Structural Dynamics
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Lecture - 25
Free Vibration for Multi Degree of Freedom Structures

Hello there, last time we started we developed the equations of motion and then started of on free vibration.

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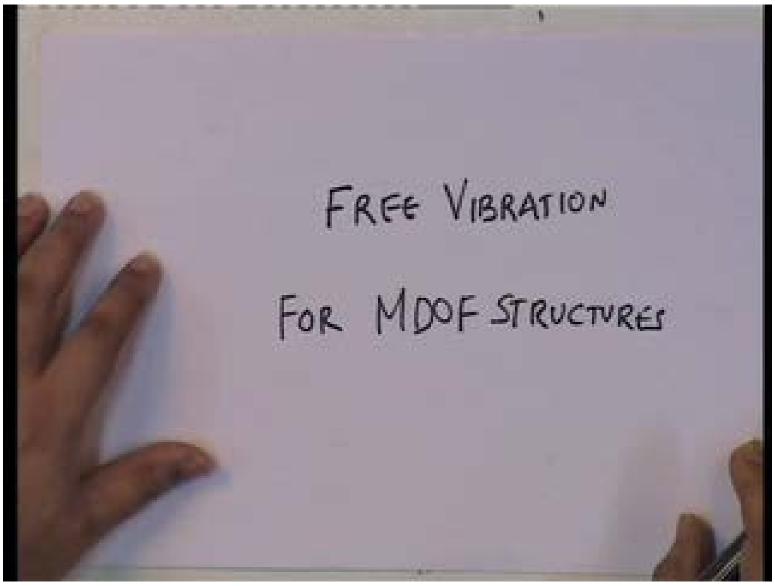

So, let me continue with Free Vibration from Multi Degree of Freedom Structures. So, this is the topic that we will be dealing with in this lecture. Just to review what we had done last time we started with the fact that.

The free vibration problem essentially became a solution to this problem this was the solution and we saw that since both of these are symmetric and positive definite. Therefore, you get both the solutions to this are n positive Eigen values and so these are given as omega one square which is the as I said of course, when I put this it automatically means that I am these are the Eigen values and the natural frequencies are ordered so this is the lowest frequency and the lowest frequency.

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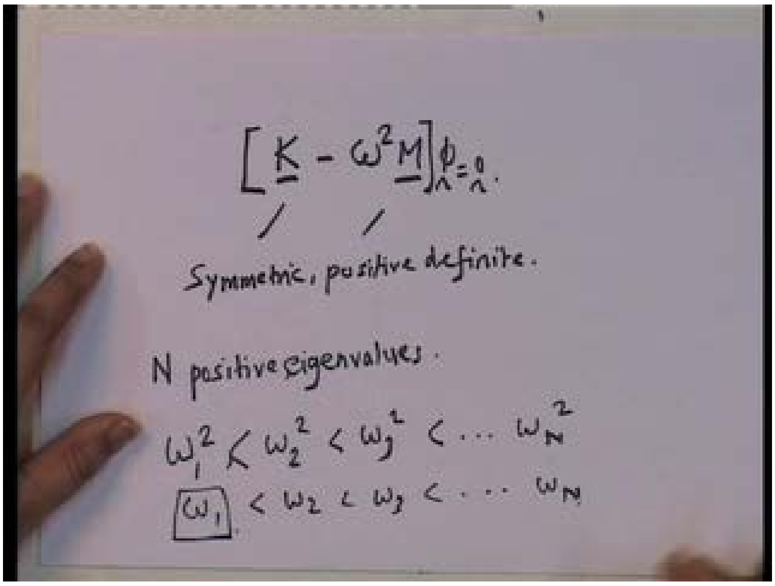

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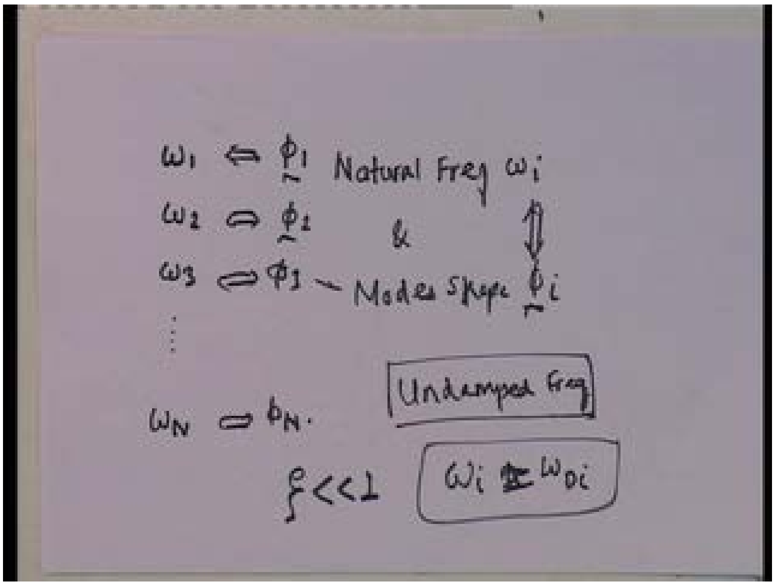

So therefore, what we have then is the following is that you have omega one then you have omega two omega three on to omega n ordered in this manner and corresponding to omega one you have phi 1 phi 2 phi 3 and phi n these are the frequencies natural frequencies and mode shapes. Now, it is obvious we have not stated it and that is that these are automatically un damped frequencies mode shapes these are the mode shapes natural frequency omega i and mode shape and corresponding mode shape.

