Structural Dynamics
Prof. P. Banerji
Department of Civil Engineering
Indian Institute of Technology, Bombay

Lecture - 27
Practical Free Vibration Analysis

Hello there, we been talking about the Eigen value problem and free vibration and we have solved some problems on free vibration analysis also.

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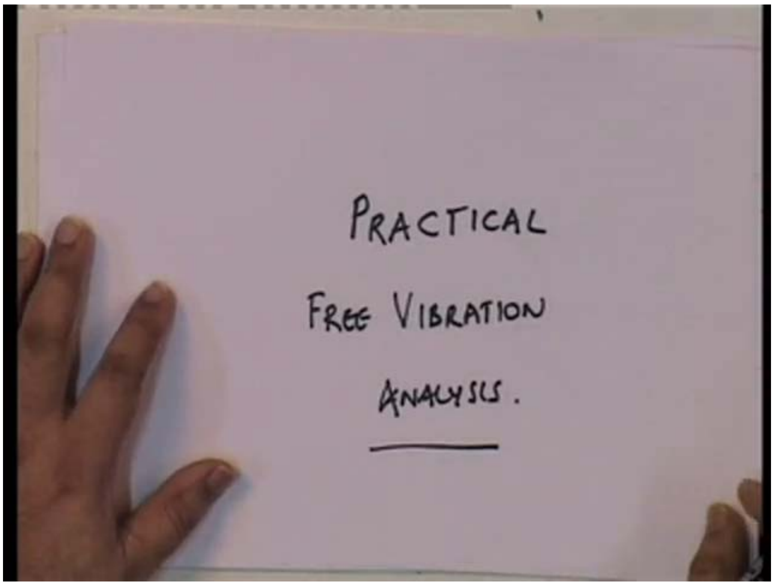

Today, what I will talk about is something to do with practical free vibration analysis. Now, this is a large class of problems last time I actually introduced to you to the concept when we looked at that. Look, after all we saw that this was, this is the problem that you solved. So, this is the free vibration problem this is the Eigen value problem. Then we saw that look essentially what the determinant of k minus omega squared M was equal to 0 that there was where if you put omega squared equal to lambda. Then we got a basically a function and it is a n th order polynomial and one of the ways of practically finding out frequencies is from doing essentially root finding algorithms right and root finding algorithms there are many.

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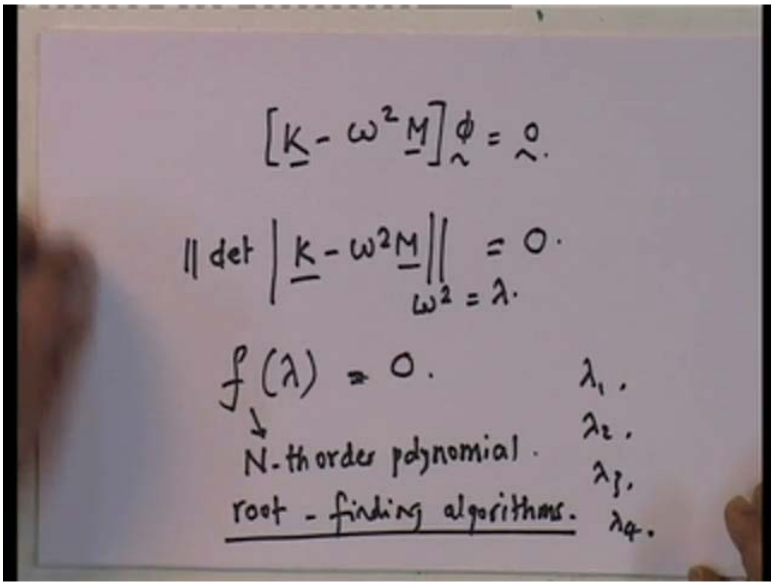

So, you basically you have to find out the roots of this function and from that you get lambda 1 lambda 2 lambda 3 and then there are there are test to show how you do not miss any root. Now, this is these this is something that is done typically in a numerical methods course. So, I am not going to talk except me the fact that this writing on the polynomial and finding out the roots of the polynomial is one procedure for doing practical free vibration analysis.

Now, today I am going to talk about another method and this actually before the advent of computers, this was the best way of finding out the frequencies of a system. Before advanced computers, I mean you still have to do metrics analysis, but metrics is much easier to do in a computer even way back in those days.

This method is either called the inverse iteration method or it is called the to give, it is old name the Stodola method when is this method start the method started from the fact that if you looked at that you have a situation where the problem becomes k phi is equal to omega squared M phi. Now, if I were to rewrite this I could rewrite in this format one omega upon omega squared phi is equal to k inverse into M into phi an k inverse k inverse is actually the flexibility matrix.

