Structural Dynamics
Prof. P. Banerji
Department of Civil Engineering
Indian Institute of Technology, Bombay

Lecture - 24
Multi Degree of Freedom Structure Equations of Motion and Free vibration

Hello there, in this lecture I am going to continue looking at the Multi Degree of Freedom Equations of Motion.

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And then what will do is we look at the free vibration also of multi degree of freedom systems, so now to continue of where I had left off, last time I had started with, I mean I ended with say talking about the lumped mass formulation. And how nicely it makes the problem better, so let us just go and review that again, so what you have is you have in the consistent mass formulation.

Now, note that the consistent mass formulation, and the lumped mass formulation, note that these are only at the member levels, they are distinguished only at the member level and not at the structure level, these two approaches are only at the member level. So, this we saw and all that happened was that in the consistent mass formulation, you have a full metrics, because at the member level v 1, v 2, v 3, v 4, there are four degrees of freedom, just lets revisit that a little bit.

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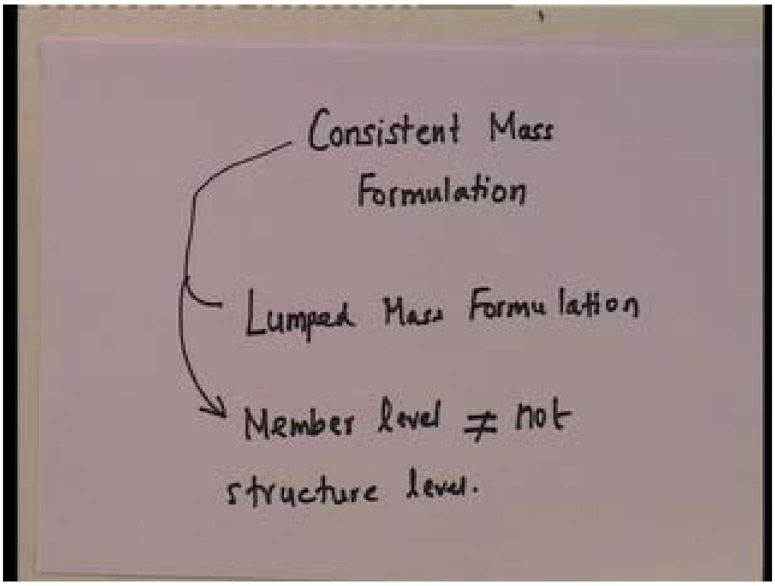

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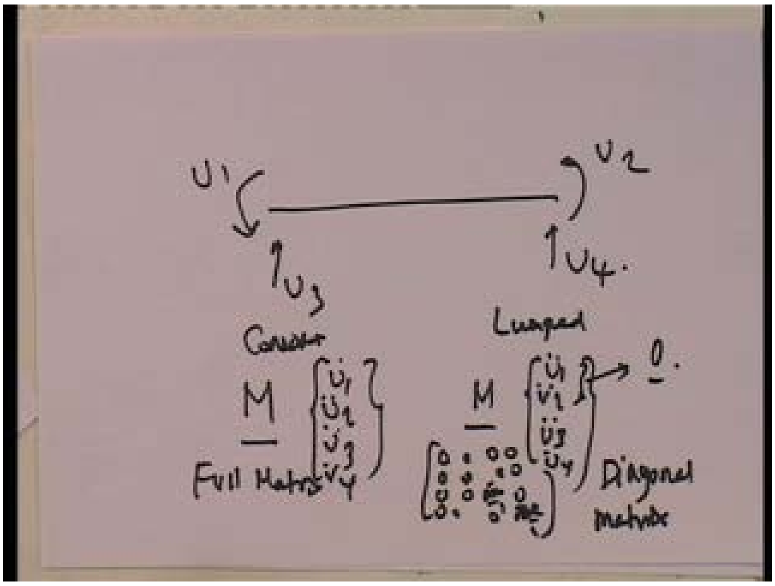

At the member level you have v 1, v 2, v 3, v 4, so in the consistent mass metrics, the mass metrics corresponding to v 1 dot and v 2 dot and v 3 dot and v 4 dot full metrics. And this is consistent lumped M, since a lumped mass is lumped at the two ends, v 1 and v 2 it is completely 0 and the only thing that we have is, so it is not only a diagonal metrics, but corresponding to these two it is a 0 metrics. So, the entire lumped mass I want to keep on emphasizing this point, because I disgusted me today to see that people do not understand, whether lumped mass formulation came from. And they continue using this lumped mass formulation for kind of situations, where you do not, it is not valid, let this will be very, very clear, the lumped mass formulation was created for only framed structures.

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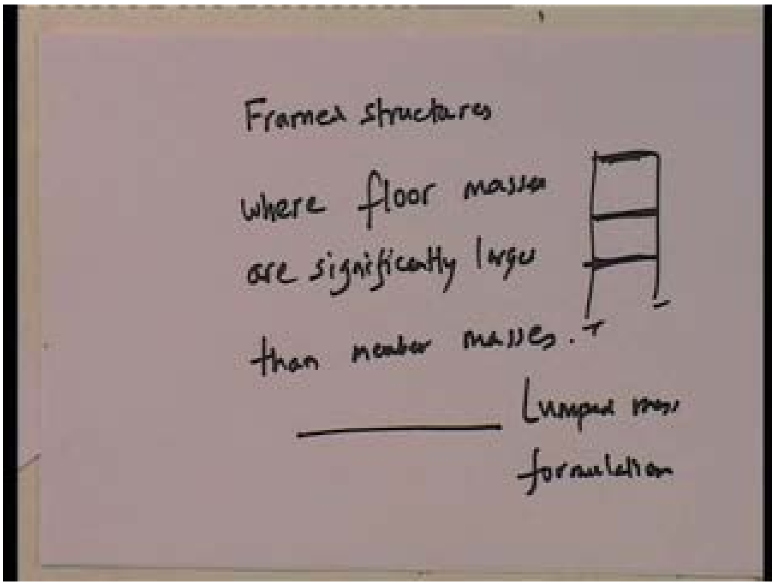

Where floor masses are significantly larger than member masses, this is very very clear, so in other words frames, where floor masses are significantly larger than the member masses. For these lumped mass formulation is now, I will show you how ridiculous the lumped mass formulation becomes, if you do this.