These are un damped frequencies and the issue then becomes is a why are we spending so much time in getting un damped frequency and that is because we accept here that since we are looking only at multi degree of freedom frame problems. Therefore, the basic fact is that for these damping significantly less than one and therefore, omega I and omega d i are practically the same.

So therefore, we do not really need to find out omega d i we are happy with finding out omega I, and going with it primarily because you know you cannot really find out damped frequencies until and unless you have a c matrix also established. Damping matrix now, note that the mass matrix as the stiffness matrix are properties that can be established for a given you know structure given the properties of the structure given the dimensions of the structure we can get the mass matrix and k matrix.

The damping if you remember if you think back to your single degree of the freedom damping you see how did, we define we said that well it was equivalent to viscous dash spot. So, that is why called viscous damping and we said well that is proportional to the velocity and so we identified a dash spot constant c and say it is c v dot that is how we introduced concept of energy dissipation.

Now, the problem becomes that you know in that case we saw that psi was equal to c upon 2 m omega and we said that look we instead of looking at c identifying c for a structure. We are going to identify psi and we saw that and you know another point is that during this course, you will also have some laboratory experiment where you will c that psi is essentially a material related constant.

So therefore, there for single degree freedom it will very easy to define 1 psi and then you know psi c was equal to 2 psi m omega and from that we got c n we know it we could write it down all we could actually write it down in terms of this thing. For a multi degree of freedom structure the problem becomes not a single c it is actually a c matrix if you write it in that form you will have to develop a c matrix and now that is not such an obvious thing as we will see later on in this class itself.

So, the question then remains is that as far as multi degree of freedom system are concerned we actually stop at in general in general we stop at developing the un damped frequencies and accept the fact that the un damped frequency and the damped frequencies are very close to each other. So, in other words we are not really treating this that there is no damping in the system we were just developing the un damped frequency problem and we are getting that.

So therefore, the natural un damped frequency, but natural frequency we just calling as the natural frequency. In this particular case and there are certain important accepts here that you are likely to see and that is that these mode shapes satisfy certain properties and the property is the following it called the orthogonality property and the orthogonality property says these are the this is called as mass orthogonality and this is called stiffness orthogonality.

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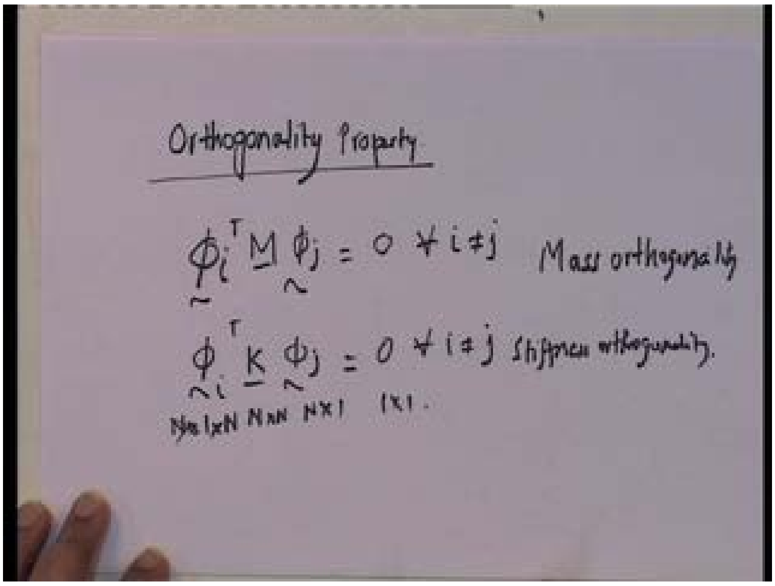

What says is that phi i m phi j is equal to 0 for all i not equal to j and phi i into k phi j is equal to 0 for all i not equal to j note that this is sorry this is not this way this is phi i transpose. So, this is a one by n this is a n by n this is a n by one when you multiply it you get a scalar one by one so this is the orthogonality property now, how do we prove this orthogonality property.

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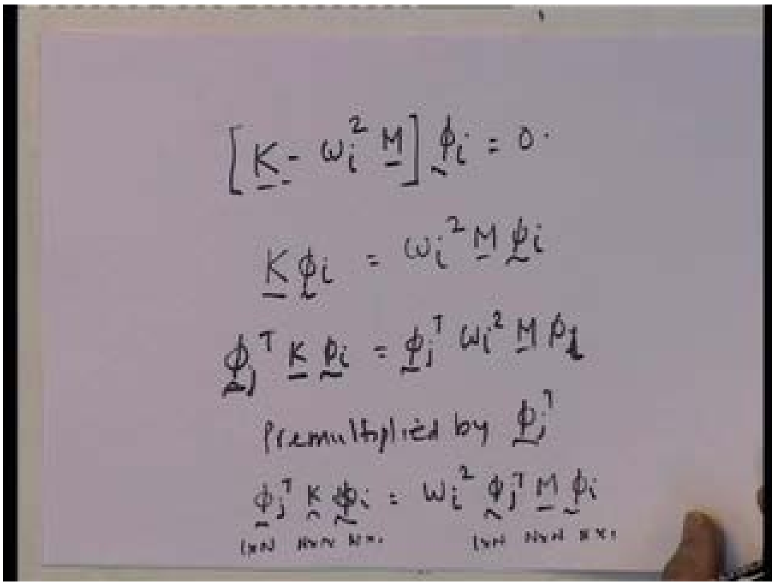

