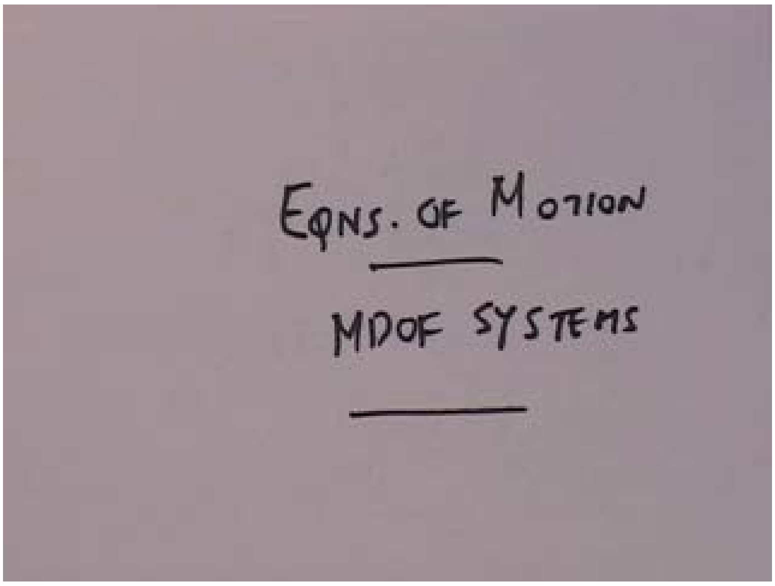
Structural Dynamics
Prof. P. Banerji
Department of Civil Engineering
Indian Institute of Science, Bombay

Lecture - 22
Equations of Motion for Multi Degree of Freedom Systems

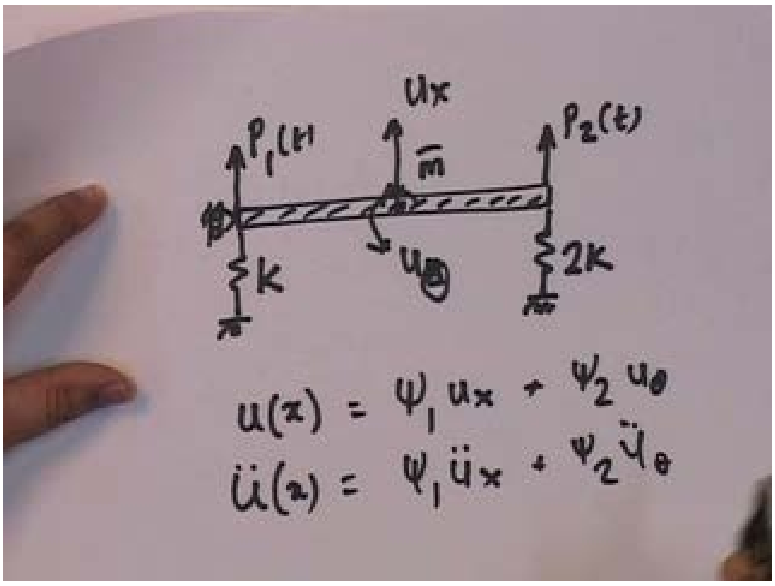
(Refer Slide Time: 00:29)



Hello there, we were discussing on my Equations of Motion of Multi degree of Freedom Systems. And we shall continue with the same topic, which is equation of motion for, you know last time when you looked at it, you saw that it became 2 single degree of freedom system problems. And you may well ask me what is the big deal, why do not we solve it as single degree of freedom, because it is just 2 degrees of freedom. Well actually, today I am right now I am going to take you up on a particular problem, in which I will show that because of certain specific points that I try to do in that system. We ultimately landed up getting uncoupled equations, and so therefore we could make it into like 2 single degree of freedom problems.

But what I am going to look at now is the same problem that same the bar means exactly the same, except that now… So, it is still not allowed to go in the in this direction, however instead of a beam and elastic foundation, I have 2 discrete springs and what I am going to do is I am going to say that this is equal to k and I am going to make this equal to 2 K. This is still m bar and the loading well I am going to say, it is going to p 1 and p 2, of course please note that if these are not time dependant, then the whole problem will not time dependant.

(Refer Slide Time: 01:32)

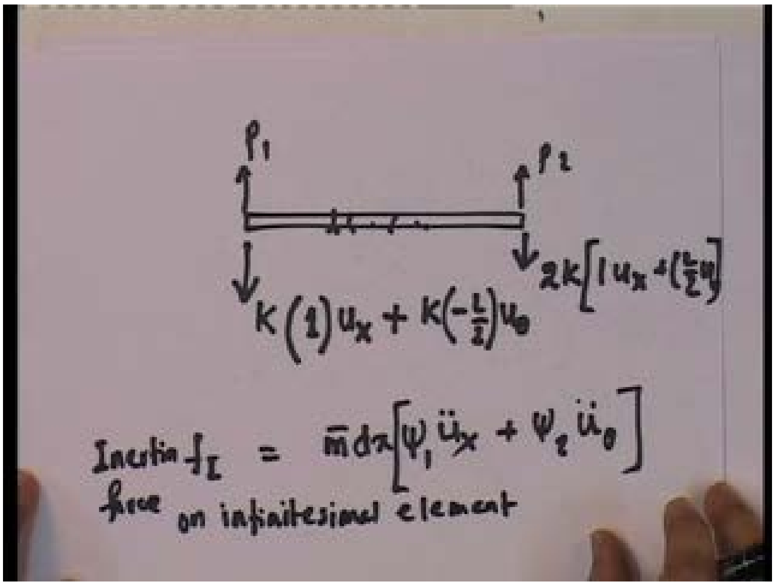


So, there is a you know p 1, p 2 are time dependent loads, so this is the problem; same problem in the same bar, excepting that I have discritize the load. Now, let us let us try to solve this problem. So, if I give it an arbitrary displacement and I still define my degrees of freedom at the geometric center as u x and u theta, so they are the geometric center.

And so then if those are my sorry, u x and u theta u theta then any arbitrary displacement you know displacement at any particular fine, can be written as psi 1 u x plus psi 2 u theta and similarly u x is equal to since psi, the shape functions are not functions of time, so it becomes this way right. So, this was the basis that, we used in the last lecture and so if I use this right, so therefore, if I look at it my the inertial force.