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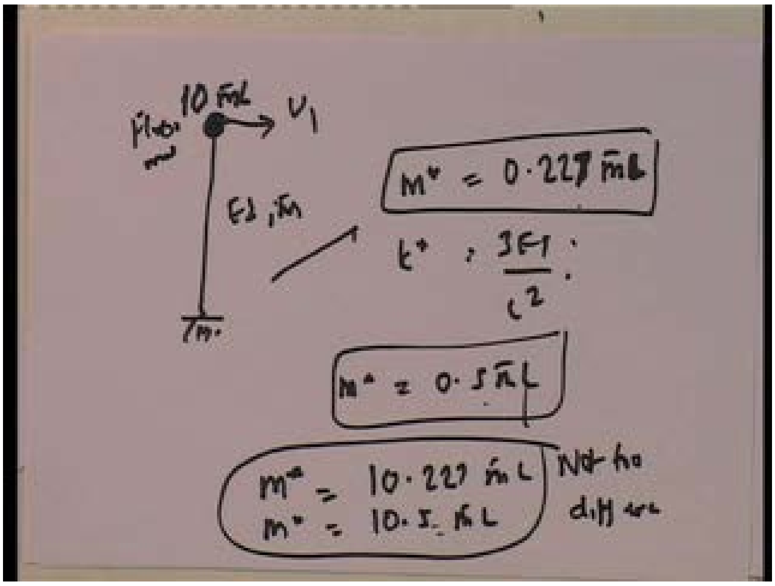

If you remember, when we have this, remember this as the degree of freedom that single degree of freedom with E I and M bar we have done it, and we got what, we got that m star was equal to 0.228 m bar l. And a k star was equal to 3 E I by l square, look back at your thing and this actually this 0.227, 0.227 and 3 E I by l cube, now if I want to do this and I will say that look this mass, which is over here a lumped half here and half here.

If I do that what does my m star become, my m star becomes 0.5 m bar l obviously, when I do this half the mass goes, so you see how ridiculous this is, if you use it this kind of situation, this becomes completely ridiculous. However, think about it, here let me put this I put a point mass add this point, and I say that this is 10 m bar l, if that is 10 m bar l then what is my m star, if I use this my m star becomes 10.227 m bar l. And if I use lumped mass I get 10.5 m bar l not too different, now you see something that looked ridiculous becomes non ridiculous, when you have floor, mass.

And this is the floor mass, because this mass is now very small compared to the floor mass, which is what happens in typical frame buildings, the floor weights that you have are significantly larger in the member weights always.

I mean which is why in gravity loads you pretty much always consider, the weight that comes from the floors as part of the gravity load. I mean you will see that your reactions, you pretty much look at the reactions, you will see that if you add up all the floor weights, you will see that the floor loads and you will see that is almost the reaction very well. So, if you see this one does not look ridiculous, so therefore, let us be very, very clear that the lumped mass formulation is only valid for framed structures, but having said that now why is it useful for framed structures.

We saw that, that for this particular problem that we looked at, I do not need to put it, you know we got this is my v 1, v 2, v 3 and for this we developed the k metrics. And then, we developed the consistent m matrix and the lumped m matrices, and in the last time we saw that ultimately what happened that, if you use the consistent metrics, then you have a again, I am not still not talked about loading. So, I am putting at a 0 load, if you do this you have in this particular case, 3 coupled linear ordinary deferential equations to solve.

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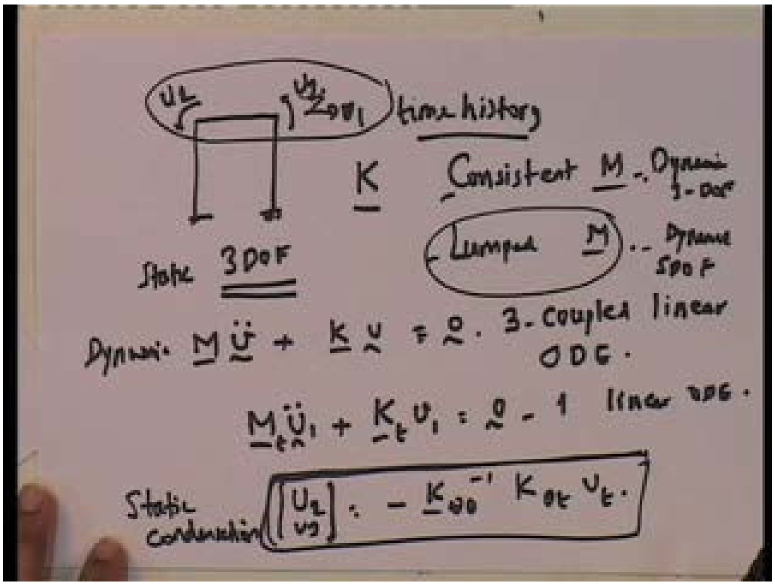

And if I use the lumped mass, I showed that all you needed to do was solve, which is one linear ODE and you needed to solve, then once you solved for v 1, v 2 and v 3 you could get from minus k theta theta inverse k theta t v t. So, in other words we still it is not as, if this is still 3 degree of freedom structure, nobody can take away from the fact, that this is a 3 degree of freedom structure.

If you still have to find out v 1, v 2, v 3 and the time histories of v 1, v 2, v 3 to be able to solve this problem, because once you know v 1, v 2, v 3 you can always find out what are the internal forces etcetera, this is a standard a procedure. I mean which is essentially note, that if you have time history, you can find out the Benny moment running point, because there it becomes essentially a static problem. Static problem, stereo static problem, because at any instant of time t, you can find out what v 1, v 2, v 3 are and you can find out what the other forces are.

