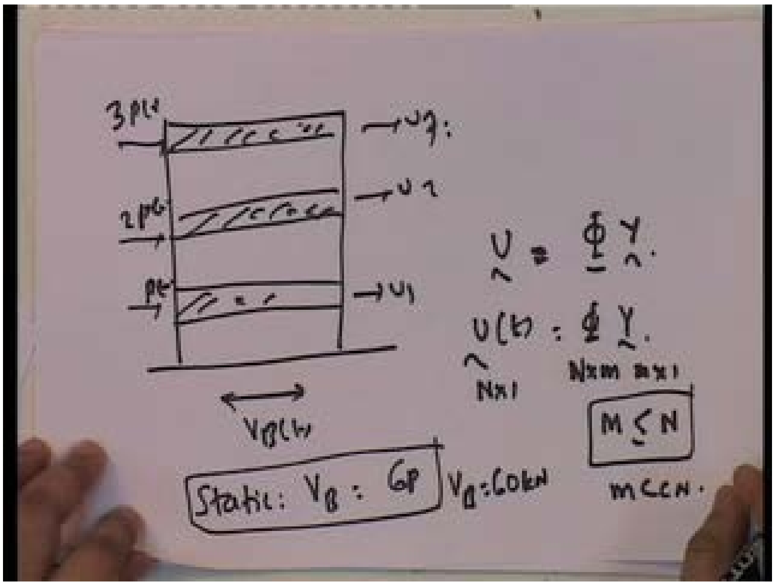

Now, typically how would I find out the force response well let us see let us take this particular problem the specific example problem,, but this is true of any kind of a problem that you have v 1 v 2 v 3 now I have found out v from phi y right I have found out my v. So, my v of t I have found out by doing phi transpose y and this is typically m modes and this is n by 1 and note m is less than or equal to n I am typically we know that m is significantly less than n,, but let us say that that is my v of t now how do I for example, suppose I want to find out the bestial due to right now note that these are if it there was no time dependence if there was no inertia what would v b be equal to if it was static what would v b equal to v b be equal to six p right and p is equal to ten kilo Newton’s.

So, v b would equal to 60 kilo Newton's I would need to have solved all of these,, but note that these are time dependent loads and. So, therefore, how do I find out my static. So, what I need to do is I have got my v. So, what I say is that look under this loads under this load I have my v of t.

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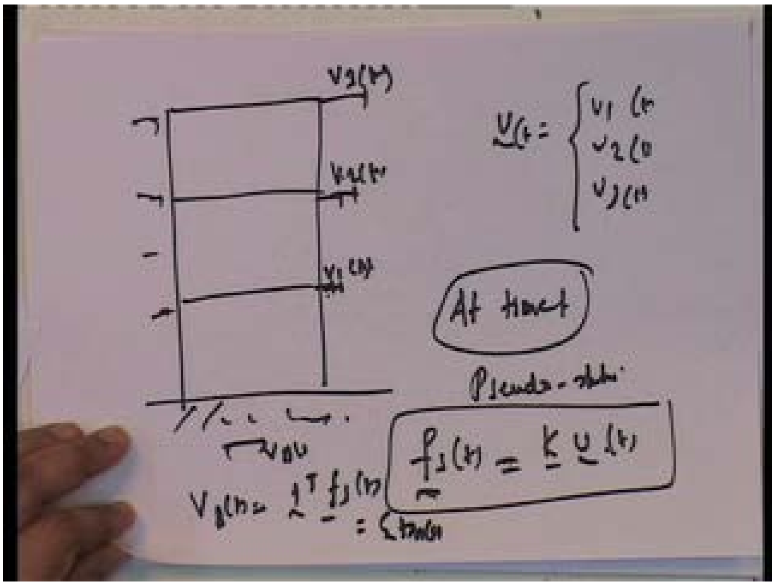

So, I have p1 of t v2 of t v3 of t. So, based on that what do I get I say that look it was like this I am just drawing line and these were. So, now, what has happened is at any instant of time this is v 3 of t at a instant of time this is at an instant of time t this is v1 and v2 and v3 now we say that look how do I find my bestial I say that look let us take the situation suppose it was the static loads.