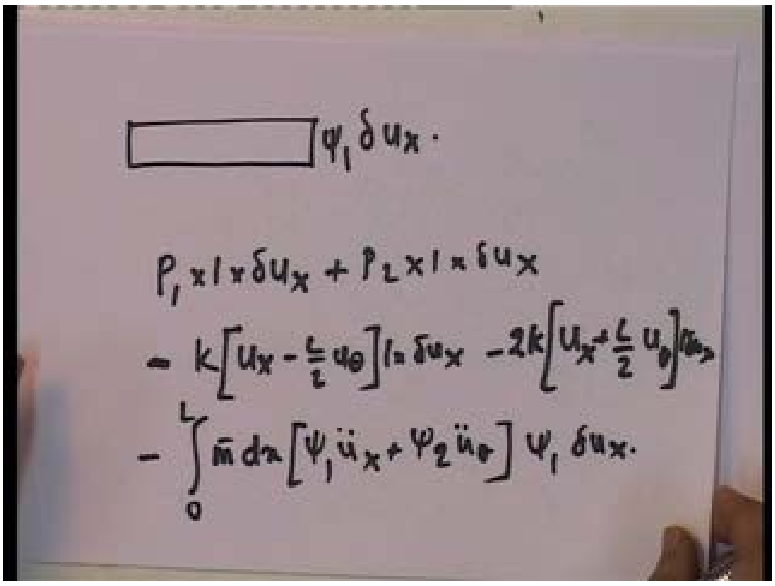
So, if you look at the forces on my structure right, the forces is on my structure are going to be the following here, it is going to be net force where, the force is going to be equal to k into now, what is psi 1 at this particular point, it is going to be minus x by L. So, it is 1 into u x and then plus K into minus L over 2 u theta over here, this is going to be equal to 2 K into 1 into u x, plus L over 2 u theta.

(Refer Slide Time: 04:14)



So, what we have then these are the 2 forces then you have load p 1 load p 2 and on top of that, you have these inertial forces as inertial force at any particular length at point x is going to be equal to m bar d x into psi 1 u x double dot plus psi 2 u theta double dot, this is inertia force on infinitesimal element. So, this is the inertial force on the infinitesimal element, so now, to solve for u x and u theta, we required 2 equations, so for the writing the 2 equations.

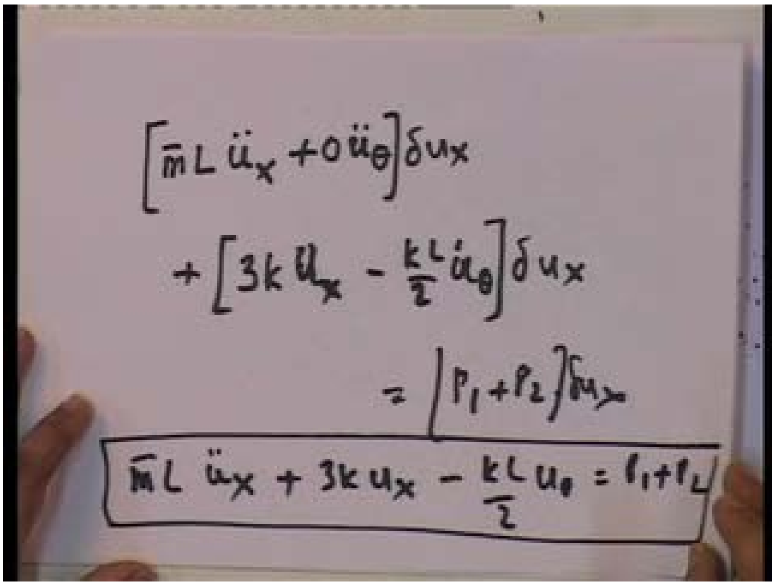
(Refer Slide Time: 06:56)



The first equation becomes, I am going to give, it psi 1 into del u x, that is the virtual displacement that, I am going to see and you know all I have to do for getting 2 equations is write down virtual work equations for 2 different or independent linearly independent. And since my degrees of freedom independent, I always take displacements that are corresponding to the degree of freedom, I know that, they are independent then I can get 2 independent equations.

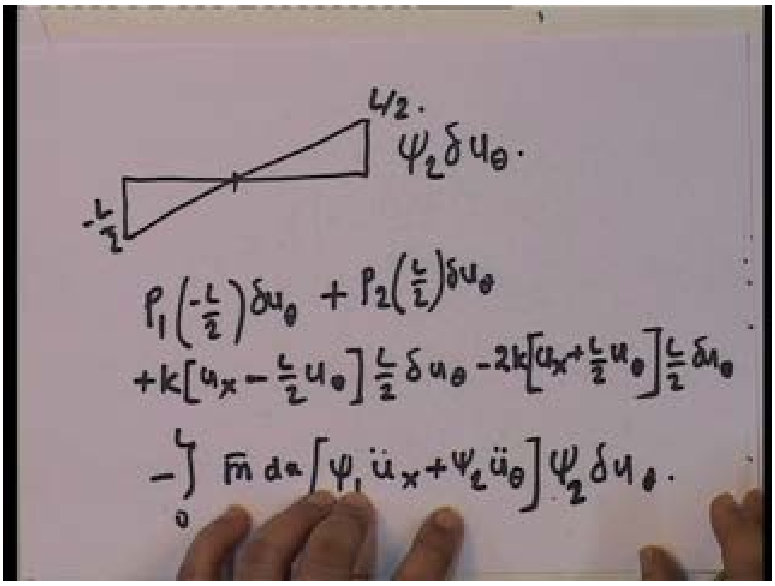
So, if I look at this is my psi 1, so then what do I get, I get the following the force p 1 undergoes 1 into del u x positive p 2 into 1 into del u x then I have minus, because these are negative work K into u x minus L by 2 u theta into 1 into delta u x, then minus 2 k into u x plus L by 2 u theta into 1 into del u x, that is also the kind of thing, then I have the mass. And the mass is the following, it is m bar d x psi 1 u double dot x plus psi 2 u double dot, this is the force into psi 1 into delta u x, this integrated over the whole length right. So, what do we get then I am going to put these equal things together and put them in a proper format and so what I get, then is I get the following.