Let us look at it lets consider that k minus omega i square m phi i is equal to 0 so if you look at this we are seeing that k phi i is equal to omega m phi i you accept that now, I will just kind of written it you know you can write this I have just taken this on the other side so it is k phi I minus omega i m phi i is equal to 0. So, therefore, k phi I is equal to m omega phi I accept that. So now, let us look at this I am going to pre multiply both sides see I can do that as long as I you know if these are equal if I pre multiply them by some I can do that I am sorry this is i phi i.

I can do this I just what I have done is I have just pre multiplied by phi j transpose both sides both sides of the equation I have pre multiplied by phi j. So, this becomes what this becomes the following that phi j transpose k sorry k phi i is equal to omega i square phi j transpose m phi i what is the this thing this is one upon n into n into 1 so both of these are scalars both sides are scalars you agree to that so now, so this is one part.

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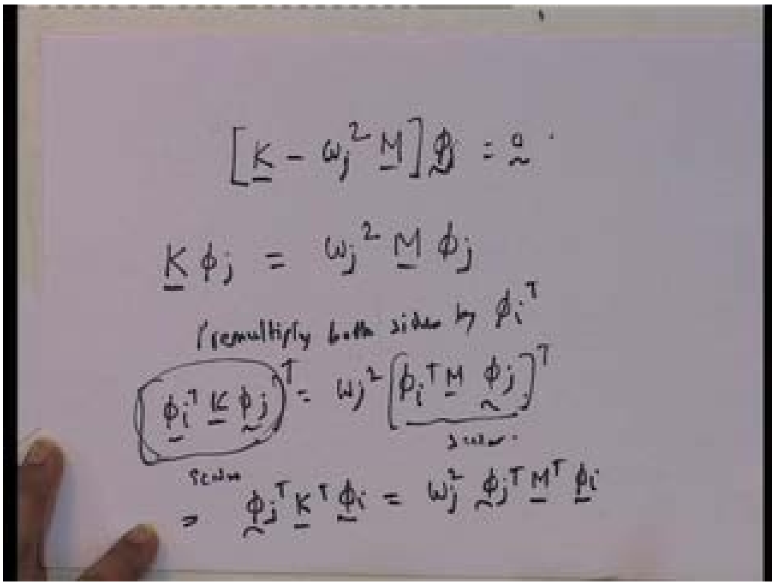

Let us look at this situation that k minus omega j square into m into phi j is equal to 0. So, again rewriting this we get it as k phi j is equal to omega j square m j so now, I am going to pre multiply both sides pre multiply both sides by phi i transpose if I do that this becomes phi i transpose k phi j is equal to omega j square phi i transpose m j. Do you agree to that now these two are what are these are scalar so now, what I am going to do is this part I will take its transpose if I take its transpose since these are scalars the transpose will be themselves.

So therefore, if you look at it this way I can do because this is scalar a scalars transpose is the same is itself a scalar transpose is what one transpose of one is one it is a scalar. So therefore, if I do that if I take the transpose of this then this becomes phi j transpose k transpose phi i transpose transposes itself is equal to omega j phi j transpose m transpose phi i transpose transposes itself.

So, that is that is what we get because a scalars transposes itself. So, therefore, this is what I get so now, from the two things if I look at what I have what do you get well one we got the previous one we got that look phi j transpose k phi I is equal to omega i phi j transpose m phi j so let us write that down.

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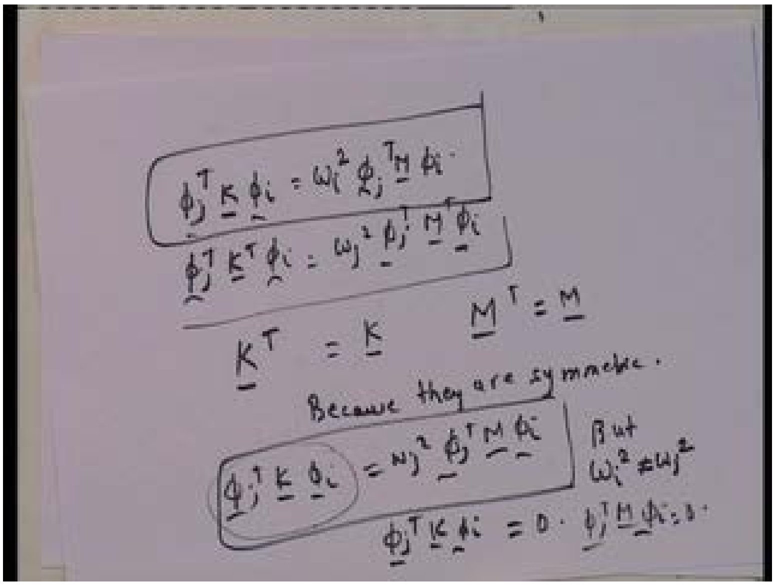

So, if I write that down this is turning out to be equal to phi j transpose k phi i is equal to omega i into phi j transpose m phi i and if I look at it from this one I get it as phi j transpose k transpose phi i is equal to omega j square phi j transpose m phi i m transpose this is what we get from the two sides now, what is k transpose the transpose of a symmetric matrix is itself. So, k transpose is equal to because they are symmetric.