So, in other words you still have to get v 1, v 2, v 3 the only thing that happens is that, the dynamic problem which is ODE is, so therefore static still 3 degrees of freedom, consistent mass dynamic also 3 degrees of freedom. Because, you still have to solve three linear ODE is and there coupled, is a lumped mass formulation, then dynamic single degree of freedom, you see the advantage. And this is in a sense, what essentially is why people tend to use the lumped mass formulation for framed structures, because it simplifies the problem.

In the dynamic problem we statically condense out, this is the static condensation, we use a lumped mass formulation, all rotational degrees of freedom are statically condensed out from the equations. And the only dynamic equation, the dynamic degrees of freedom are all the translational degrees of freedom, only the translational degrees of freedom are the dynamic degrees of freedom. So, therefore, if I have a single story building, I have only 1 degree of dynamic degree of freedom.

And so therefore, even if this beam, remember if this beam was rigid, then this v 2 and v 3 would be 0, see note that if this beam is rigid v 2 and v 3 are 0, and it is a true single degree of freedom structure. But, if you use if it is not a rigid beam, this beam is not rigid, then v 2 and v 3 are not 0, if v 2 and v 3 are not 0, then what happens then we can still solve a dynamic problem, as a single degree of freedom problem. The only thing is that to the static condensation procedure, we statically condense these out, so that once we solved for the translational degree of freedom.

Then you use the transformation metrics to get the, which v 2 and v 3 which are the rotational degrees of freedom. So, the difference is that you still solve a dynamic degree of freedom only a single degree of freedom, but then you have to go back and use this to get these three. So, you have to find out these three is just that, you have to find out the time history of all of these to be able to get any response, you need to do that. Only thing that happens here, the fact that you only dynamically solve for this, and these two then become just a pseudo static relationship, rather than a dynamic relationship.

So, that is why the lumped mass formulation is a preferred formulation to the consistent mass formulation, the advantages of consistent mass formulation is that, it gives you a kind of a specific, in a whatever frequencies that you get, we will now start solving the free vibration equation. But, whatever a frequencies that you get, that they are upper bounds, if you solve the free vibration equation solving the lumped using the lumped mass formulation, you too cannot say whether frequencies that you get from the lumped mass formulation are upper bounds, the close to them.

If you use the lumped mass formulation properly, then it will give close, if the consistent mass formulation, the lumped formulation if reasonably close answers. But, the fact remains that you are essentially saying that, it is an approximation we do not know whether it is a frequencies higher or lower. Whereas, the consistent mass formulation you know that the frequencies that you get, from the consistent formulation is going to be higher than the true frequency, for the lumped mass you cannot, but does not matter.

As long who cares whether it is in upper bound or a lower bound, is only mathematicians who are interested in that, as engineers we could not care, whether it is in upper bound or a lower bound as long as it is close enough to the true value, where it is who cares. So, that in a sense it turns out, so therefore a reviewing whatever I have done, I have formulated the equation of motion for a rigid bar kind of a situation. And I have also solved it for a flexible, where I have flexible bars and it is an assemblage.

So, therefore, I have looked at equations of motion, I started off by looking at multi degree of freedom equation of motion, using rigid bar assemblages, which are more than 1 degree of freedom, to establish certain basic premises. And then, I looked at assemblages of flexible bars, so in a way I am following the same principle that I used, in the generalizing the degree of freedom, accepting that I am using it in the multi. With rigid bars and springs etcetera, it becomes straightly easier to establish certain rules and then, we looked at it does not have to be, it can be flexible. The only thing that I must add is that, even if it is made up of uniform members, the k metrics and the m metrics is are approximations of the true thing.

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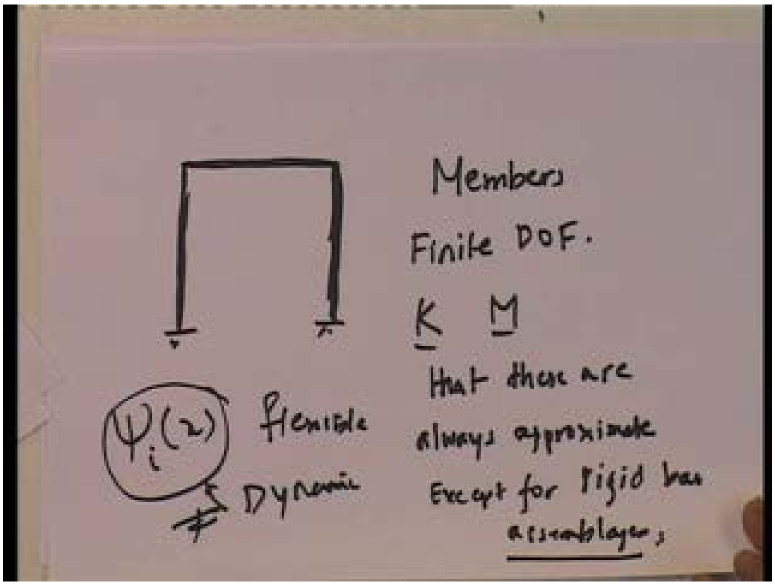

Purely because if you look at any structure, I say that look by defining this as a member, this as a member and this as a member, I made it in to a finite degrees of freedom problem. But, actually if you look at this, this is a continuous member, so it is actually has infinity degrees of freedom, we just did what did we do, we actually kind of created members, from that we got finite degrees of freedom.

And once you got the finite degrees of freedom, we find out k and m, and please note that these are always approximate, except for rigid bar assemblages, is in rigid bar assemblages the k and m are actually you can get them exactly. But, if you have flexible, because of the fact that we used psi i x as a cubic omission polynomials, these do not represent the dynamic behavior at all. And so therefore, k and m are always approximations, and which is why this k and m are new approximations, the lumped mass formulation has this beauty about it.