(Refer Slide Time: 09:24)



I get see I have already done m bar x psi 1 squared, I have already done that right. So, what do I get, I get m bar L into u dot plus 0 into u double dot theta, the entire thing into del u x plus now, let us put u x and u theta a separately, so I get K. So, I get 3 K u x and I get plus and then I get minus, so I get minus K L over 2 sorry, this is u x and this is u sorry, this is u theta into del u x is equal to p 1 plus p 2. So, in other words the finally, the equations becomes m bar and you know into del u x, del u x is 3 everywhere. So, finally, we get this is the first equation that we get by putting, the first this one right guess, so this is the actually a work equation, but since delta u x goes out, it looks like a equilibrium equation.

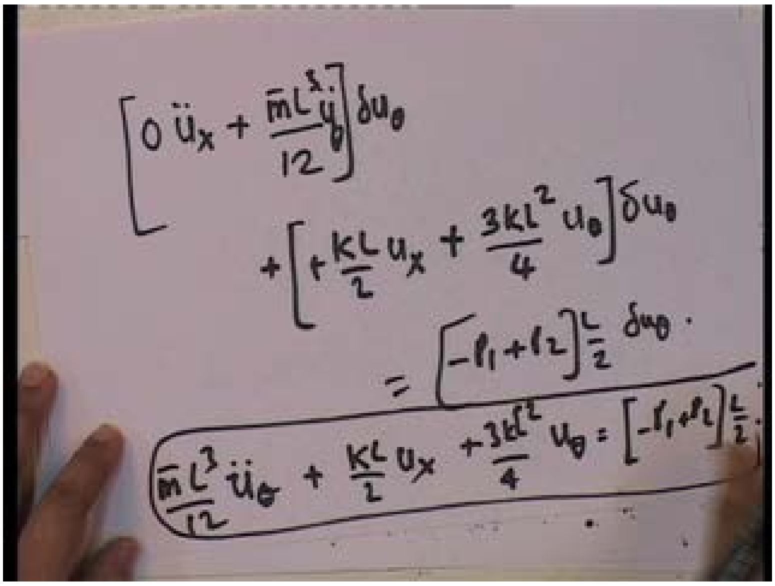
(Refer Slide Time: 11:23)



So, we got the first equation and now what we need to do is we need to define another virtual displacement pattern, a which is not is not included in the other pattern. So, if we do this the obvious 1 to do is a virtual displacement pattern, which is like this and where you know this is minus L by 2, this is L by 2, so if we get this pattern then what is the virtual displacement virtual work equation look like let start, the first one is the p 1 here.

So, p 1 is being subjected to this plus p 2 into right, then we have plus K into that is this force right and I am getting plus because it has minus L upon 2 right. So, it is actually minus into minus makes it plus L upon 2 u theta into L by 2 delta u theta, then we have the other 1, so that one is being subjected to minus so it is minus. So, its minus 2 K into u x plus L by 2 theta into L by 2 delta u theta and the final is the mass equation and if you look at the mass equation, what you get is minus now integrated over the whole length m bar d x, that is the mass per unit length into the acceleration, which is this and so this multiplied by the right. So, the here you get psi 1 psi 2 integrated, we know this, we have already evaluated this in the last equation, so if I put that in what do I get.

(Refer Slide Time: 13:52)



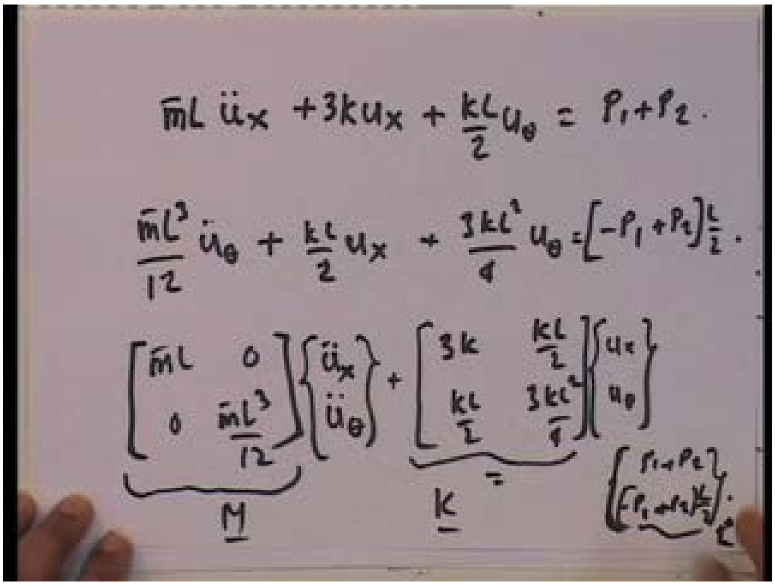
I get the following 0 u double dot u x plus m bar L cubed by 12 into I am sorry, this is u d theta into del u theta, then we have now if you look at the other one, I am going to put the K u x and u theta part together. So, from the u x, if you look at this, K u x is this one and on this side and that is multiplied by L by 2, so its K L by 2 plus and then here, I have minus 2 K L by 2, so I get I am going to put plus here minus K L by 2 u x, so that is the first one, that we get.

And then if you look at this right, you have the other one, this one turns out to be minus K L squared upon 4 right minus K L squared and this one comes out to be minus 2 K into L square by 4. So, both of them are minus so you get minus K and then minus 2 K both into L squared by 4. So, what you get is minus 3 K into L squared by 4. So, that basically becomes what it becomes equal to minus 3 K 3 upon 4 K L squared, so that is what I am going to put down here, one aspect that, I need to look at very carefully and that is that, I need to put this part, if you look at it a minus K all of them are in the same side. So, when I take this minus K and I take this K and minus K L over 2, I get K minus K L over 2, but if you look at this essentially becomes what it essentially becomes on the when I take it on the other side.

I get it equal to just like on this one, actually when I took it on the other side, if you look at this right. This is plus K L squared and this is minus 2 K L, but this is on this side when we take it on the other side, it becomes plus K L by 2. So, this is plus K L by 2 and this is plus K L by 2 and if I look at this one going back. This one again is on the other side, so I need to put it, if I put it on the other side.I get it equal to plus K L by 2 and this one becomes 3 K L squared by 4 u theta into del u theta is equal to. Now, if you look at this becomes on the if I this is on this side. So, it becomes p 2 L by 2.