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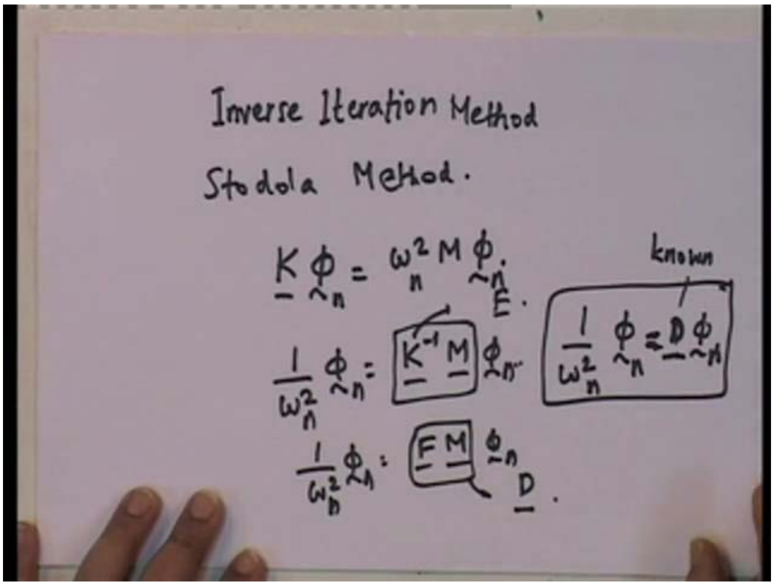

So, it will not as if you are doing k inverse you can actually find out the flexibility matrix directly, you do not have to do k inverse. So, you are essentially doing something like, so call this matrix d matrix, so then this problem can be written in this format this is the basic concept behind this problem.

Now, D matrix is a property matrix you can derive it, this D is known now this is equal to this if this displacement that you get is the actually let me put it this way that right. So, this is phi n, phi n, omega n omega n phi n into phi n, so this is omega n phi n d phi n. Now, this is true if this displacement or you know this is displacement after all phi is nothing but the displacements corresponding only thing here is that it is a shape. So, that is that does not matter you can take a any displacement this is true, however Stodola method say is that look you start you start with a specific vector.

So, you start with a vector and we will call this vector v 1 and you start with a vector that is the 0 th. So, this is a n by 1 vector it is a column vector right take any vector you think of a vector and think of it. So, then what we say is that look by definition and you assume this two be the fundamental kind of an approximation of the fundamental motion. Then we can say that look it is equal to this one 1 is equal to d into v 1 0, now what you do is you do not actually do this, so what you do is the procedure.

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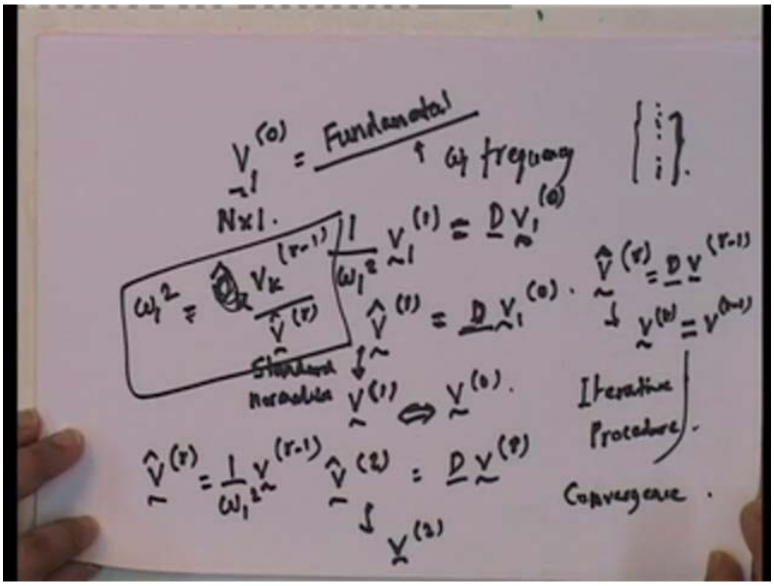

Now, this is this procedure now you do not know omega 1 is right you do not know omega 1, so you call this as and now the trick is to compare. Now, what you do is you normalize this v 1 you normalize because every time you multiply with D you need to bring it back. So, let us say you start with something like one, so after multiplying by D you get some value, now this is a shape this is displacement. So, what you do is you normalize this and bring it back into this format, so you are essentially bringing it then you are comparing v 1. So, the way you go from this to this is through a standard normalization, so compare if they are not same then you do this.

Then, standardize again to get v 2 compare, so this is the iterative procedure and since this is a inverse iteration procedure you will get to a situation where let us say r some r th iteration and this is equal to r minus 1 and from that you find out v r. Now, let us say that v r is equal to v r 1 convergence now, now the tricks becomes that look if you look at it. Since, this is equal to this we know by definition that look is equal to 1 upon 1 omega squared it is actually one upon omega 1 squared v, but we do not want to find this out.

So, what we do is, we do this and how can you find out omega 1 well you can take any k and if the two of them are equal you will see that any k, sorry it is omega 1 squared is going to be equal to v k r minus 1 upon v r r. So, that is my omega 1 and this omega 1 is actually the fundamental frequency.

Now, you may well ask how that happened well again the convergence of this procedure to the first mode the proof of the convergence actually lays in a very fundamental aspect and that is that you can rewrite. You can write it in this format, I will prove it you in a in a rather simple manner. Actually, why should we, because it is not really very relevant for us to solve it and because again this is a numerical method it can be shown that this converges. It converges to the first frequency you can look up any of the books that I have given you as the reference for solution to this particular procedure. So, now the trick is to, so we can find out the first frequency.