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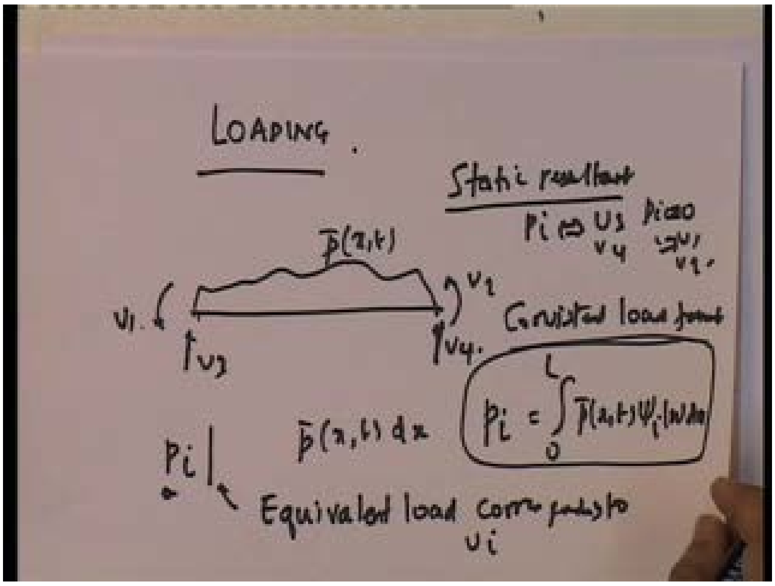

So, now let us look at finally loading, how do I get loading well, I already done it for rigid bars let us look at flexible bars, in flexible bars let us take any arbitrary loading on the system. So, this is given as p bar x of t, so if we say p bar x of t and these are my, what do I have, I have the situation that again I take slices. So, p bar x t d x is the load, and so this is the load, and so if I want to find out p i, what is p i, p i is the equivalent load corresponding to v i, the degree of freedom v i.

So, if this is the load per unit length, then if I use the formulation you will see that p i is nothing but integrated over the whole length p bar x of t into psi i x d x. Again this is known as the consistent load formulation, this is the consistent load formulation, I can also use another approach which is static resultant, I consider this as a simply supported beam. If I find out the simply supported beam, I only get reactions I do not get any moments, and so if I use static resultant I only get corresponding to p i, corresponding to v 3, v 4.

P i corresponding is equal to 0, corresponding to v 1 and v 2 this what happens, so therefore, again where does this thing happen. So, therefore, now once you develop this at the member level, you can put them together corresponding to the this thing by using principle of virtual displacement, I am not going to go there. These are things that we have already covered many, many times in earlier courses, and I do not want to go there, please use principle of virtual displacement to get the load.

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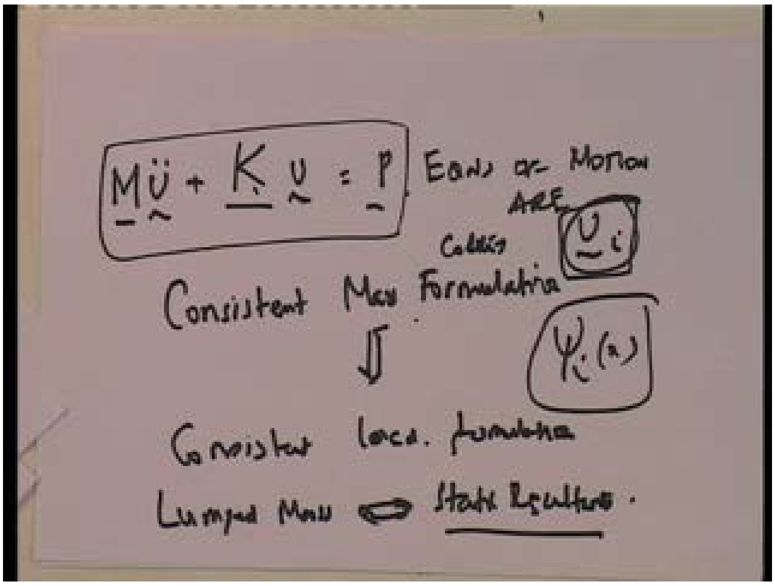

So, now the question becomes that I finally, have this problem M v double dot plus K v is equal to p, now if I use the consistent mass formulation, then it is obvious that I have used to use the consistent load formulation. In other words, psi i is are used consistently for K, for mass and for load, however if you use the lumped mass, see lumped mass you get to do what, to do use static condensation. If you do static condensation, then you should use the static resultants approach, because what happens in static resultant approach saw that in member forces corresponding to rotational degrees of freedom are 0.

So, if you use this, then you can do static condensation, because even at the global level the p i is corresponding to the rotational degrees of freedom are 0. So, that in essence is the procedure to develop the property matrices, corresponding to any set of degrees of freedom. And again and again I want to establish that the equations of motion are what, are corresponding to a set of degrees of freedom. So, if you use different degrees of freedom, your mass stiffness and load masses stiffness matrices and load vector are going to be different.

I have showed you that in the two lectures ago, so I do not want to continue that, I just want establish that if you are using the consistent mass formulation, you combining consistent formulation with static resultant, or using lumped mass formulation with consistent load formulation makes no sense worth so ever. So, when you use consistent mass formulation, use the consistent load formulation and use the lumped mass formulation, use the static resultant formulation for the load.

And then, you can get, therefore ultimately the whole idea of the lumped mass is that dynamic degrees of freedom, become different from static degrees of freedom, this want take away. Why because since the rotational degrees of freedom do not have any inertial associated with them, I needed do they have any load associated with them, then the rotational degrees of freedom are related to the translation degrees of freedom, through a static equation.