So, what is this equation become this equation becomes phi j transpose k phi i is equal to omega j into phi j transpose m phi i do you agree to that. Now, but you see this is the scalar that so if you look at this scalar then if you look at it what does happens then that this divided this if you look at this one, and this equation are identical, but omega i square is not equal to omega j square do you agree to that because they are two different natural frequencies.

So, they are two different frequencies so if you have a situation like this where omega i and omega j are not equal to each other, but both sides are identical what does that mean by definition that means, that phi j transpose k phi i is equal to 0. So, that that is the first proof now, in similar way because you see once we put it in that way you know we can always say that look if this is equal and you know omega i square and omega j square are not equal to 0, then automatically it is also mass orthogonal. Now, this is actually a very important property the property of the orthogonality is a very important property and this we will see is actually going to help us the orthogonal.

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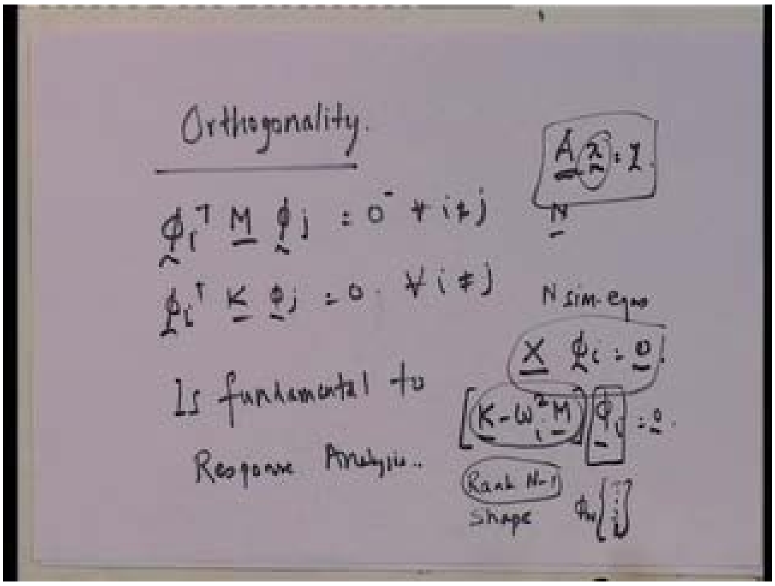

The fact that phi i transpose m phi j is equal to 0 and phi i transpose k phi j is equal to 0 these property for all i not equal to j for all i not equal to j, so these this orthogonality property is fundamental to response. We will see you know later that how to get you know how to get how to use this orthogonality. Now the question becomes that you see one of the major issues that happens is that since mode shapes are just a shape you know you can only the this matrix k minus omega square m is equal to 0 since this is determinant is 0.

So therefore, as I said rank is m minus 1 and what you do is this you cannot evaluate explicitly what you get is that you will get a shape which is in terms of some like for example, I can put it this way I can find out n in terms of 1. So, that is why this is a shape rather than a specific value because this is rank n minus 1, so you can only you only have n minus 1rank n minus 1 what does that means, it just means that it is essentially n minus 1 you see you have n n n equations here, there are n equations to solve for phi i because this is a matrix this becomes a matrix something you know I mean an x matrix.

So, x into phi i is equal to 0 essentially represents n simultaneous equations. Now, ideally if x is determinant is not zero, then x is of rank n and you can explicitly get it is like this A x equals to y this is simultaneous equation as long as a is of rank of n you can and y is given you can always find out x by solving simultaneous n simultaneous equations the problem here is that this determinant is indeed 0 and it is of rank n minus 1 so essentially what it means is that you only have n minus 1 independent equations.

So, when you have n minus 1 independent equations, you can only get n minus 1 values so that means, the what does that means essentially you are getting you do not know one and you getting all of them in terms of the other. Now, this brings in a certain amount of randomness in the mode shape and why is that? Because essentially well you may choose phi n to be one I will choose phi 1 to be 1 it will be the same shape, but it will look completely a different a mode shape.

So, the thing is that now lot of people say well what do we do well you look at the mode shape and you choose the highest value as one so that every other value is less than one now, my question is these are all fairly random processes. So therefore, there are a specific types of mode shape which everybody would get the same mode shape and that mode shape is called as an orthonormal mode shape.