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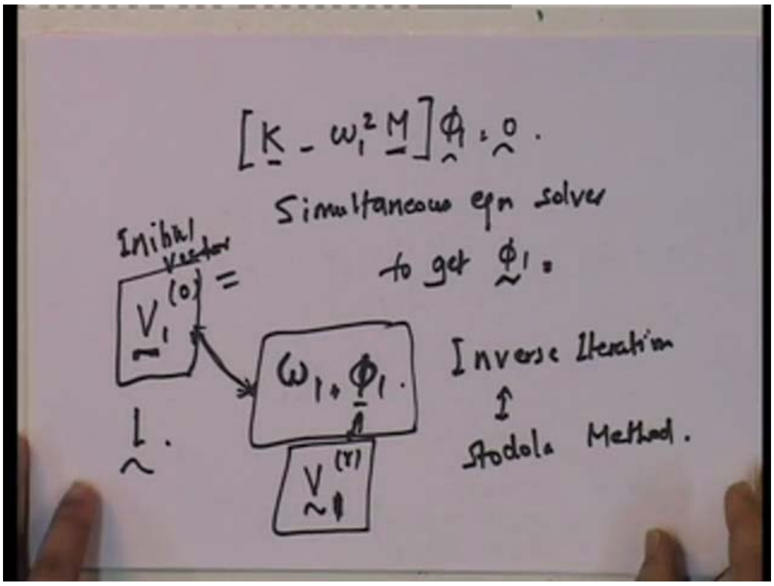

Once, we find out the first frequency please note that to get the first mode shape you still go back there is no other procedure then simultaneous equation solver to get phi 1 there is no question about that. Now, the question then becomes, so you can you can that that this is not an issue at all. So, we found out omega 1 and phi 1 using the inverse iteration or the Stodola method I hope you understood what the method is after I am done. You know we will we will actually try to solve problems with this and I will show you what I will do is, I will have solve the problem that I have done last time and we will try to see wither we can get to the equation.

Now, the question then becomes is that and whatever we want you start from and you do the inverse iteration use iterative procedure. You will always converge to this it does not matter, it can be shown it just that if you use bad initial vectors, this is the initial vector.

This vector is chosen any arbitrary fashion for example, a classical initial vector is just a vector of once it does not matter what your initial vector is, you will always converge to get omega 1 and phi 1. Note that by the way the phi 1 is essentially the v 1 r is really the phi 1 because it satisfies this. So, in other words whatever you start from you will always land up here. So, how do I use the inverse iteration procedure to get the higher modes.

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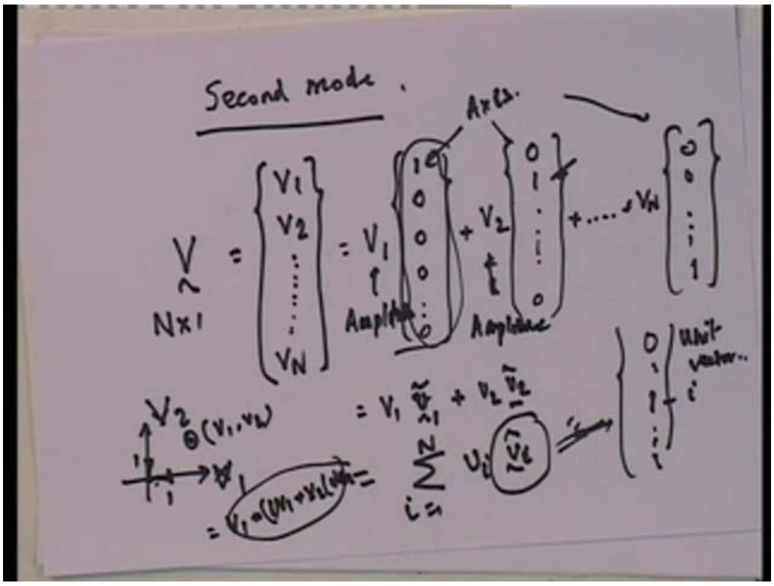

For example, let us say I want to find out the second mode how do I find this out. Now, this becomes the crocks of where I have to go back and look at a specific form and that is note that any displacement vector v is, what it is actually if you look at it this vector would be v 1 v 2 v n. Now, this is equal to I can say is equal to v 1 plus v 2 plus plus v n do you agree to this this can be written this form I mean after all it is nothing but v 1 plus v 2 plus and that is what this is. So, in a way we can say that this is equal to a kind of a kind of this is the unit vector right this is the unit vector.

So, this can be then obviously written as summation i going from 1 to n v i into where this is the unit vector where the i th one is 1 and everything else is 0 that is, what this is, this is a unit vector. So, we can write it so in other words any n by 1 and what are these, these are like the amplitudes this is like an axis, this represent axis. So, in n dimensional space what do these represent this represent axis think about it in two dimensional space what are the axis represent 0 0 1.

So, x is 0 and this is 1 is one axis and one 0 because y is 0 represents the other axis, so in other words these represent the axis all these represent axis and these represent. So, if you have like let us say you know v 1 v 2 space i represent this, this is represented by v 1 v 2 which is nothing but v 1 into the unit vector along one plus v 2 into the unit vector along v 2. So, these represent if you look at these, these represent amplitudes along this axis.