And therefore, we can condense out the rotational degrees of freedom, and get the dynamic degrees of freedom in a framed structure, are only the translational degrees of freedom. And therefore, the size of the problem becomes that much smaller for example, in a one by one story frame, what do you have, you had 3 degrees of freedom being, 3 static degrees of freedom being taken into one dynamic degree of freedom.

So, that becomes the essential issue that we roam around, so now I am done with generating degrees of freedom, for equation of motion for multiple degrees of freedom. And of course, I have concentrated on rigid body assemblages, and flexible body assemblages are only looked at framed buildings, framed structures. And in fact, buildings purely because lumped mass formulation, consistent mass formulation give exactly similar results, when you have buildings where flow masses are significantly larger anyway, so much for the equations of motion. Let us now move on, because ultimately equations of motion have to be solved, so to solve the procedure, the next thing that I look at is free vibration.

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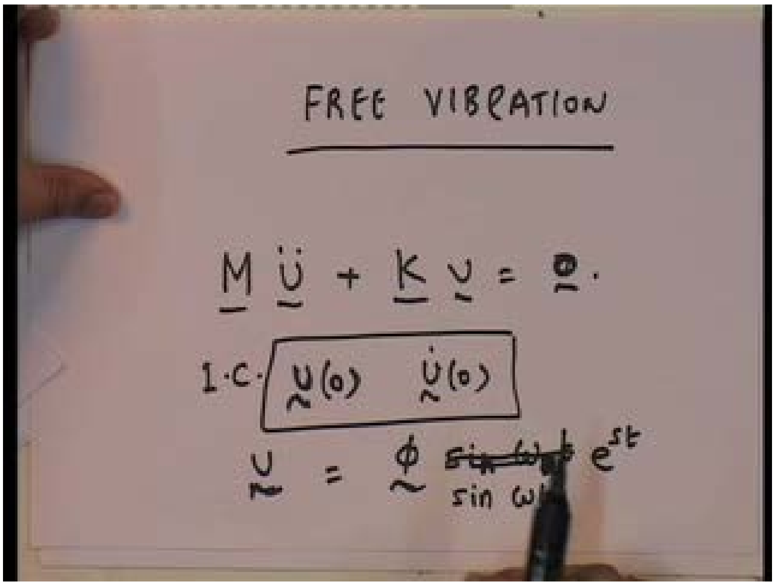

And free vibration the equations are, because there is no load, free vibration means its freely allowed to vibrate where you are given, now this is the issue, that this the vector. So, you have to be given all displacements at time tau equal to 0, and corresponding velocities are given, so here the difference becomes that, there it was just given initial conditions. Initial condition means, you have to be given displacement corresponding to each degree of freedom, and velocity corresponding to each degree of freedom, initial displacement, initial velocity those are the initial condition.

Now, mathematics says that look this kind of a situation, v is equal to actually v omega n sine, so I am not going to put it as omega n, I am going to put it as omega. So, therefore, you see it still remains the same, although again in a classical sense, I am putting sin omega t it actually is e to the power of s t. But, then we know s, and so I am just going with sine omega t, all of this is irrelevant, I am not go through the same procedure that I have gone through. The only thing that says is that, for this to be solved v has to be of this form, where phi is the amplitudes of displacements at every point, and this is given in this form.

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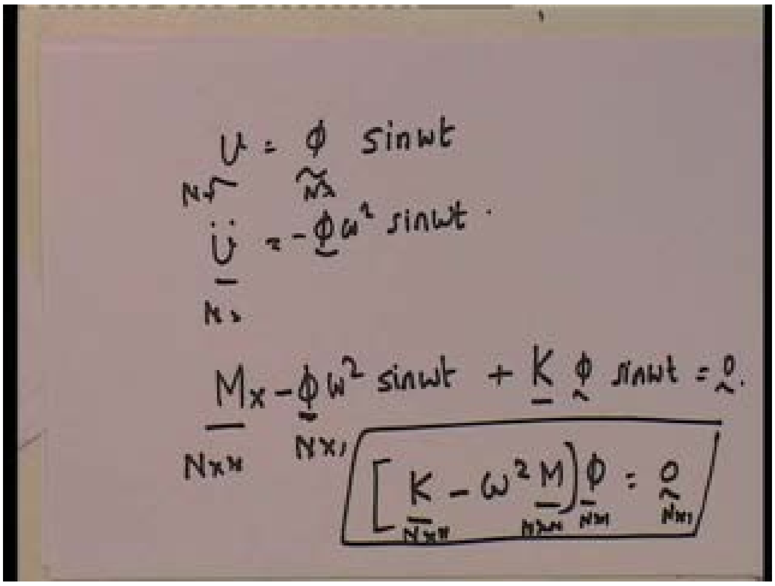

So, if I substitute this into the equation I get the following, I get v is equal to the displacement, so v double dot is equal to phi minus phi omega square sine omega t. So, therefore, what this becomes, then is the following M into v is nothing but M multiplied by minus phi omega square sine omega t, note this is N by N and this is N by 1, because this is N by 1, this is N by 1, so this is N by 1.

And then, plus K into phi sine omega t is equal to 0, now note that since sine omega t is not equal to 0, this equation is only satisfied when I have K minus omega square M into phi is equal to 0, this has to be satisfied. Now, if you look at the, this is just look similar to what I had done there it was the single K, it was minus omega squared M and so omega is equal to root K upon M, is a single degree of freedom we could do that. But, note that these are N by N, N by 1 and this is also N by 1, these are N by N matrices and so therefore, this problem is not a problem which is tractable.

So, this problem essentially becomes, I am going to re write this problem and this problem is K, you see this is not a problem that we are un familiar with. If you look at your matrix algebra remember that, any matrix a square matrix, it always has to be a square matrix. As you were always asked to find out the eigen values, and eigen vectors of this matrix A, and how did I find out the eigen values and eigen vectors, these were done by solving A minus lambda I, were I is identity matrix into phi is equal to 0, where this is the eigen value and this is the eigen vector of the matrix A, and this problem was known as an eigen value problem.