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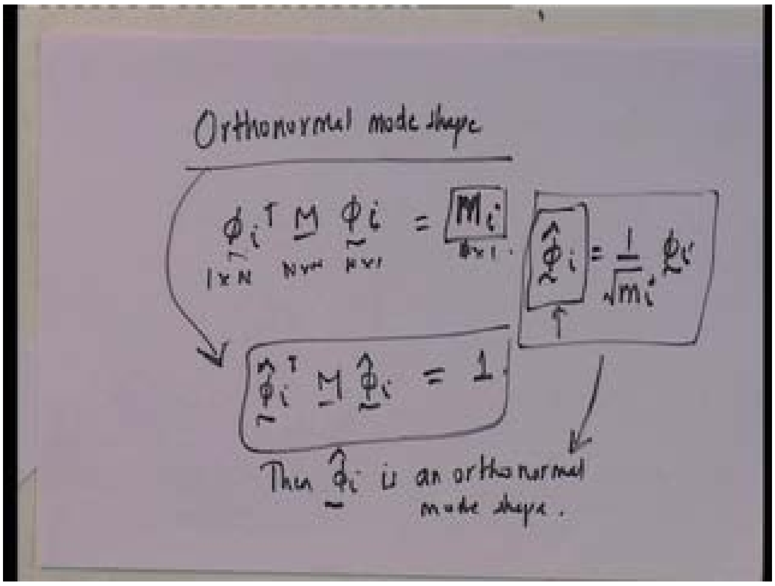

What is orthonormal mode shape typically see phi i transpose m phi j is 0, but phi i transpose m phi i is not 0 this is some I will call it as M i because this is after all since these are dimension less and this is the mass you know this scalar quantity note that this is 1 by n n by n n by 1 so this is a 1 by 1 this is scalar. So, if if I do this then an orthonormal mode shape is one where I will call that orthonormal mode shape by a hat phi i transpose M phi i hat is equal to one this is the definition then phi i hat is an orthonormal mode shape.

So now, given so I find out any one how do I make it orthonormal well very easy if you look at it this becomes very simple phi i hat is equal to 1 upon root m i into phi i this phi i use any phi i. So, I mean no it does not matter how you get your phi i you know once you get phi i you do phi i transpose m phi i you get your this value and then the orthonormal is one upon root m i into phi i this everybody will get the same shape and if you know it is obvious because this is the case just plug it in you will see one upon root m i phi transpose and one upon root m i phi i transpose.

This phi i transpose giving m i one upon root m i into one upon root m i into because one upon m i one upon root m i becomes one. So, automatically this is your orthonormal mode shape how to evaluate an orthonormal mode shape and this orthonomal mode shape is unique this is unique, so now the question becomes is that we have looked at mass orthogonality we have looked at stiffness orthogonality phi transpose k phi j is equal to 0 for all I not equal to j.

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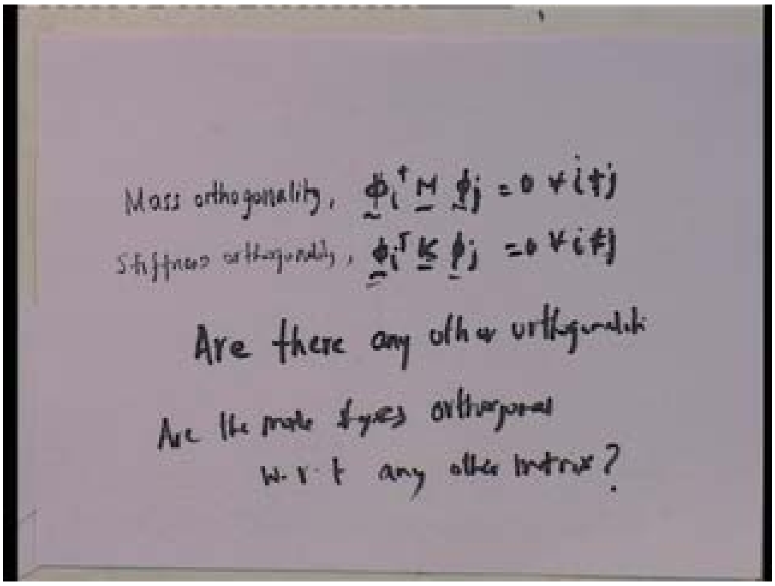

Now, the question becomes are there I will just so that you can see it for all i not equal to j for i not equal to j are there any other orthogonality in other words are the mode shapes orthogonal with respect to any other matrix that is. That is the question that comes up are the mode shapes we have seen mass orthogonality we have seen stiffness orthogonality and the question then becomes is that are there any other orthogonalities in the system and this was something that caughey.

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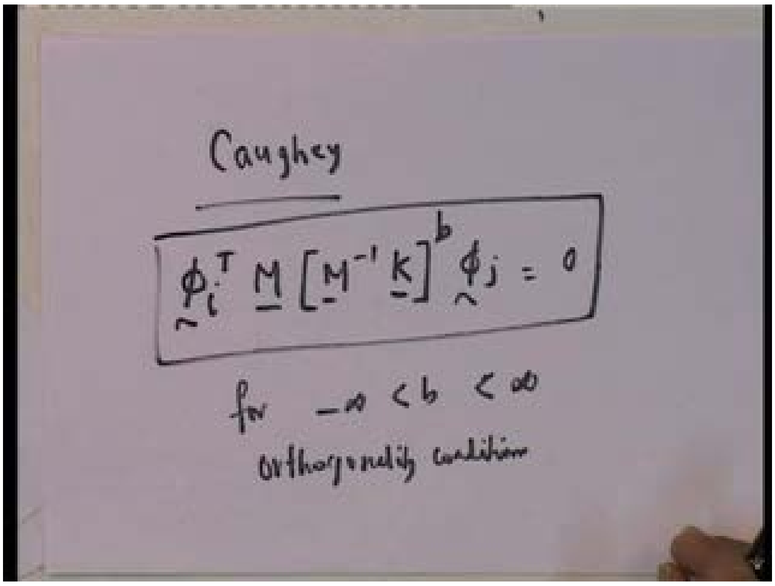