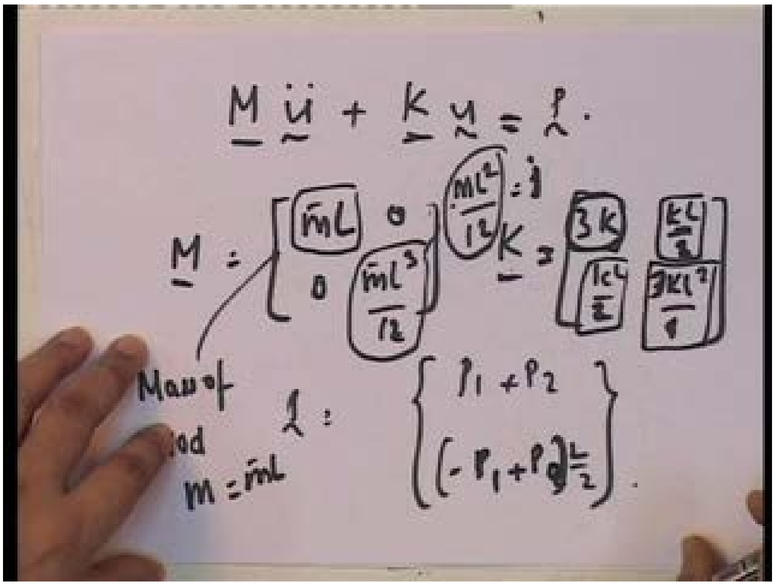
So, it becomes minus p 1 plus p 2 into L by 2 into delta u theta. So, this delta u theta exists everywhere, so finally, what do I get, I get it equal to the following, I get m bar L cubed by 12 into u theta double dot plus K L by 2 u x plus 3 K L squared by 4 u theta is equal to p 1 plus p 2 into L by 2. So, putting this is the second equation, so putting together both equations together, I am going to rewrite, the both the equations together and if I write both the equations together, I get the following.

(Refer Slide Time: 18:36)



I get it equal to m bar L u x K L by 2 u theta is equal to p 1 plus p 2 and the other one is m bar L cubed by 12 u theta plus K L by 2 u x plus 3 K L squared upon 4 u theta and minus p 1 plus p 2 into L by 2. So, if I rewrite this in a equation, in a matrix format, what I get is this m bar L 0 0 m bar L cubed upon 12 equal to p 1 plus p 2 and on this one, I have p 1 plus p 2 into L by 2. So, this is what, we get on this particular equation, so this is the mass matrix, this is the stiffness matrix and this is the load vector.

(Refer Slide Time: 20:54)



So, finally, what does it look like well, m into u double dot plus K into u is equal to p and note this time, that since you see, you see this equation this equation has u x and u theta and this equation has u theta and u x. So, they are not uncoupled, since they are not uncoupled, its actually 2 simultaneous differential equations and that is why you have this if m or k or both are not diagonal matrices, then automatically the system becomes coupled.

And in this particular case m is equal to m bar L 0 0 m bar L by 12 K is 3 K K L by 2 K L by 2 3 K L squared upon 4 this is 3, so that is K and the p vector is equal to p 1 plus p 2 and minus p 1 plus p 2 into L by 2. Now if you look at this the mass matrix is a diagonal matrix why, well there is a reason, because the center of the if you remember, you define the degrees of freedom here, which is also the center of mass, when you define equations at the center of mass, then the mass matrix always becomes uncoupled.

However, the stiffness is no longer uncoupled, because the lateral and the torsional degrees of freedom here, the lateral and the rotational degrees of freedom are coupled to each other for this particular case. For the case of the sub grade beam and elastic foundation where, K was a constant and m bar was a constant, you see you got uncoupled why, because the center of stiffness, what is the center of stiffness. Center of stiffness is when you apply a static displacement, the net force goes through the to center of stiffness.

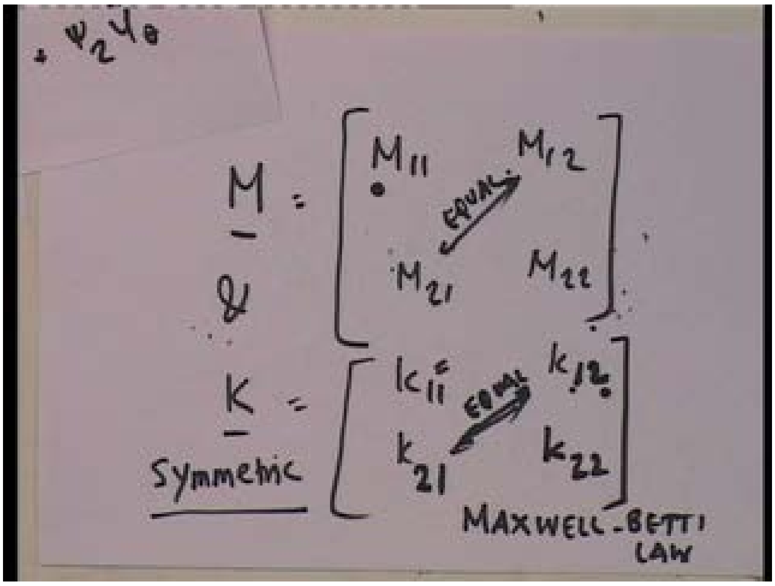
If you looked at it the center of stiffness, if you add it uniform, when we add uniform over the entire thing, the center of stiffness one at the center of mass and in the center of mass and the center of stiffness are at the geometric center, where are define my degrees of freedom I want coupled. But, since in this particular case, since this is K and this is 2 K understand that where is the central stiffness well, if you really look at this is center of stiffness is somewhere over here 2 3rd from this and 1 3rd from this, that is the sent that is the force, that is the net force due to this to unbalanced forces.

So, since the center stiffness is here, so therefore, the since the degrees of freedom I define here and it does not coincide with the center of stiffness what we land up getting is that the stiffness matrix is coupled. Now note something very interesting and that is that, what is this term m bar L, m bar L is the mass of the rod and what is this m bar L squared, this if I say mass of rod is m, which is equal to m bar L, then this is actually m L squared by 12.

Incidentally this is the mass moment of inertia of this bar about it center of mass. So, this also has a specific and that is known as the mass moment of inertia j, if you look at this, what does it represent? This actually represents that, if I were to give it only a u uniform, these k plus 2 k would add up and this is what you have here whereas, if this is the center of stiffness, which is the net stiffness is 3 and this distance, if you look at it is L upon 6 3 K into L upon 6 is K L upon 2, that is what we get and again, you know since this becomes 2 K into L upon 2 plus K. So, this is like the rotational stiffness. So, this is the coupling term and this is the rotational stiffness, due to these 2 about center of mass, so these have all physical and if you look at p 1 upon p 2 like.