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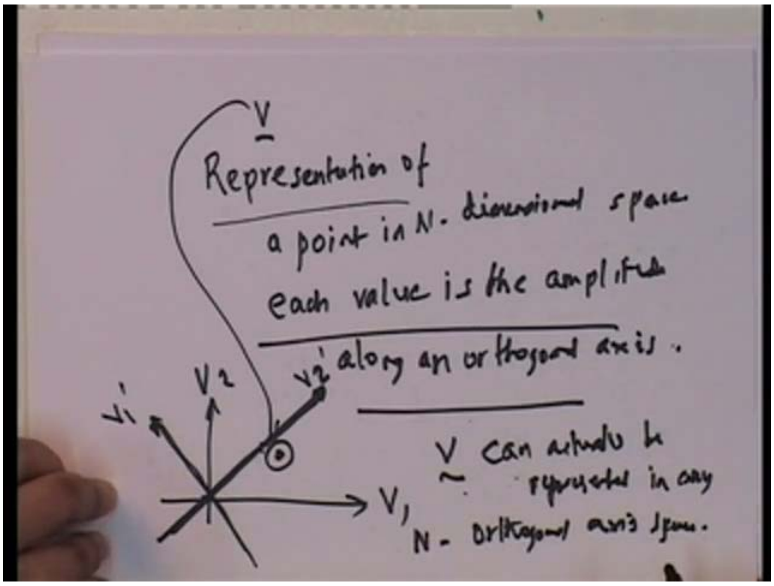

So, in a way any displacement is a representation is a representation of a point in n dimensional space. So, that is what it is a n degree of freedom displacement is essentially a representation v representation of a point in n dimensional space, where we know that these represent orthogonal sets. These are orthogonal this one does not contain this one does not contain this one, so none of them contain each other they are orthogonal. So, that is why we say n dimensional space, because each value is the amplitude along an orthogonal axis, do you agree to this? I have already shown it specifically if I show v 1 v 2 and I should look at this point this point represents a displacement pattern.

Of course, n dimensional space I cannot show it to you, but I can at least tell you that look this is how it looks. Therefore, that is very important now if you look at it a any orthogonal any orthogonal. So, this point this v can actually v represented in any orthogonal axis space as long as it is an orthogonal axis space. Now, let us look at one space one such space that I know once such space that I know is and we know that phi 1 and phi 2.

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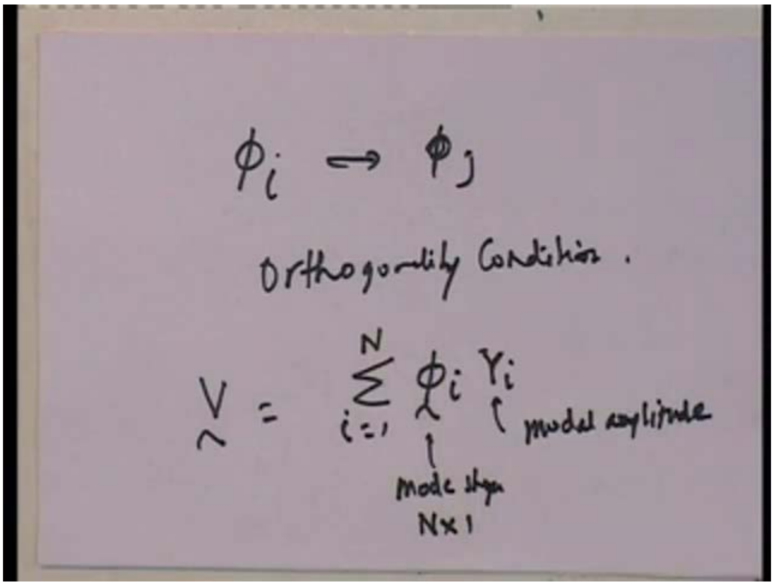

I mean I am talking about you know phi i is orthogonal to phi j i know that because they satisfy the orthogonality condition. Therefore, I can represent the displacement just like I represented, I can represent it in this fashion I going from 1 to n phi i. Now, I will call this is this phi i represent well it represents a direction in n dimensional space which is orthogonal with respect to every other space and what is this. So, this is the mode shape and this is model amplitude and note that this mode shape is a n by 1. I know that this one does not contain any of the other modes because they satisfy the orthogonality condition.

So, what I have done is if you look at what is this line represent. Let us say one line and this represents the 1, minus 1 line, now if you look at it remember the problem that we solved earlier, did you remember that we got v 1 and v 2. We got 1, 1 and 1 minus 1 remember that when we solved it in v x and v y we got 1, 0, 0, 1. So, that was almost like saying that unit vector along the x direction and you know along one which was u x and 1 0 was along u x and 0 one was along u theta.