Tom caughey who is a who was a professor of structural engineering at k tech he discovered that indeed there are other orthogonality properties that you can get and he proved that there are a class of orthogonality which are given by phi i transpose into M M into M inverse k and to b into phi j transpose is equal to 0 for minus infinity to infinity in other words orthogonality conditions exist for an infinite number of values well let us look at a specific value.

Let us see I will show you what this one is how this one is proved by looking at a particular value see put b equal to 0 what is b equal to 0 mean that means, this one to the power of 0. So that means, that is just my mass so you know if you look at this b equal to 0 I have phi i transpose M into M inverse k to the power of 0 phi j which is nothing, but phi i transpose M phi j which we know is 0 b equal to 1 phi i transpose M into M inverse k into phi j what is M into M inverse i i into k k.

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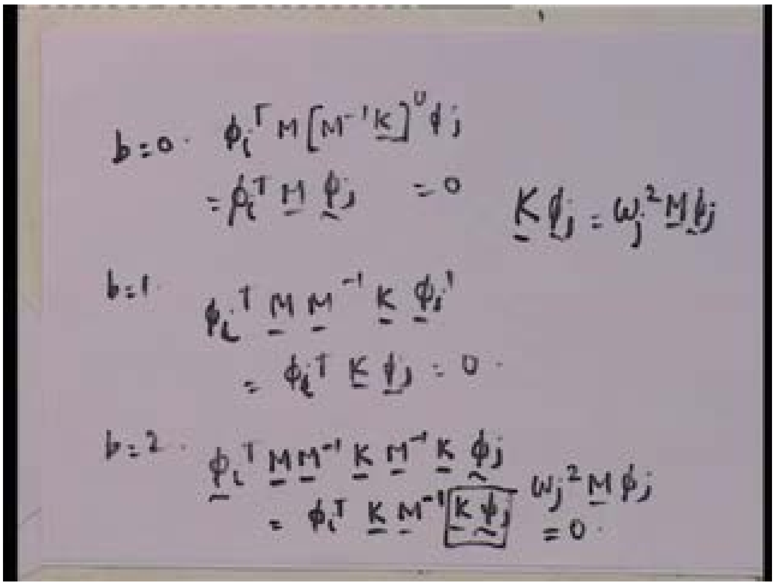

So, this is equal to phi i transpose k phi j which we know is equal to 0 let us look at b equal to 2 b equal to 2 is phi i transpose M M inverse k into M inverse k into phi j let us see what this is so if you look at this M into M inverse is nothing, but i into k is this so this is equal to phi i transpose k M inverse k into phi j. So, let us look at this what is that is equal to what well, let us see let us look at this particular one and that is let us let us look at the term k inverse k into m inverse.

Now, if you notice the following that what is k phi j note that k phi j is equal to omega j square M into M phi j do you agree to that we have already seen it. So, if I substitute this by omega j square into M phi j then what does it become if you look at M omega j is a scalar so I can take it outside so I get m inverse into m which is i into k is k into i is k so again I get phi i transpose k phi j into omega j square which is non zero.

So, obviously, this is equal to 0 and in this manner we can keep showing that indeed you can you know I mean, so I can show b equal to minus one. I do not want to go get into that because minus one all it means is the inverse what is b minus 1 let me just show it because you have to understand what this essentially means.

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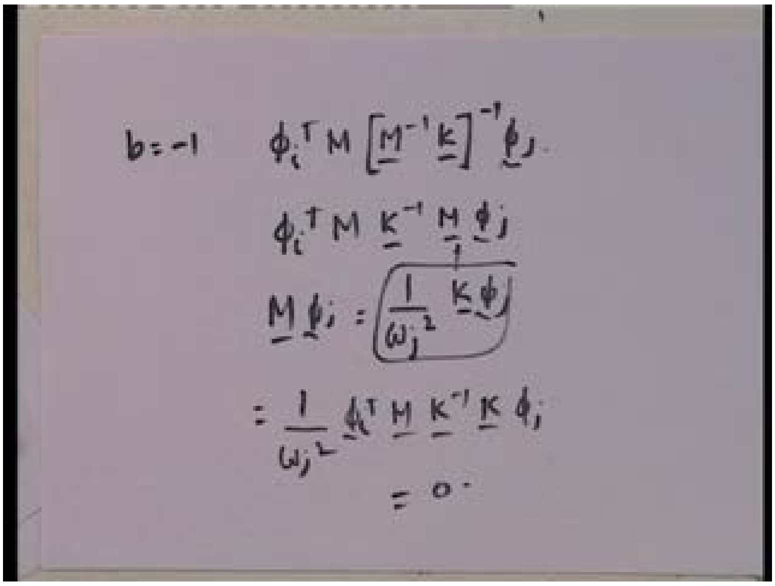

This essentially means is that b is equal to minus 1 all it means is phi i transpose M M inverse k to inverse and what is the inverse of k inverse it becomes phi i transpose M k inverse M inverse which is this now, I can say that look M j M into j is equal to 1 upon omega j square into k j. I have just put this omega j square which was here on this side so I can then put this in here plug this in here and so this will become one upon omega j square phi i transpose M k inverse into k phi j note that k inverse is basically i.