What does this p 1 plus p 2, if I look at p 1 p 2 and if I have the center p 1 plus p 2 is a load and what is the net moment net moment would be what p 2 see, this one has negative this has positive, so this is p 2 into L by 2 minus p 1 upon L by 2 is a net moment here. That is what, we have you see all of these terms have physical significance in the entire scope of this thing and therefore, coming back to it, if we look at it.

(Refer Slide Time: 27:15)



What do this defined, if you look at this definition says that, lets look at K 1 1, it is the load then the force corresponding to degree 1, if you apply a displacement equal to 1 corresponding to degree 1. So, if you look at this what this becomes is the following, if I apply a displacement 1 at u 1, which is u x, if I put 1 what do I get, I get force K and I get force 2 K. So, what is the net force here 3 K, that is K 1 1, what is K 1 2, this is the force at 1, due to a displacement corresponding to 2.

So, that means, it is the force corresponding to this, if I give a unit rotation here, give it unit rotation this one becomes this way 2 K L by up 2. So, it becomes K L by 2, I mean sorry, K L and this one becomes, which way if it goes this way, this one goes this way. So, the this one is coming this way, this one is going this way, this is K L by 2 sorry, K L and this is K L by 2, so what is the net force here, K L minus K L by 2, so that is the force, that is K L by 2.

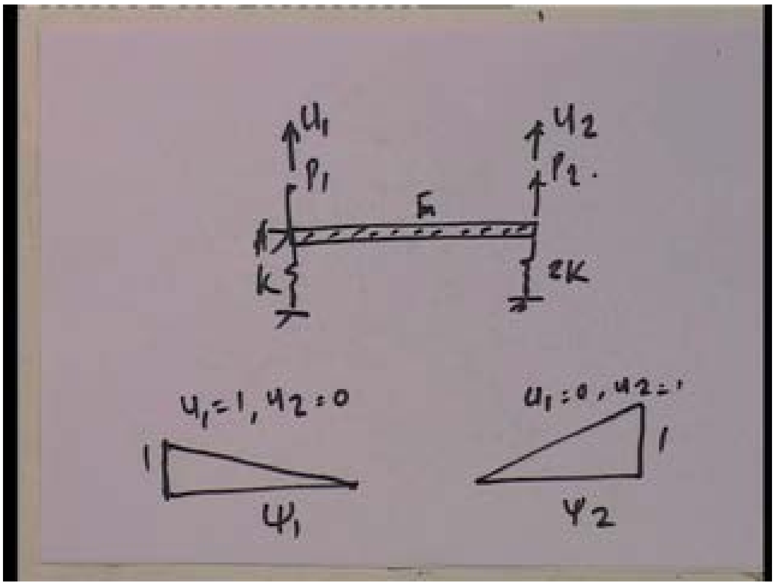
So, you see it is a force corresponding to degree 1, due to a unit displacement corresponding unit 2, what is K 2 1 K 2 1 is the force corresponding to degree 2. In this particular case, what is the degree 2 rotation, what is the force moment. So, it is the moment due to a unit displacement here, let us look at the unit moment that, we get here, you see if I put a unit displacement here, what do I get over here, I get k l what do I k l which way this way. If I put it, if I put you know displacement, what do I get over here I get. Please note that, what am I saying, this K 2 1 is the displace wrote, I mean force corresponding to, which is the moment due to unit displacement 1. So, if I give you unit displacement 1 that is this way.

This becomes K l and this becomes 2 K L sorry, 2 K 2 K and K right. So, these are the things, this are the forces, which is why I had 3 K, so let look at the moment let us take the moment about this point what do I get, I get this way anti clock wise 2 K into L by 2, that is K L and here, this is counter clock wise K L by 2. So, that means, the net counter clock net clock wise moment applied is equal to K L by 2. So, that is the equivalent load over here is equal to K L by 2.

Note this is something very interesting and similarly you can do 2 by verses 2, there is no problem ok. Now, now this becomes very important and the important thing that, becomes is that, I can actually start looking at, getting everything very, very easily and that is that these 2 are equal. Force corresponding to degree 2, due to a unit displacement at 1 is equal to the force at 1 due to a unit displacement at 2, which law does this remind, you of this is the max well Betti law, which says what, which is essentially says, that force at 2 due to displacement at 1 is equal to the force at 1 due to displacement at 2, so they have to be equal, these are always equal in this particular case, they where K L by 2 K L by 2, in this particular case. Similarly, we can show that, since this is also a force mass in to acceleration, that these also have to be equal to each other. Although strictly speaking its really mass into acceleration that makes it equal, but so they are also.

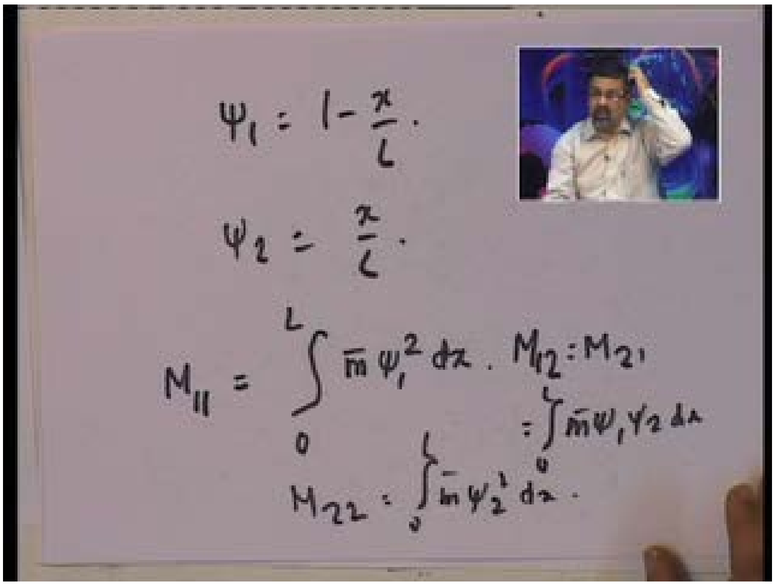
So, in other words both mass and stiffness matrix are symmetric matrices, this is a fundamental property of the mass and stiffness matrices. Now the point becomes look I have already develop the equation of motion, what are the equations of motion look like well, I saw that till time. Mass was equal to this, stiffness was equal to this corresponding to what, corresponding to u x and u theta defined at the degrees of at the center point. Let me now try to look at situation where, I have the same problem and I want to show you that, this is indeed that the mass and stiffness matrices are indeed not property matrices, but they are property matrices corresponding to the degrees of freedom. This is the very important point to be noted and the way I will do it is by solving the same problem by defining degrees of freedom somewhere else, as long in the understand this is the 2 degree of freedom problem.