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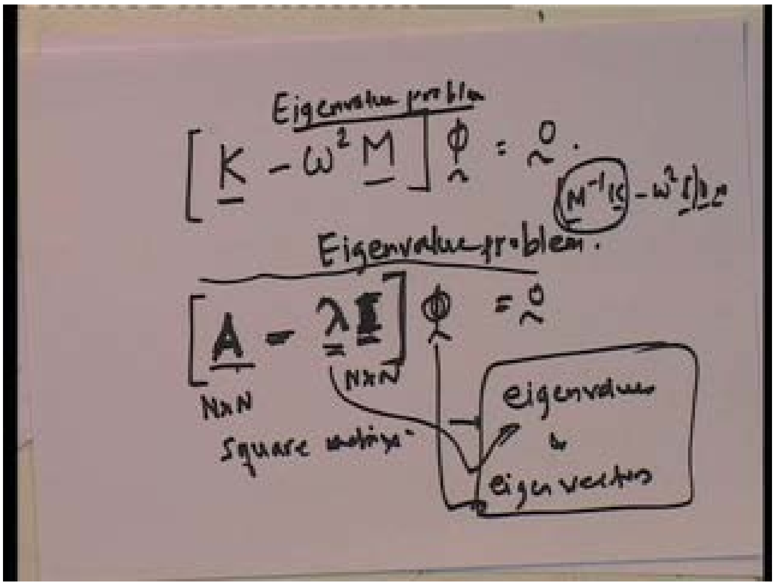

So, if you look at this, this kind of a situation K minus omega square M it is pretty much similar, all I have to do is, if I pre multiply both of them by M inverse you will see, that this will become then, the problem will become M inverse K. And M inverse into M omega square I into phi is equal to 0, so you see the A matrix is this matrix ((Refer Time: 3532)). And omega square is like an Eigen value, so this problem is also an Eigen value problem. Now, how do we solve this Eigen value problem, let us look at it, I am going to just put down the procedure.

Solution of eigen value problem, so what do we have, we have the eigen value problem is this problem, this is the eigen value problem. Now, for this to be valid, since phi is non zero, it implies that K minus omega square M, the determinant, this is the determinant is equal to 0, determinant is the scalar, so that is equal to 0. Now, there are certain aspects associated with this, and this comes from linear algebra, matrix algebra. And that we know, these are symmetric matrices we know this, we already shown this, we will also show you that indeed, these are what are known as positive definite matrices. What is a positive definite mean, positive definite is the following.

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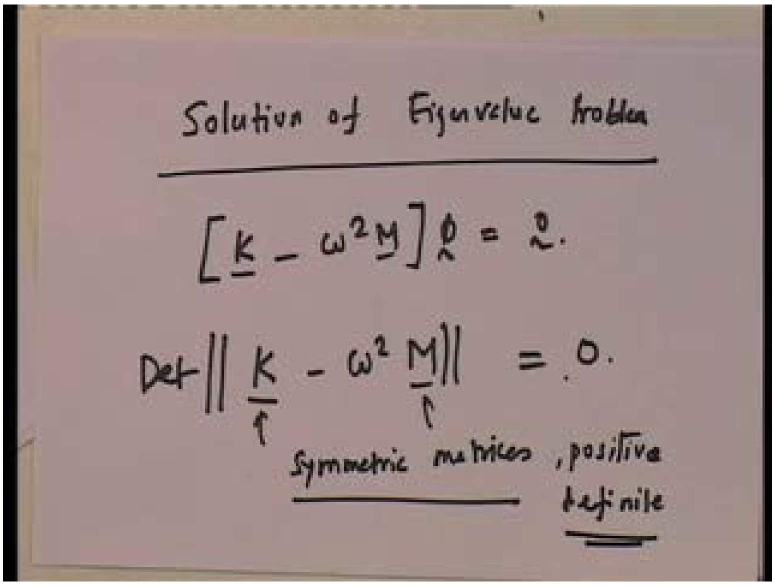

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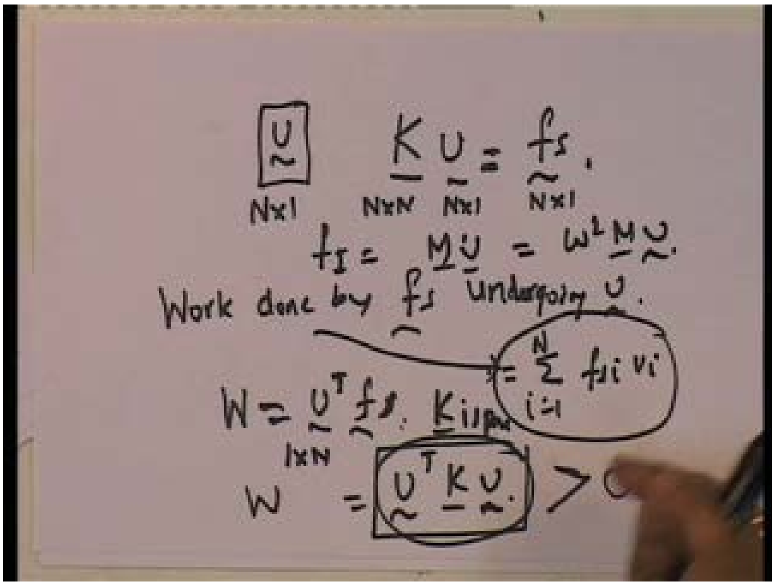

Let us take any vector v, displacement vector v, now what is K into v, K into v is the elastic force vector by definition is the elastic force vector. So, now, let us see, so therefore let us look at this, this is N by 1, this is a N by N, this is a N by 1 and this is obviously, N by 1. This is the displacement and this is the elastic force vector corresponding to those displacements. Now, I want to find out the work done by f s undergoing v, now work done is the scalar, how do I find this W is equal to v transfers f s.