So, this become M into i 0. So, you see I have shown to you this was I mean you know caughey actually developed it in a in a much broader sense, but I have showed it to you that it is indeed a possible to show that all values of b from minus infinity to infinity. Actually are have an orthogonality condition built into it now, this makes it very interesting and that is that.

We can now start talking about damping now, you may ask that why am I talking about damping in this particular state because you have to understand that we have to develop the concept of a damping. So, because ultimately it goes down to the fact that if you have damping you need to know the damping otherwise if you do not incorporate damping into your system there are issues, now let us look at this how so now, the question becomes how do I get a c matrix.

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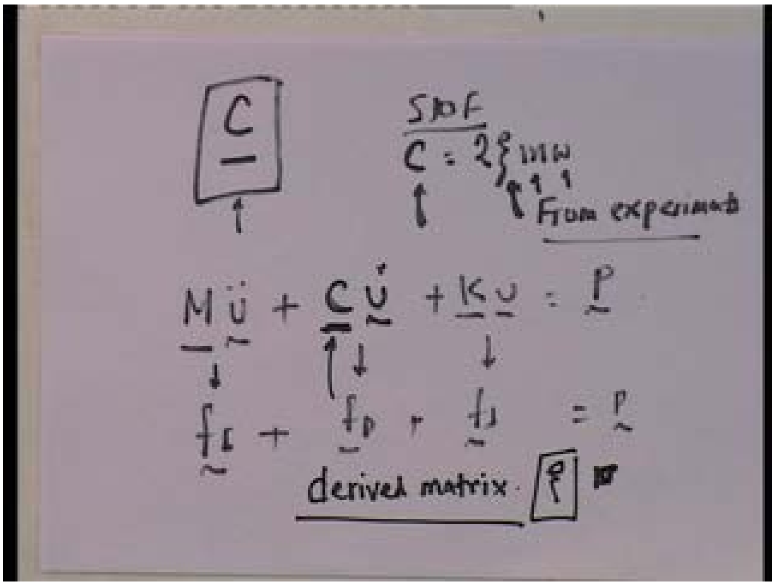

Now, you see there are various reasons behind it and that is that why get a c matrix well you need a c matrix so that you make it look like this where this is these are the inertial forces corresponding to the degree of freedom these are the damping forces these are the elastic forces and so this plus this is equal to the applied load. So, I mean you know it is just a question on later on I will show you that you really do not need to find out c, but you know the question comes out is that well you know I mean suppose I need to find out c how do I go about it.

Note that in the earlier case in the single degree of freedom case c was actually evaluated from two psi m omega and I knew this from experiments I know this from the experiments. So, once I do that I know this I know this and therefore, this was evaluated for a single degree of freedom system so essentially it comes back to having to come from that you get this from the experiments and this you get.

So, understand that this is a derived quantity it is a derived quantity. So therefore, even in a single degree of freedom this was derived and so in here c is again a derived matrix what is it derived from psi it basically comes from psi. Now, the question here is the following that there was only one natural frequency and therefore, only one psi in this particular case what do we have omega 1 omega 2 omega 3 omega n.

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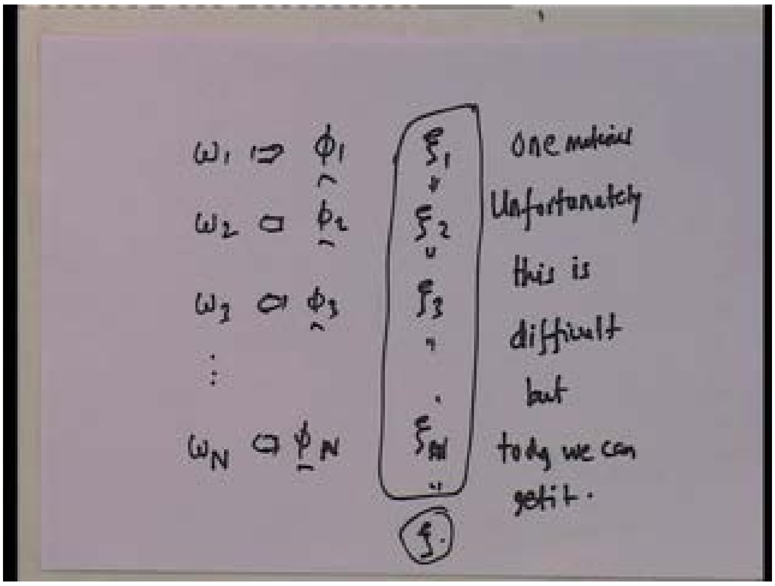

So, I have n frequencies and the question then becomes is that well corresponding to these. I have phi one I have mode shapes and these mode shapes satisfy the orthogonality condition, but in addition the question, now remains is that well this is the mode is there a psi one can I find out from experiment can I find this out for experiment unfortunately this is difficult, but today we can get it we can experimentally determine this.