(Refer Slide Time: 33:57)



I then say this that look, I have the same 1 2 K K p 1 p 2 right and now instead of and this of course, I is m bar uniform, now instead of defining degrees of freedom at the center u x and u theta, I will define u 1 and u 2, these are 2 degrees of freedom note that, it cant move in this direction. How is this different well, you see when I put u 1 equal to 1 and u 2 equal to 0, what kind of it displacement pattern do I get, I get u 1 equal to 1 u 2 equal to 0, this is the displacement pattern.

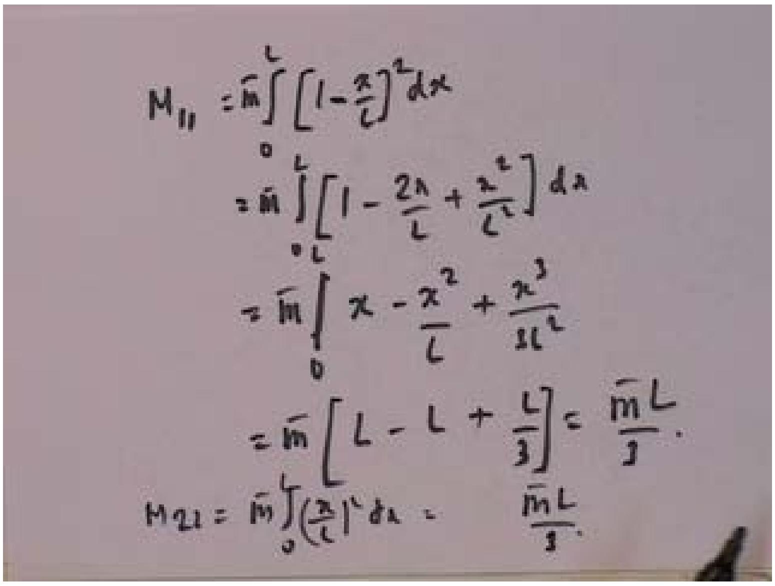
(Refer Slide Time: 36:00)



This is my psi 1 and when I put u 1 equal to 0 and u 2 equal to 1, this becomes 1 1, this is my psi 2, you see as soon is that define my degrees of freedom very differently, my psi 1 and psi 2 become very, very different. And so when so psi 1 and psi 2 become very, very different, what happens well you know, I am not going to waste to much of time on getting this all, I am going to do is you know, what is my psi 1, I am not going to define my x not from the center as I done the last time, I am going to define my x from here.

So, if I define my x from here, psi 1 is equal to 1 minus x up on L and psi 2 is equal to x by L. So, I know my m bar, my M 1 1 what was it M 1 1 is due to thus thing and we saw that, if you look at it, this is equal to 0 by L m bar psi 1 squared d x plus M 1 1 look back, I am you know, I am already derived it, look back and you will see that, M 1 1 u x in the first equation, if you look at it was m bar psi 1 x squared. And if I look at this, this is going to be and then if I look at M 1 2 and M 2 1 look at them, it is going to be 0 to L m bar psi 1 psi 2 d x and M 2 2 is equal to 0 to L m bar psi 2 square d x.

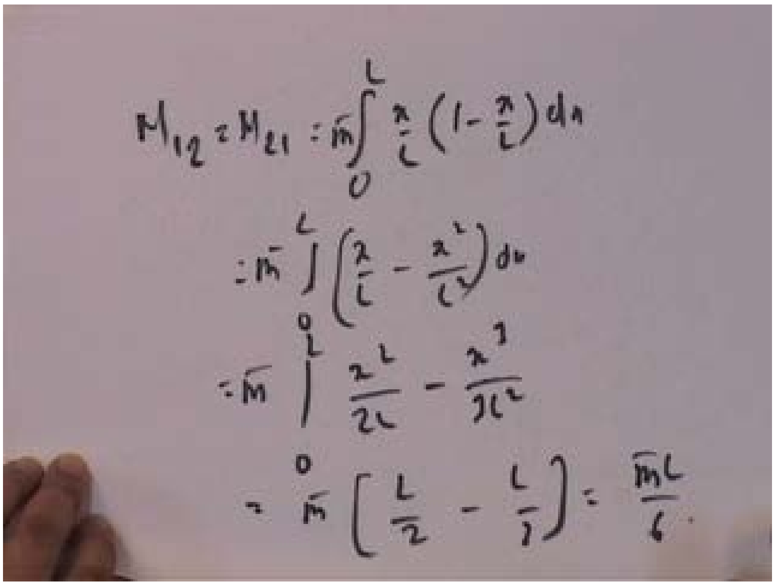
(Refer Slide Time: 37:25)



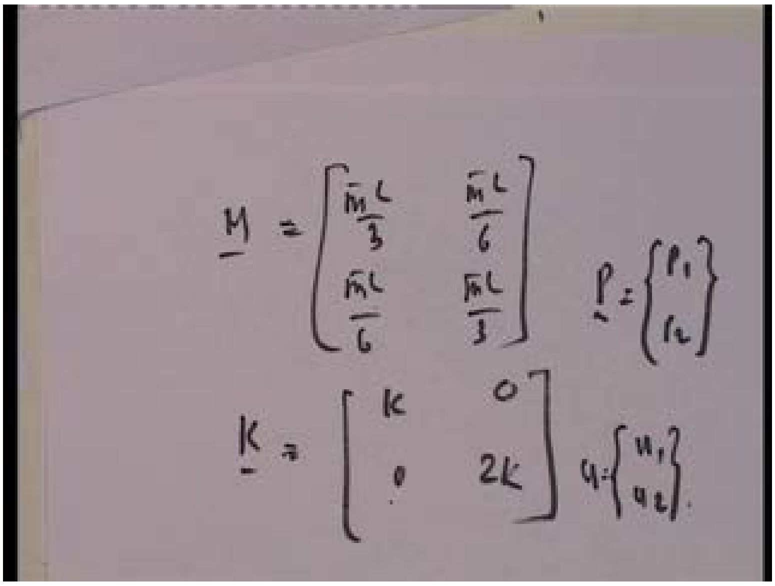
So, now given these facts, let us find out what M 1 1 and M 1 1 is equal to 0 to L, I can put m bar outside and inside, I get 1 minus x upon L whole squared, that is what 1 minus x upon L, the whole square. So, how do I get that then you know if I can do this, derive this m bar 0 to L 1 minus 2 x by L plus x upon L squared d x. So, this is equal to m bar, so this is going to be equal to what 1, so that is going to be equal to 0 to L, this is going to be x, this is going to be x squared up on L and this is going to be x cubed upon 3 L squared 0 to L, the all of them have 0.