So, this is pseudo static at time t it becomes a pseudo static problem to get this displacement what are the forces that I have to apply static forces pseudo static; obviously, which are going to give me this displacements well we know that what would be those forces those would be k into p is it not? It see in other words at any instant of time t to get a displacement given by this displacement quantity.

What would be the loads that I have to apply that is nothing,, but k into v that is a known I mean that something in a static sense that is what it is right. So, what we have is and now look at it how do I find out v b t it is this plus this plus this,, but in other words I can write v b of t as note see this if I write this one transpose this is a vector of one's vector of 1111 see it is a three degree of freedom 111111 into f s look at this 1 into f s1 plus 1 f s 2 plus 1 into f s 3 which is nothing,, but a summation of this is equal to summation of f s n. So, see understand. So, v b of t once I get f s of t I can actually for that load I can find out all the force responses that I need to find out ok.

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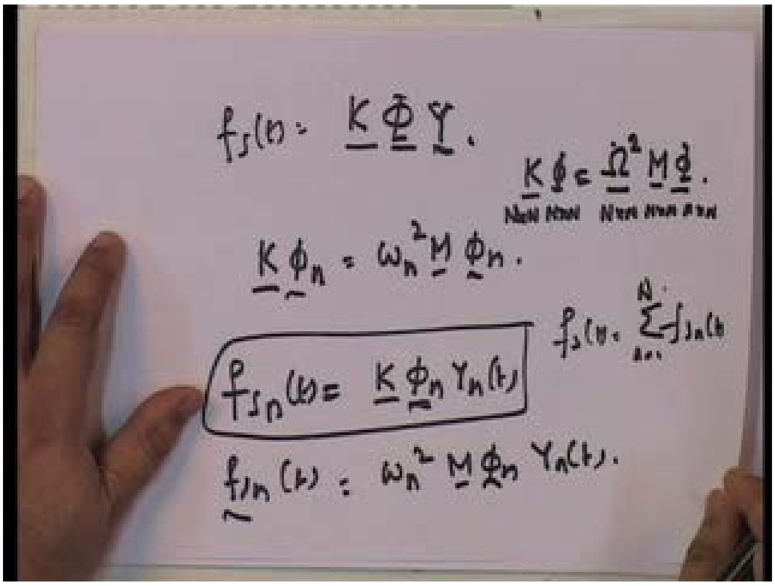

So, f s is equal to k into v let us see what happens then when I have f s equal to k into v f s of t is equal to k into if you look at it phi into y. So, if you look at this let us see what k phi is equal to if you look at it k into phi n is equal to omega n squared m phi n ok. So, if I were to not look at it then that I look at f s n of t f s n of t would then become what it would become equal to let us see what would f s n of t it would be equal to k into phi n into y n of t this would be this is the if you look at it well put it put this down. So, if I were to drew this k I can say that k phi is equal to k phi is equal to omega squared into m into phi well this is nothing,, but look at this is n by n this is n by n note that this is n by n this is n by n.

And this is n by n where this is nothing,, but if you look at it is a diagonal matrix where the first omega 1 omega 2 omega 1 squared omega 2 squared omega3 squared. So, if I were to rewrite in this fashion that I will say that look f s of t is nothing,, but summation of all modes going from 1 to n then f s of t becomes this and if you look at this k phi n of t is omega n squared. So, I can write it in this form that f s n of t is equal to omega squared m phi n into y n of t now if I were to look at the bestial what is bestial one transpose ok.

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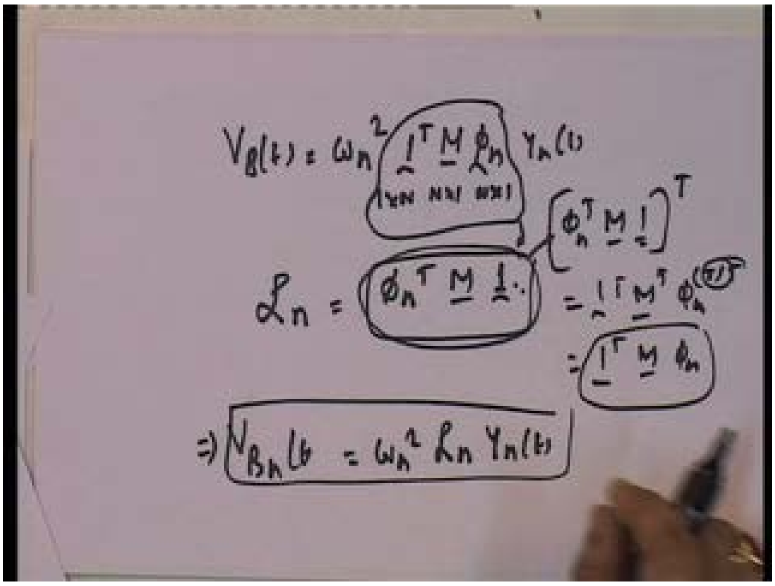