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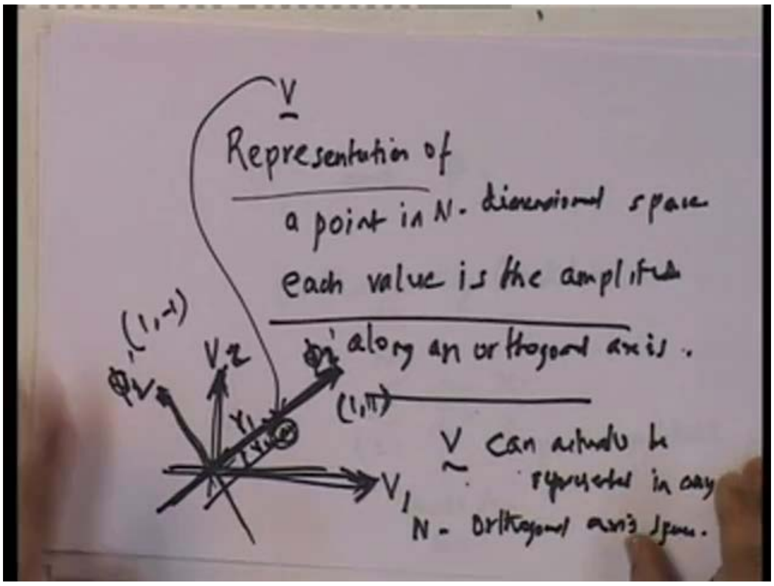

Now, we defined that by v 1 and v 2 we got phi 1 as 1, 1 and phi 2 as 1, minus 1. So, you see this is the original space for representing v 1 and v 2 which are basically v x and v theta and I can also represent them in this axis which is 1, 1, 1, minus 1. They are orthogonal with respect to each other and this point then will have what it will have this as my y 1 and this as my y 2. So, you see those are the amplitudes, so those are the modal amplitudes now the advantage of this is the following that if I were to do the following.

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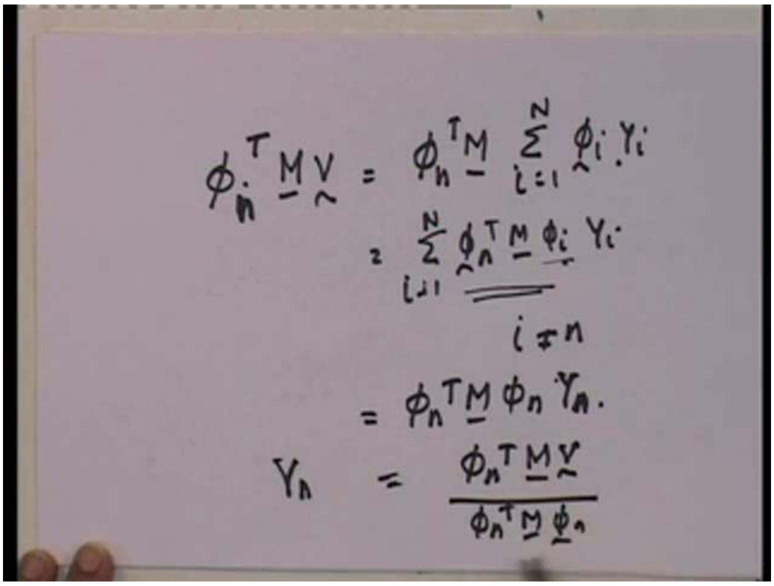

If I were to pre multiply both by let us say phi n transpose M v then I pre multiply the right hand side also with phi n transpose m integration from 0 to i phi i y i. Now, if you put this in is equal to summation i going from 1 to n phi n transpose M phi i into y i, this we know that if i is not equal to n this is equal to 0 this is equal to 0.

Therefore, what does this become this then becomes the following only the term phi n transpose M phi n and y n. Now, what is the modal amplitude equal to, it is equal to phi n transpose M v upon phi n transpose M phi M this is the modal way, I can if I give a any displacement quantity. I can always find out the y n through this procedure I can always find it out and now the trick becomes that let us now look this.

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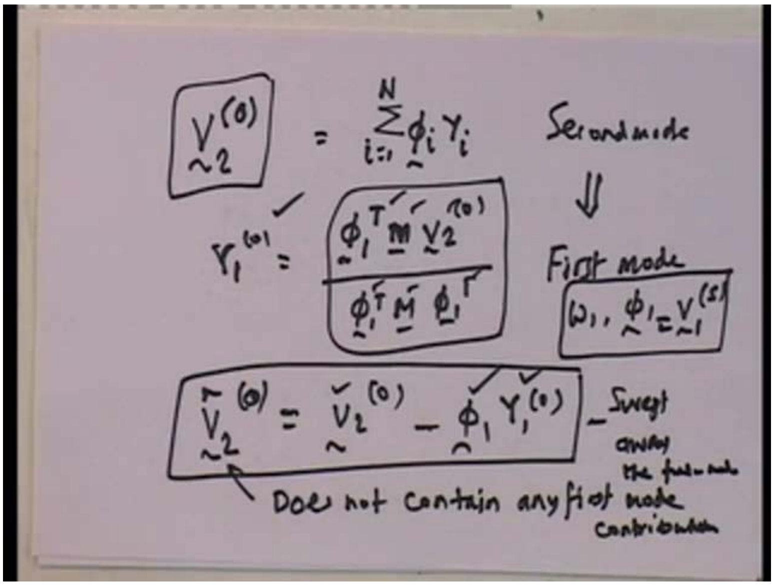

Let me start with a displacement quantity v 2 0 and Now what I am going to do is, I am going to find out, I am going to do what I am going to do the following. I am going to ensure that this is equal to summation i going from 1 to n phi i y i. Therefore, I can always find out y 1, how can I find out y 1 by just doing phi 1 transpose M v 2 0 and differentiate it with phi 1 transpose M into phi 1 transpose second find out. Now, note that I am trying to find out the second mode, I can only find out the second mode. If I have already found out the first mode omega 1 and phi 1, I have already know those is that clear I have already know that and what is phi 1 phi 1 is actually the v 1 s this is equal to this.