So, the other one has L, so this becomes m bar L minus this become x squared, so this become L and the here, I get L cubed upon 3 L squared, so this becomes what, L upon 3. So, I get my equation to the m bar L upon 3 do M 2 2, which is m bar 0 to L x upon L, the whole squared d x and you will see that, this is also equal to m bar L upon 3.

(Refer Slide Time: 39:22)



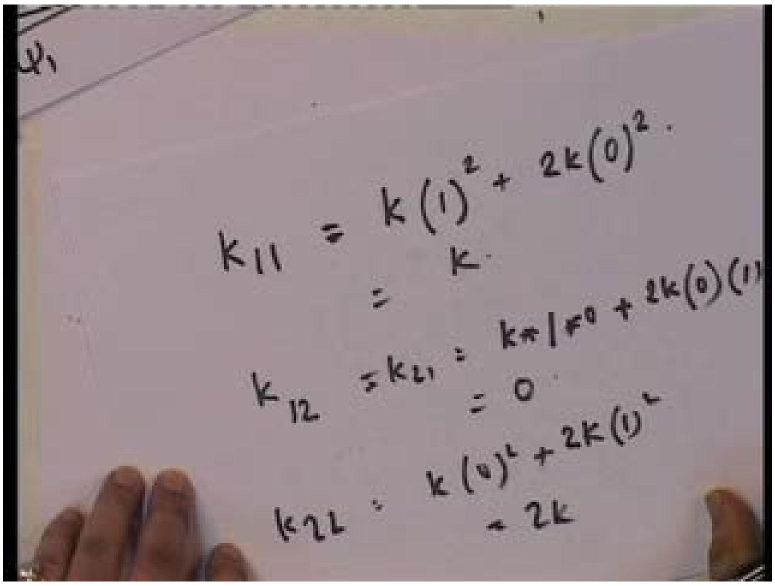
(Refer Slide Time: 40:15)



Let us derive m bar 1 2 and 2 1 both of them are the same m bar 1 2, which is equal to m bar 2 1 is equal to m bar outside 0 to L x by L psi 1 and psi right minus x by L d x. So, this becomes m bar 0 to L x by L minus x squared upon L squared d x, so this is equal to m bar 0 to L x squared upon 2 L minus x cubed upon 3 L squared, so this basically, becomes m bar L upon 2 minus L upon 3, so this basically becomes m bar L upon 6.

So, if you look at my mass matrix, my mass matrix for this thing looks like m bar L by 3 m bar L by 6 m bar L by 6 m bar L by 3 completely different from the previous mass matrix isn't it. And so therefore, the mass is. So, I have proved that the mass matrix is indeed, this thing and then let us look at the following and that is let us get the K. Now for the K, lets look back what are the thing that I require for K, so let us look at that, if you look at this one right. So, if I do what is my load, if I have a random thing, it is going to be equal to.

(Refer Slide Time: 41:11)



So, my K 1 1 is going to be equal to load this and this you know virtual displacement this 2. So, K 1 1 is going to be K here, if you look at it, it is going to be equal to K into 1 squared plus 2 K into 0 squared. So, that is equal to K 1 2, which is equal to K 2 1 is equal to this is the real 1, so this is going to be K into 1 into what, let us look at this into this. So, this is going to be 0 plus 2 K into this, which is 0 into this, which is 1, so what is K 1 to 0 and K 2 2 well, it is going to be equal to K into 0 0 squared plus 2 K into 1 squared.

So, that is equal to 2 K, so what does the stiffness matrix look like, if you look at this the stiffness matrix looks like, K is equal to K 0 0 2 K, interesting indeed the previous 1 at the mass matrix uncoupled and the stiffness matrix coupled. And here, the mass matrix and stiffness matrix are uncoupled and if I look at the p vector well, I am multiply each 1, so in 1, I get p 1 into 1 plus p 2 into 0, so that is p 1 then the other one, I get p 1 into 0 plus p 2 into 1, so this is my p matrix.

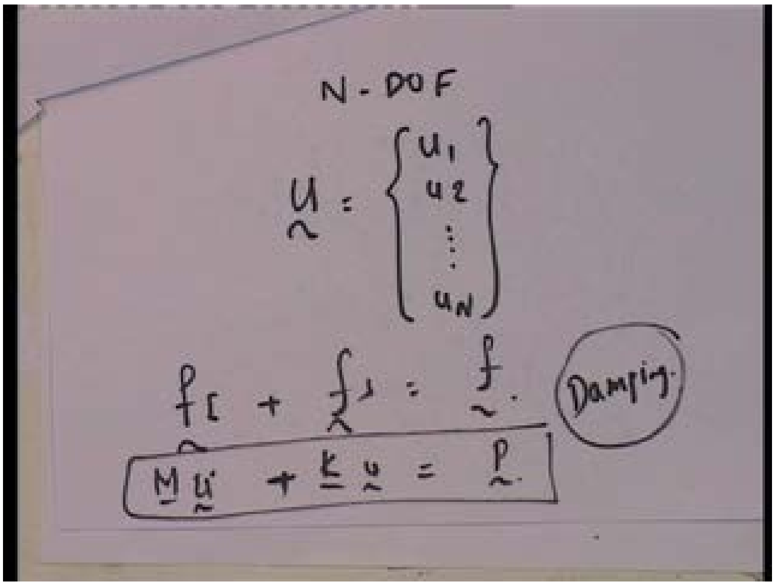
And obviously, my u's in this particular case are u 1 and u 2, so the mass stiffness and load vector are completely dependent on how, I have defined my degree of freedom. When I defined it at the center of at the center, geometric center with u x and u theta, I got m bar L, m bar L cube by 12 0 0 and I got what did I get here 3 K K L by 2 K L by 2 3 K L squared by 4 and p's where, p 1 plus p 2 and the other one was p 1 L by minus p 1 L by 2 plus p 2 L by 2.