Note that, what we are doing here, work done it is actually this is going to be equal to submission I going from 1 to N f s i into v i that is what it is, so if you look at this in a matrix format it can be written as, if you look at this transpose, basically the vector N by 1 becomes 1 by N. So, the column vector of displacements become the row vector of displacements and then, if you multiply you will see that v i being multiplied by f s i which is really this ((Refer Time: 39:43)).

So, the work done is this, which is also equal to this, so if I take this is equal to substituting this it becomes this ((Refer Time: 39:53)), so work is this. And we know that work done is always greater, I mean if work done was less than 0 it will be beautiful situation, by making forces undergo displacements, we would actually get work which we could use energy, which we could use somewhere else.

So, therefore, this if you look at it, this is like an energy term, an energy is always positive, so this implies if this is true, if any arbitrary displacement into K in this format is greater than 0, then K is positive definite. So, obviously, K is positive definite, similarly we can show that M is positive definite, how do we do it well in exactly the same way f i is equal to M into v, which is equal to omega square M into v. And from there you can solve exactly the same way, work done by the inertial forces and you get exactly that this is positive definite. So, we see that ((Refer Time: 41:15)) these are symmetric positive definite matrices, if they are positive definite matrices, what is the beauty of it.

So, K, M are symmetric positive definite matrices, now the linear algebra says, that if this is true and you find out the determinant, and that has to be 0. You see determinant what will happen is, this is a N by N, what they say is that all omega squares will be greater than 0. And you will have N distinct, so if K and M are symmetric positive definite, all omega squared values, the omega n squared there are N distinct omega n square values, for which the determinant is equal to 0. And all of them are greater than 0, what is the beauty of this, this means that you have omega n real, that is what it means. And of course, omega n by definition is the frequency of natural vibration and therefore, it normally real it has to be positive.

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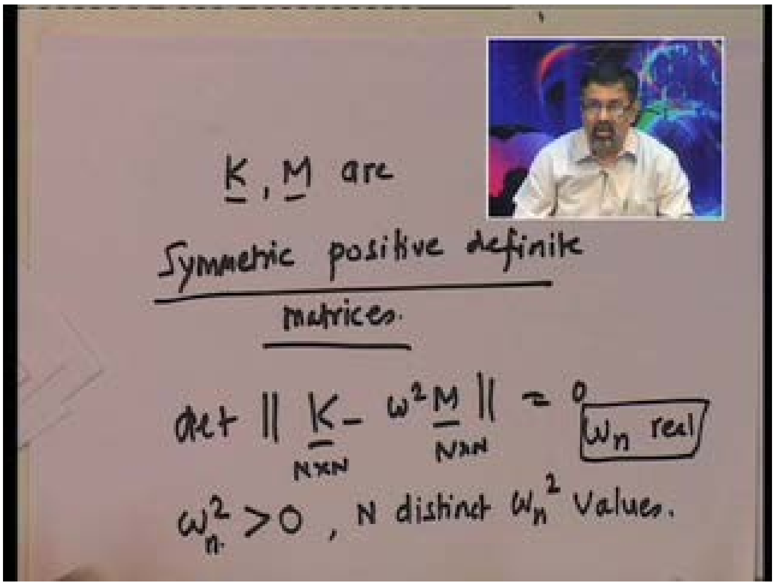

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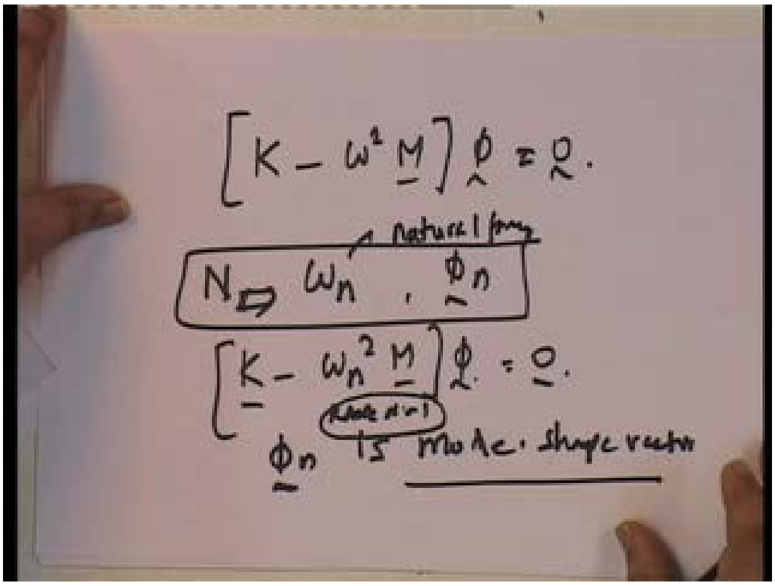

So, there are N distinct so that means, what we get finally, is the following and that is that you have by solving this eigen value problem, this eigen value problem is solvable for N, omega n values. And every omega n, for a specific value of omega n you find out phi n, so every omega n has it is own corresponding and there are N, such omega n and phi n. Now, this omega n squared are known as eigen values, and these are known as eigen vectors, note that it is always true that the eigen vector, because the determinant is equal to 0.

That means, it becomes one less rank, so phi n is not a vector, is not a displacement vector, it is a mode shape vector, why because if this determinant is 0, this problem becomes of rank N minus 1. If the rank of the matrix is N minus 1, you can only get phi n cannot be obtained directly, it will always be obtained in such a way that you can get N minus 1 values in term of another value. So, this is like a mode shape vector and these are my natural frequencies of the structure.