So, this is equal to the v b of t is equal to omega n squared one transpose m phi n y n of t this is 1 by n this is n by 1 this is n by 1 and this quantity is now I will define a quantity l n l n is nothing,, but phi n transpose m phi 1, now let us look at this what is this is this is nothing,, but this phi n transpose m1 transpose is equal to look at it 1 transpose m transpose phi n transpose which is nothing, but itself and m transpose is one. So, this become 1 transpose m into phi n note that since it is a scalar it does not matter scalar transpose of itself is this. So, therefore, v b n of t is equal to omega n squared l n into y n of t where l n is this quantity ok.

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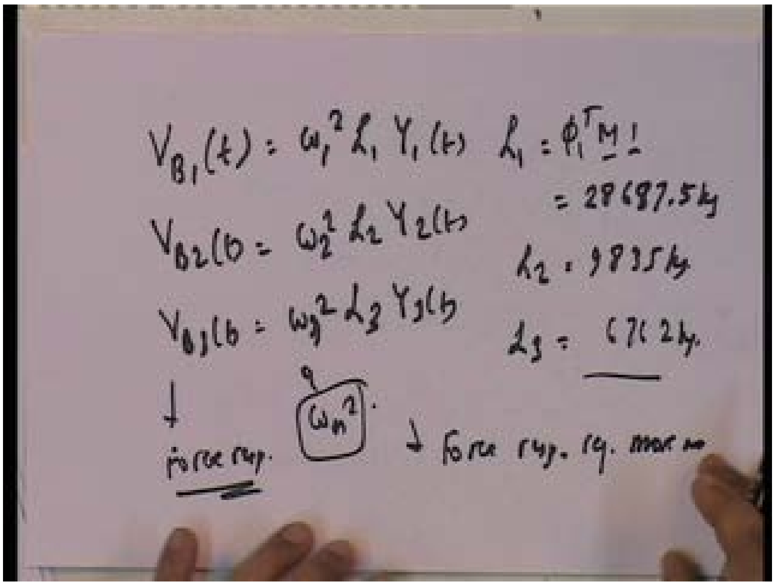

So, for this particular problem that we were solving for the particular problem that we were solving what did we get look at it let us look at it we are going to get it equal to v b 1 t is equal to let us look at one. So, it is going to be equal to omega 1 squared l 1 into y 1 of t v b 2 is equal to omega t squared l 1 y 2of t and v b 3 is equal to omega 3 squared l 3 y3 of t now l 1 is phi 1 transpose m into 1 and doing that we see that this is equal to 286687.5 kg's l 2 is equal to 9835 kg's l three is equal to 6762 kg's note something very interesting and that is that l is also a quantity which decreases with,, but this is a force response note that even if these goes down they multiplied with omega n squared and that is why for force responses, you will see that if you put this in and you will plug it in and put it in you will see that v b3 also contributes significantly in this particular case with your this thing and therefore, yeah.

So, you know the force response requires more modes because they are actually multiplied with omega n squared. So, you know this one is lower this one is higher,, but this one is lower,, but this one is higher. So, as your omega n is being multiplied the contribution of the higher modes to force responses is more than to displacement responses this is another very important aspect that you need to consider. So, therefore, if you looking at the force responses you need to consider many more modes then you need to consider for example, typically I will just give you some idea of some problems that I have solved that I had a problem where I had 15 degrees 1,500 degrees of freedom in a structure fairly complex structure. So, I had 1,500 degrees of freedom.

To get accurate estimates of the displacement responses I required only three modes to get accurate estimates of force responses. For example, banding movement bestial storestia all of those I needed about 50 points. So, very less number of modes if you are looking at displacement response more, but still 50 is significantly lesser than 1,500 modes. So, that is the beauty of the mode superposition method, I am going to stop here and see you next time bye.