Completely different, that the what is that mean does that mean that, these are 2 completely different equations of motion, well later on I will show how, we will come back to these problems and I will show you that indeed again. We used 2 different degree you know, degrees of freedom independent degree of freedom set, they seem to look like generating completely different equations of motion. So, therefore, the mass this mass matrix, the stiffness matrix and the load vector are property matrices, but they are not invariants, they are corresponding to the degree of freedom.

And but you know just the degree of freedom define different things, we will see later on when, I solve multi degree of freedom problems, you know dynamic problems I will show you and we will come back to this problem, to show you that know, this is not they do not give me different solution. Only thing is that, I will get u x t and u theta t from that previous equation in this one, I will get u 1 t and u 2 of t.

And they will not be identical how can, they are 2 different degrees of freedom, but you know you look at u 1, what is u 1 equal to in terms of u x and u theta u 1 is equal to u x minus L by 2 u theta and u 2 is equal to u x plus L by 2 u theta. So, incorporate that and you put them in, you will see u 1 and u 2 are exactly, the same whether, we use u x and u theta as my degrees of freedom or whether, use my u 1 and u 2 as my degrees of freedom, that is what matters. So, overall what we seen is that, we have seen that ultimately, a multi degree of freedom and now I am going to define, this in a fundamental sense a multi degree of freedom and N degree of freedom.

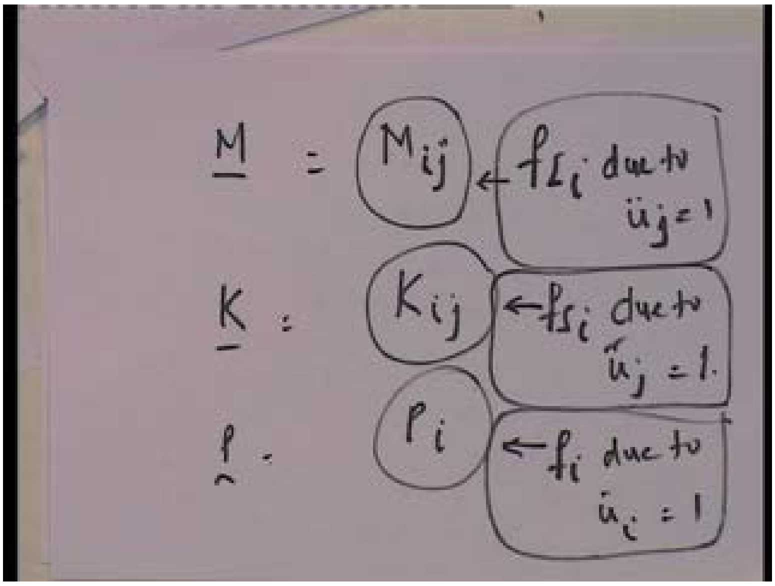
(Refer Slide Time: 46:26)



Suppose, I have an N degree of freedom structure, its defined by a displacement vector or note the displacement vector, last time when I defined u x and u theta, that was the degree of freedom displacement vector, but only thing is that u theta was the rotational displacement you see. So, it does not matter what I define, so u 2 u N degrees of freedom. So, I need to define all these, I need to find out all this displacements to be able to define, the displacement of the structure.

And then I have f I vector plus f s vector is equal to the outside vector and so that, this one is equal to m into u double dot plus K into u is equal to p. Of course, note that, I have not defined damping, well we will as I said that m and K are property matrices damping is a completely different thing, we will see how, we look at multi degree of freedom problems. So, ultimately what we see is that, this becomes our overall definition for a multi degree of freedom system and the entire problem then becomes nothing but defining.

(Refer Slide Time: 48:11)



The mass matrix and all in mass matrix all, I need to do is find out M 1 M i j and if I find out if M i j, each of them I got the mass matrix K, I just to need find out K i j and p, I need to find out p i. These are once I find, so the entire problem then becomes not for every complicated problem, I do not need to have to write down the entire equation, develop the equation, all I need to do is find out the elements of the mass matrix.

The elements of the stiffness matrix and the elements of the load vector and if you look at this, what is this? This is the inertial force corresponding to displacement degree of freedom degree u i, due to a unit displacement, I am sorry, unit acceleration. So, this is the force corresponding to degree of freedom i, due to a unit acceleration corresponding to j, that is this. So, all we need to do is find out f at i due to unit, that is the whole problem, this one f s at i due to sorry, no there is nothing this is displacement unit displacement and this well this is the load at i due to unit displacement i.

So, this becomes this problem, so ultimately this is what the problem boils down to and once, we find out each one of these, how do I find this out well, I give a unit see the whole thing becomes a very simple problem. A problem becomes give a unit acceleration and how do I find out a f i well give a virtual displacement corresponded to i and find out all the work done by the inertial force and I have got got it. Similarly, I can find this out by how do I find this out, well give it a unit displacement find out all the elastic forces and then give a virtual displacement corresponding to i and find out the work done by all this spring forces elastic forces and that is your k i j.

So, the entire and what is f i well, I give unit displacement corresponding to this and find out the work done, u virtual displacement corresponding to this and find out the work done by all the applied loads if I do that I am done. So, I have broken down the entire problem into small small parts, which can actually be solved and next time onwards, I am going to show, you how this can be done for more complex structures, I started with a very very simple structure to establish the fact 2 establish 2.1.

The fact that M what is M i j, another fact that M i j is equal to m j i, K i j is equal to k j i, so therefore, the mass and stiffness matrix is a symmetric, that is one thing that, we know and second that mass K and p are corresponding to degrees of freedom, they are not in variant matrices. So, this is very different from the previous one where, M and K were property, here this correspond to the degrees of freedom.

And in this way the very similar to the M star and K star and p star, that we generated for the generalizing single degree of freedom system, you see how it slowly building up building up the blocks to gum to it. And from here on out all I am going to do is find out m i j k i j p i and that is all, I have got my mass matrix, stiffness matrix, p matrix and then I have got my equation of motion, which is given by M u double dot plus K u is equal to p and so I have got my differential equation, which are them solved.

Thank you see you next time, bye, bye.