Of course, you know if we say that the frame was made from one material we have stated that look if it is made from one material we say that for the material your psi was a function of the material. So, if you are using one material then we should say that these are all equal to each other and equal to psi which is the materials psi value and in fact, why not you know although you know we can get for every frequency.

I will not get into that now how we do it, but we can and the question then comes down to the following and that is that essentially, if we say that the frame is made of from one material then a material has a damping that was one of the first things that we say otherwise you will go to get very difficult. Because for every material I mean for every structure we have to do an experiment to find out psi and you know to fill the structure to find out it is psi.

Now to design the structure you need the psi it is a chicken and egg kind of situation we got out to the chicken and egg situation by saying what well if it was made from concrete we would know what psi would be remember. I said that experimentally is being shown that if concrete is subjected to low strains psi is about 3 percent 0.3 if it subjected to relatively high strains psi is equal to 5 percent.

If it subjected to very high strains psi is equal to seven percent for steel well it is about one percent for low strain it is about two percent for medium strain and up to about may be 3 percent 2.5 to 3 percent for as long as my building is made from one material I have no problem I know the psi and I can put that psi in every mode. So therefore, you know damping is being incorporated only problem is it is been incorporated in every mode and how does that help me well ultimately if you look at it I need.

You know I can actually remember I developed numerical method the new mark beta method the linear acceleration method well look at this you know this you know all that does is it sets up a kind of a delta k till the into delta b is equal to p. Now, you know so that was you remember we did that we said that delta k I mean sorry s not k delta k k till the into delta b is equal to delta p and that is how we found out delta b remember we solved that now here also if we do that same thing all that we will get is that we will get k till the into delta v into delta p all will have is n simultaneous equations.

So, the o d can be transferred into a simultaneous equation which can be solved using any simultaneous equation solver. So, therefore, you know it goes back to the fact that to get this we need c matrix so how do we get the c matrix now, the question here is the following that damping as I showed you that is single degree of freedom damping had a very eclectic beginning in multi degree of freedom.

Damping had a very interesting concept there was this business when rayleigh we will show later why proportional damping proportional damping was important I will tell you what proportional damping is I am just introducing the concept here rayleigh said that look take c is equal to now, you can take mass proportional damping mass proportional damping would be c is equal to alpha m stiffness proportional damping c is equal to beta k. So now, and Rayleigh said well this is the combination it is a combination of mass proportional damping and stiffness proportional damping.

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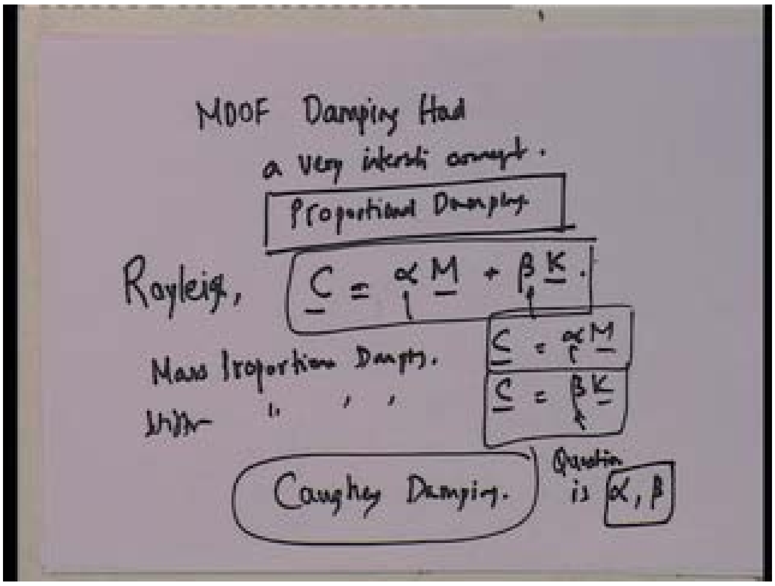

So now, the question becomes so what we still do not know let us look at let us look at first this these then we will look at this and then we will slowly look at a broader kind of damping which is called as Caughey damping. I will come to this Caughey damping later on in the in the next lecture the question here then becomes is that how do I find alpha and beta. Because or you know if it is mass proportional damping how do I find out this proportionality constant.

So, the question is alpha later how do I find these out and multi degree of freedom problem I am just identifying this and I will come back to this much later I will come back to this not in the next lecture will come back to it later to show that getting alpha and beta is easy provided. You define the damping in each mode remember this that this is necessary this is the definition and of course, typically we define them to be the same in all modes, but this is definition is essential to getting c matrix and this in essence we will see is overall how damping will be taken into account.

But I am not I am going to get out of damping now and come back to the concept that essentially you know damping is important because dynamic characteristic in each mode damping is the integral part of damping. You know the dynamic characteristic these are dynamic and has we have seen that the response of a structure is determined by it is damping characteristics.

So, it is very important for us to define dynamic characteristics very appropriately and that is why I brought in the concept of damping. I want to end with one final comment and that is there is a reason why we do this that you know we put it the highest frequency and the lowest frequency as a first mode why well this mode the lowest mode is also known as the fundamental mode fundamental mode of vibration of the structure I will talk about this mode in the next lecture.

Thank you, bye.