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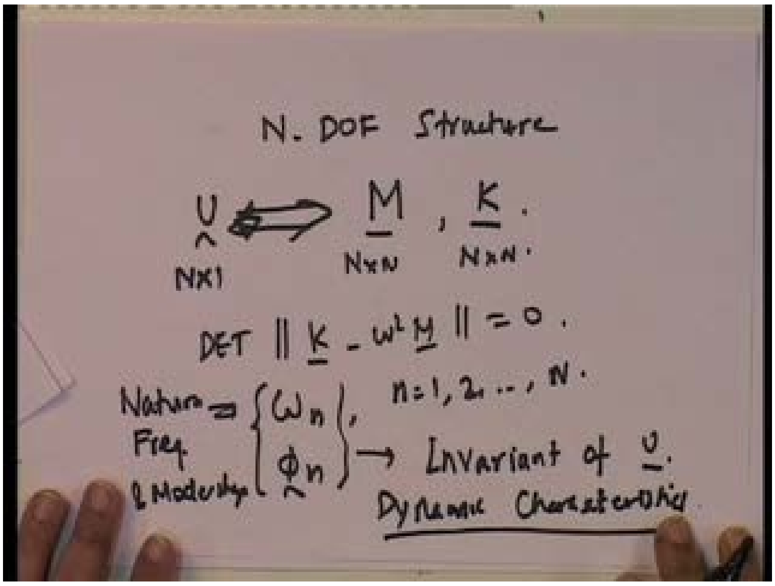

So, ultimately what do I have, if I have an N degree of freedom structure, where I define v as my this N displacement quantities, then corresponding to them we get the property matrices K and M. So, note that these correspond to the degrees of freedom, these are size N by N, and we already developed how to get M and K through proper this thing. Then by solving determinant K minus omega square M you get omega n, n equal to 1, 2, 3, 4 to n, and corresponding to every omega n you have a phi n.

And these are the natural frequency and mode shapes of the structure, now note this is very, very important, these are in variant of degrees of freedom why, because these are dynamic character restricts of the structure. So, now, the question becomes this that M and K are dependent on the degrees of freedom that I define, but the omega n and phi n that I get, the phi n would correspond to the degrees of freedom. But, omega n does not degree, so the phi n gives the shape, but the mode shape, once you give as shape that shape is an in variant. So, the frequency and mode shapes are in variant of how the degrees of freedom are defined.

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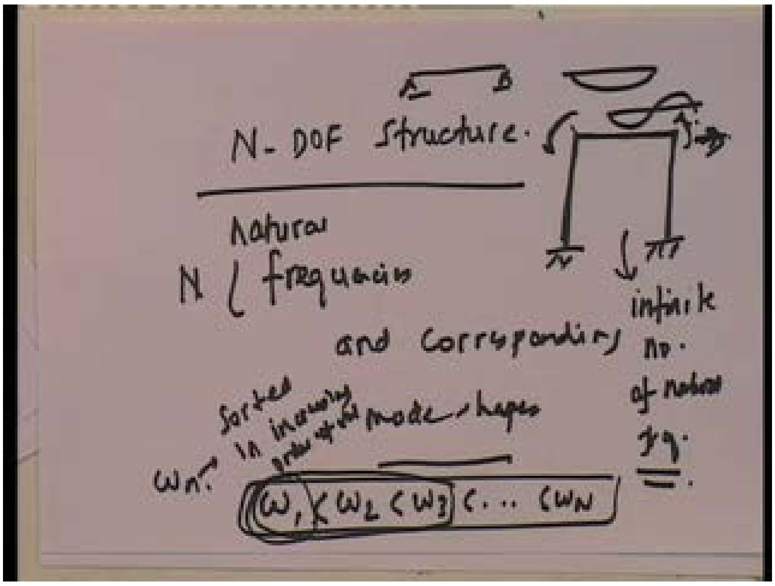

So, therefore, the question then becomes is this, that if I have N degrees of freedom structure, I get N natural frequencies, and corresponding mode shapes. Now, the question, so if I have a N degree of freedom structure, I have N natural frequencies, and corresponding mode shapes. So, and we have to solve the Eigen value problem, I am going to start solving the eigen value problem, how to solve we will solve examples, in the next lecture.

But, now for example, it becomes very interesting, that if I have N degrees of freedom and this is dynamic degrees of freedom mind you, because ultimately it is always the dynamic degrees of freedom, that you deal with, and this is very, very interesting. And that is that if I have N dynamic degrees of freedom, I have N natural frequencies, now you ask me this problem, how many frequencies does this structure have in reality, in reality infinite number of natural frequencies.

And we will show this later on in this course, I am going to actually solve continuous systems and I will show you that in reality, there are infinite number of natural frequencies that any continuum structure. If it is a system of rigid body assemblages, they do not have infinite, but any continuum structure remember a framed, building is a continuum. So, therefore, when we take it as a 3 degrees of freedom, I get only 3 natural frequencies, which frequencies do I get, we can show that we actually get an approximation of the three lowest frequencies.

Because, note another thing that very important and that is that, when you have omega n, you have different omega n, and the way you put them together, is in this format omega 1, is less than omega 2, is less than omega 3, is less than omega N. So, in other words they are sorted in increasing order if value, so therefore an infinite degree of freedom, we made into the 3 degree of freedom, which three natural frequencies are found. Well if you look at this the continuum, you will see that you get an approximation of these frequencies.

Now, I said that if I use the lumped mass formulation, I only get a single degree of freedom, these two also disappear, which one do I get I actually get only this one, approximation. And that is the beauty of this specific thing, and that is that all we do is we get a good estimate of the natural frequencies, and the corresponding shapes of vibration. Because, whenever you have a specific continuum it has a shape of vibration, each vibration remember for example, if I have ((Refer Time: 52:50)) this tube it can vibrate this way, it can vibrate this way, so each has a shape associated with it.

So, I have completed my equations of motion and I have introduced to you that ultimately the free vibration problem is nothing but an eigen value problem. And we also saw that since M and K are positive definite symmetric matrices, the eigen values are all positive and distinct. We will continue with this free vibration problem in the next lecture.

Thank you very much, bye.