So, I have to use inverse iteration to first find out the first mode, now once I know this and I take any mode, I can find out y 1 and what I do is I find out a new v 0 which is v 2 0 minus phi 1 y 1 0. If I subtract out this, now I know this I have found this out from this where I know this, this, this, this, this and this so I can find this out once I find this out I know this.

This I know this because that is what I started with I can get a new value a new value this one does not contain any first mode contribution. So, in another words what we have done is we swept away the fundamental mode we have shifted away if we shift it away then what we do is the following. Now, what happens over here is that you get now note that if you look at this particular one let me rewrite this in in a specific way.

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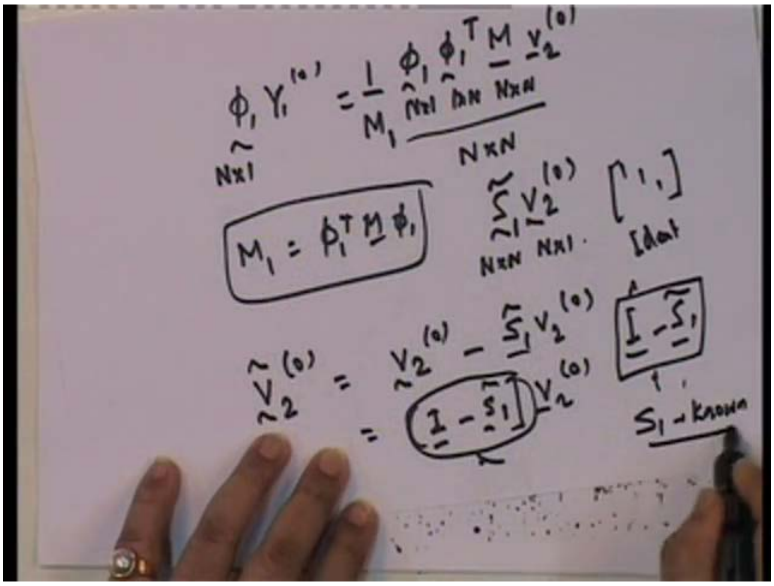

So, what we are doing is you have phi 1 y 1 0 this is equal to phi 1 into phi 1 transpose into M. Now, what are the units well let us look at this this is an n by one into v 2 0 well one upon I will call this M 1, where M 1 is equal to phi 1 transpose M phi 1, we have already gone through that that is a scalar. So, this is a scalar this is n by 1, this is 1 by n this is n by n. So, if you look at this multiplication this is nothing but an n by n, so this this n by n matrix normalized with this I will call this as my s. So, this is s into v 2 0 where s is an n by 1, this is a n by 1, so phi 1 0. So, if you look at this v 2 0 to the is nothing but v 2 0 minus s 1 this is s 1 because it swifts out the 1 s 1 into v 2 0.

So, this I can call as i minus s 1 into v 2 0 note that you know since this is n by n, I have to get a equivalent you know this thing now if you look at this, sorry I will call this as you know s, this i will call by s 1 tilde and this i minus s 1. So, what you are doing is you are taking i is nothing but the identity matrix which is nothing but 1, 1, 1 only in the diagonals. So, this into s 1 this I will call, sorry s 1 tilde I will call as my s 1 matrix, now if the first thing and this s 1 matrix is a known matrix.

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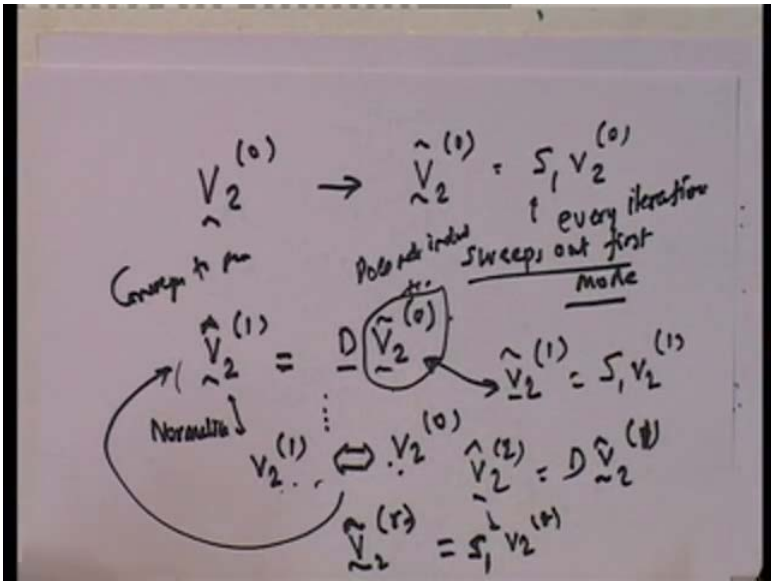

So, the first thing that I do that is I choose a any vector v 2 0 the first thing, I do is sweep this sweeps out first mode. Then if we can write this in this format that find out v 2 1 at is equal to d into v 2, next check v 2 1 this is normalize and check against v 2 0 then iterate that this becomes now v 2 2 is equal now. Now, the thing is that as soon as you find out v 2 check this first, if it does not work then what you have to do is take v 2 1 to the is equal to s 1 v 2 1 plug that in here get the next one.

Find this continue doing this every time you get a new vector v 2 r, you should sweep it to get the v 2 r sweep all the time you should sweep it every time, sweep it before you put it back into the iteration. If you sweep out the first mode then the swept this one does not include first mode converges to second mode. Ultimately, this one and the only thing is you have to sweep it in every iteration why, because if you do not sweep it some value of what will happen is if you do not sweep it some value of v the phi 1 come into this one and if you iterate it more it will converge to the first mode.

So, you have to sweep out the first mode at every instant every iteration, the first thing is check this, if this check is not then take this put it into this sweep it out take this put it in here find out with D. Find out the next one normalize check, if it does not go back sweep everyone has to be swept everyone has to be swept if you do not sweep it out. Then you are always going to converge to the first mode you have to sweep out the first mode. If you sweep out the first mode every displacement quantity that you get does not contain the first mode, it will converge to the second mode.

So, this way you can find out omega 2 and your v 2 which is effectively phi 2. So, this one this is how you find out the second mode well this procedure, you can continue using because you can always find out you know that you can you need to just sweep out. Now, you understand how to get higher modes, you want the third mode.

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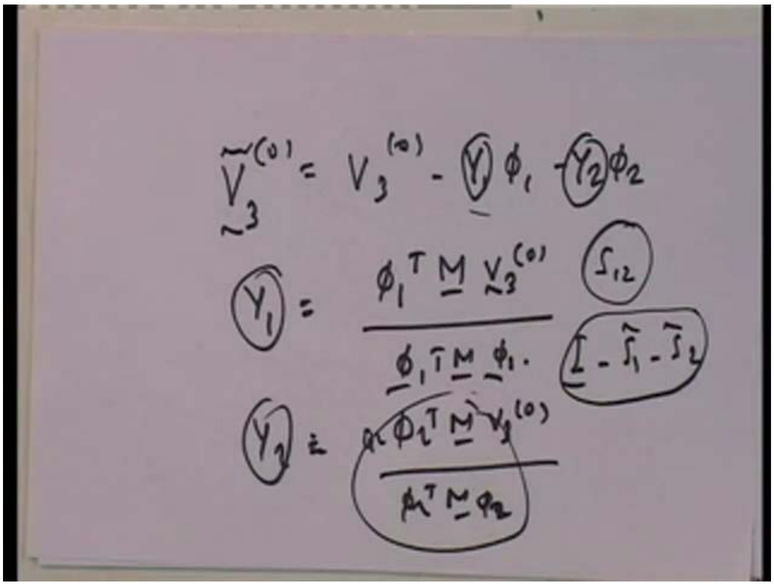

If you want to find out the third mode, the third mode is equal to this minus y 1 into phi 1 minus y 2 into phi 2. These we have already discovered, how to get these the way to get them is well y 1 is equal to phi 1 transpose M v 3 0 upon phi 1 transpose M phi 1 and y 2 is equal to phi 2 transpose M v 3 0 phi 2 transpose M phi 2. This is the way you can find out these two once you find these you can get them out. Again, you know we can write down the sweeping matrix for the you know s 1, s 2 sweeping s 1 2 sweeping, it will become i minus s 1, minus s 2 tilde, where s 1 tilde is nothing but phi 1 transpose and s 2 is going to be equal to p 2 transpose into this.

So, this is and then you just every time you get a value, you sweep it you sweep it and you get the clean value once you get the clean value. It will not contain the first mode and the second mode as it does not contain the first mode and the second mode it will it will converge to the third mode. So, this in a sense is what we need to do where you are looking at this thing, now how do we converge. Now, how do we get the highest mode, now we cannot keep doing this if you have 1,000 degrees of freedom you cannot keep getting every one of them and then going ahead and doing it. So, the way you do it is that you generally just look at this particular solution.

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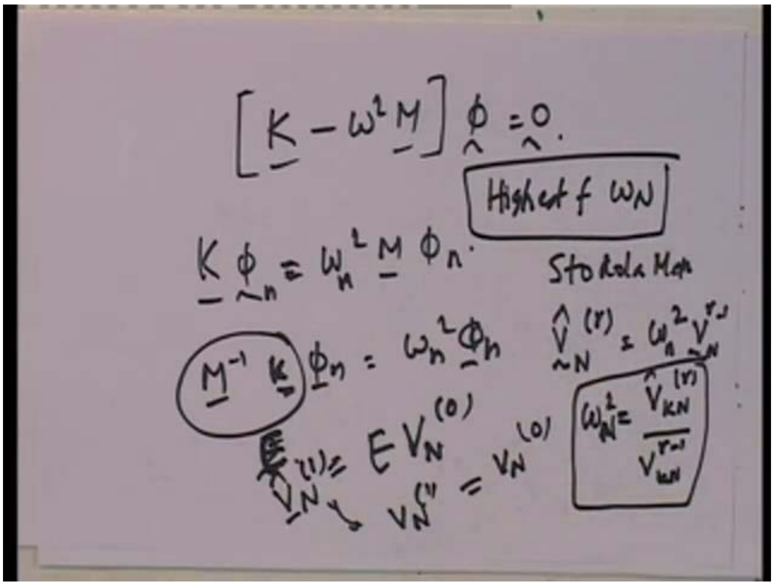

That if you go back there k minus omega squared M into phi is equal to 0. So, what you do is k phi is equal to omega k phi n is equal to omega n squared M phi n, this I mean this is only valid for a specific omega is that clear, that is why you cannot just write it arbiterly. So, it is valid only it is equal only in a particular and what I would like to do is I would like to you know pre multiply both by M inverse k phi n is equal to omega n squared phi n this is known as again the Stodola method. The only thing in that happens is that this, now becomes your D and the ways it goes is you define v n 0.

Then, you multiply it by this, I will call this E because this is different from that, so multiplied by E which is M inverse k and typically M inverse, what do we do M inverse typically will be the lumped mass M inverse is just the you know the 1 upon M at every diagonal step. So, M inverse actually very simple to get and if you multiply with k that one becomes E, so what you do is you find out by E. Then of course, you go you know v and one compare with v n 0 if it does not work go back and find out v n 1, E v n two tildes and ultimately the thing that happens is that v n r is equal to omega n squared into v r minus one v n.

So, we have omega n squared basically becomes any v k of n r tilde upon v k n r minus 1. This is my omega n and this is how this entire thing it can actually be shown that if you use this direct iteration the direct iteration. Actually, you know this is not the inverse iteration, this direct iteration if you take the direct iteration. You actually converge to the Stodola method; I mean you converge to the highest frequency. So, this one turns out to be the highest, sometime finding out the highest frequency is required because it gives you an idea of a lots of things.

I mean basically because some times what happens is considering the lowest frequency you know if you are looking at for example, frequency domain analysis. You do not need to consider a kind of a large enough a range frequencies remember that it is skip for your transform remember that Nyquist frequency, the Nyquist frequency should always be greater than omega n and that is the reason why we always try to find out this. Now, here you know I have taught you two ways, one is root finding and one is iterative the inverse iteration procedure.

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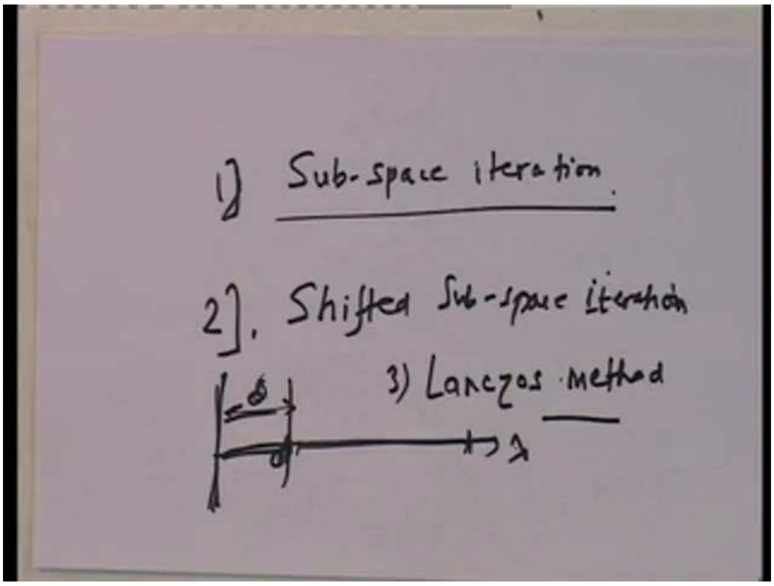

Now, there are lots of other procedures you know there are I mean, I will just write them down, I am not going to discuss them. Here, for example sub space iteration this is actually in a way this is you know here you are doing frequency for frequency each one the sub space iteration is why you take a space. So, instead of having a specific you sweep out certain once and so that you can get a block of frequencies in one shot. I am not going into this, if you all are interested in it you can always look up other methods.

Now, in certain kind of structures especially for example structures that is not stable. For example, you know for a framed building you do not get 0 frequencies, but suppose you have a satellite is also a structure. You would like to investigate, what kind of dynamic behavior happens up the top, now you know you have, what is known as zero frequencies. In another word you know, what is the zero frequency, a zero frequency is where there is only rigid body movement. It is only rigid body movement, there is no deformation, there is no strain energy, there is no strain energy k is 0, if k is 0, M is non zero.

So, obviously omega 0 omega, so zero frequencies, so any rigid body mode has 0 frequencies in that particular case, how do you eliminate zero frequencies by looking at what is known as shifted sub space iteration. You know we really look at 0 lambda, we know that these lay in this domain, but suppose it lies in this domain all that you do is take a delta shift put it here put the axis at a known delta sift away.

Now, this becomes specific one, so you can do that that is called shifted sub space iteration then there is the Lanczos method these are all methods that are used for practical vibration analysis. So, root finding inverse iterations Stodola method sub space iteration shifted sub space iteration Lanczos. These are all various ways of doing free practical free vibration analysis for you know real structures. So, I will stop here.

Thank you very much.